Single Server Retrial Queueing System with Feedback Using Matrix Geometric Approach

M. Seenivasan[1, a *] and J. Shiny Epciya[2, b]

[1]Mathematics Wing – DDE, Annamalai University, Annamalainagar – 608002, Tamilnadu, India

[2]Research Scholar, Department of Mathematics, Annamalai University, Annamalainagar, Tamilnadu, India

Corresponding Author: [1, a *]emseeni@yahoo.com; orcid ID: 0000-0003-1475-923X

[2, b] epciyashinydoss@gmail.com

Article Info

Page Number: 4254- 4267

Publication Issue:

Vol 71 No. 4 (2022)

Abstract

In the present study, we examined Single Server retrial queue with Feedback. Arrival follows Poisson process. An arrival finds the system is full, the arrival enters into an orbit. From the orbit the customers try their luck. The time between two successive retrials called retrial time, it follows negative exponential distribution. After getting a service, incomplete or unsatisfied customer choose to join the orbit for another service is called feedback or after service completion he leaves the system forever. This kind of model examined utilizing Matrix Geometric Approach (MGA). We also provide some numerical examples by taking particular values to the parameters. Using that probability vectors, we also derived some performance analysis.

Article History

Article Received: *25 March 2022*

Revised: *30 April 2022*

Accepted: *15 June 2022*

Publication: *19 August 2022*

Keywords: Retrial queue; Arrival rate; Feedback; Matrix Geometric Approach (MGA).

Introduction

Queueing systems has been examined for more than twenty years as a very useful tool for modeling and analyzing computer systems, communication networks, manufacturing, industrial organizations, production system and many other. In queueing theory in which customer arrives who finds the severs and waiting places are engaged, may retry for service after an irregular measure of time is called retrial queue. A survey of retrial queues was explored by Falin (1990).

Artalejo (2012) determined the M/M/1 retrial queue with finite population. The detailed information of M/M/1 retrial queues with pre-emptive priority service has been obtained Ayyappan et al (2010). Choi and Chang (1999) examined M/M/1retrial queues with priority calls. Aissani and Artalejo (1998) studied M/M/1retrial queue with breakdown.

The unsatisfied customer goes to the orbit for another service is called feedback. This model has been investigated by Choi, et al (1998) analyzed multiserver retrial queue with feedback& loss. Choi and Kulkarni (1992) explored feedback retrial queueing model. Chang et al (2018) studied on retrial queue with unreliable-server & feedback. Single working vacation with feedback using MGA was explored by Seenivasan and Abinaya (2021). Fig. 1 indicates the structure of this queueing system.

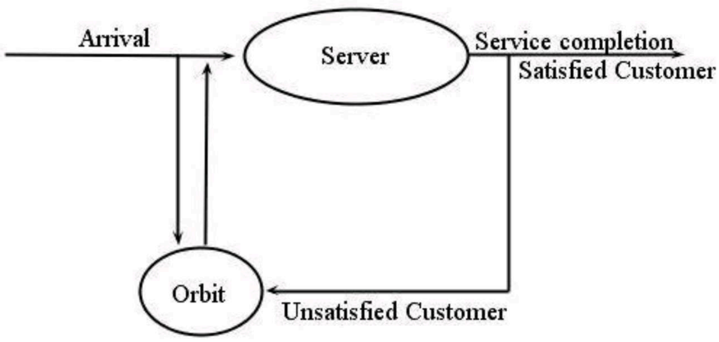

Fig. 1 Structure of the queueing model

Matrix Geometric Approach is awesome key for detecting some risky queue obstacle in queueing frameworks. Neuts (1994) investigated an MGA in stochastic process. A retrial queue with two heterogeneous servers using MGA was researched by Seenivasan et al (2021). Ramesh and

Kumaraguru (2014) investigated Performance measures of retrial queueuing model with fuzzy parameters using robust ranking technique.

In this paper, we analyzed M/M/1 retrial queue model with feedback. This queue has been examined by MGA. Arrival follows FCFS schedule. The remaining of our article is structured as follows. In 2nd division model description is given. In 3rd division some mathematical illustrations are derived. In 4th division performance analysis has been presented. In final division short conclusion about the model.

Model Description

In this article, we concentrated on single server retrial queueing model with feedback. Arriving customer follows Poisson process with rate λ. Assuming that the server is free, the arriving customer served right away and if the server is occupied, he will join the orbit. In retrial, each customer is viewed as equivalent to a primary customer. Assuming he observes the inactive server, he accepts his service right away. The retrial time is exponentially distributed with rate ν and the service time of customers are exponentially distributed with rate μ. Assuming that the served customer choose to leaves the system forever with probability $\overline{\beta}$ or he joins the orbit for another service with probability β (it is called feedback). Where $(\beta+\overline{\beta} = 1)$. The rate transition diagram is shown in Fig. 2

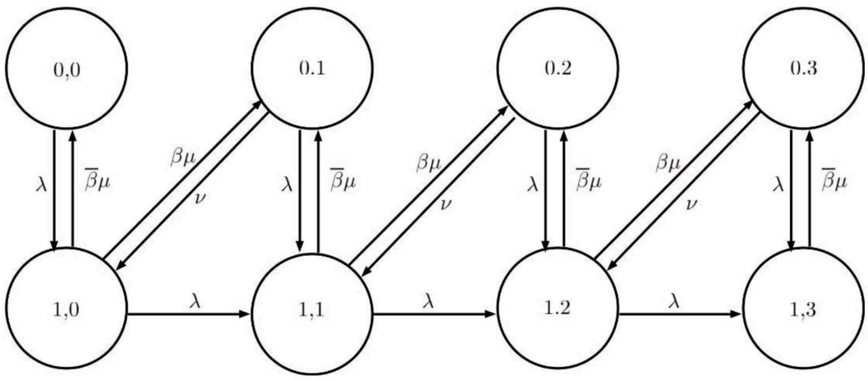

Fig. 2 Transition rate diagram

Let $J(t),\ N(t):\ t \geq 0$ be a MP, where $J(t)$ and $N(t)$ addresses state of process at t.

$J(t) = 0$, if server is idle.

$J(t) = 1$, if server is on ordinary working period

N(t) represents customer present in the orbit.

Lexicographical series given by:

$\Omega = (0,0)\ U\ (1,0)\ U\ (2,0)\ U\ (i,j);\ i = 0,1\ \&\ j = 0,1,2, \dots ,\ n \geq 1.$

Infinitesimal generator matrix Q:

$$Q = \begin{pmatrix} C_{00} & D_{00} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ E_{00} & F_{00} & D_{00} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & E_{00} & F_{00} & D_{00} & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & E_{00} & F_{00} & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & E_{00} & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Where

$$C_{00} = \begin{pmatrix} -(\lambda) & \lambda \\ \bar{\beta}\mu & -(\lambda + \beta\mu + \bar{\beta}\mu) \end{pmatrix}; D_{00} = \begin{pmatrix} 0 & 0 \\ \beta\mu & \lambda \end{pmatrix};$$

$$E_{00} = \begin{pmatrix} 0 & \nu \\ 0 & 0 \end{pmatrix};\ F_{00} = \begin{pmatrix} -(\lambda + \nu) & \lambda \\ \bar{\beta}\mu & -(\lambda + \beta\mu + \bar{\beta}\mu) \end{pmatrix}.$$

We define $\pi_{ij} = \{J = i, N = j\} = \lim_{t\to\infty} \Pr\{J(t) = i, N(t) = j\}$,

where j indicates no. of customer in the orbit and i indicates the server state.

The stationary probability vector Π is given by Π= $(\pi_0, \pi_1, \pi_{2,\dots})$ and $\pi_j = (\pi_{0j}, \pi_{1j})$, j=0,1,2,3, …

The static probability row matrix is addressed by using $\Pi Q = 0$.

$$\pi_0 C_{00} + \pi_1 E_{00} = 0 \tag{1}$$

$$\pi_0 D_{00} + \pi_1 F_{00} + \pi_2 E_{00} = 0 \tag{2}$$

$$\pi_1 D_{00} + \pi_2 F_{00} + \pi_3 E_{00} = 0 \tag{3}$$

.

.

.

$$\pi_i D_{00} + \pi_{i+1} F_{00} + \pi_{i+2} E_{00} = 0 \quad (4)$$

And $\pi_j = \pi_0 R^j$ for $j \geq 1$. (5)

Let us assume R be a rate matrix.

Using matrix equation [Neuts (1994)] it is irreducible non-negative solution.

$$\pi_0 [C_{00} + \mathrm{R} E_{00}] = 0 \quad (6)$$

$$\pi_0 [R^2 E_{00} + R F_{00} + D_{00}] = 0 \quad (7)$$

The balancing mathematical statement is

$$\pi_0 [I - R]^{-1} e = 1 \quad (8)$$

Where 'e' is section unit vector with every one of its components equivalent to one.

$$R = -F_{00}^{-1} [D_{00} + R^2 E_{00}] \quad (9)$$

The matrix R is numerically computed by using iterative procedure with $R(0) = 0$

$$R_{n+1} = -F_{00}^{-1} [D_{00} + R_n^2 E_{00}]$$

Utilizing the above condition to solve R matrix and check all values of R will monotonically expanding. Furthermore, R will converge to $-F_{00}^{-1}$and $[D_{00} + R_n^2 E_{00}]$are positive. By considering matrix geometric procedure the steady state is obtained.

Π partitioned as Π= (π_0, π_1,π_2) is a stationary probability vector of the (reducible) matrix generator from G = D_{00} +F_{00} +E_{00}. Then G can be written as

$$G = \begin{pmatrix} -(\lambda + \nu) & \lambda + \nu \\ \mu & -\mu \end{pmatrix} \quad (10)$$

G is clear and Π can be shown to be solitary such that ΠG = 0 and Πe =1.

$$\pi_1 = \left(\frac{\lambda+\nu}{\mu}\right) \pi_0 \text{ and } \pi_0 = \left(1 + \frac{\lambda+\nu}{\mu}\right)^{-1}$$

Mathematical Study

Presently we provide mathematical outcomes of our model is about in before segment. Our goal is to show the impact of the parameter on organization attributes. On fluctuating $\boldsymbol{\lambda}$ and fixed all remaining rates we tracked down probability vectors, in the underneath cases.

Case 1. Assume that $\lambda = 0.1$, $\mu = 1$, $\beta = 0.4$, $\bar{\beta} = 0.6$, $\nu = 0.05$ and $R = \begin{pmatrix} 0.3810 & 0.2691 \\ 0.5714 & 0.2542 \end{pmatrix}$

Table 1**:** Probability vectors for λ=0.1

Π_j	π_{0j}	π_{1j}	Total
π_0	0.2538	0.0423	0.2961
π_1	0.1209	0.0791	0.2000
π_2	0.0912	0.0526	0.1438
π_3	0.0648	0.0379	0.1027
π_4	0.0464	0.0271	0.0735
π_5	0.0331	0.0194	0.0525
π_6	0.0231	0.0138	0.0369
π_7	0.0169	0.0099	0.0268
π_8	0.0121	0.0071	0.0192
π_9	0.0087	0.0051	0.0138
π_{10}	0.0062	0.0036	0.0098
π_{11}	0.0044	0.0026	0.0070
π_{12}	0.0032	0.0018	0.0050
π_{13}	0.0023	0.0013	0.0036
π_{14}	0.0016	0.0009	0.0025
π_{15}	0.0012	0.0007	0.0019
π_{16}	0.0008	0.0005	0.0013
π_{17}	0.0006	0.0003	0.0009
π_{18}	0.0004	0.0002	0.0006
π_{19}	0.0003	0.0002	0.0005
π_{20}	0.0002	0.0001	0.0003
π_{21}	0.0002	0.0001	0.0003
π_{22}	0.0001	0.0001	0.0002
π_{23}	0.0001	0.0000	0.0001
π_{24}	0.0001	0.0000	0.0001
π_{25}	0.0000	0.0000	0.0000
Total			0.9994

The above table represent the probability vectors which has been obtained from using R matrix in Eq. 6 and Eq. 8, we get Π_0= (0.2538, 0.0423). Using that Π_0 in Eq. 5 all other following vectors are obtained. Overall prob. was validated to be 0.9994 ≈ 1.

Case 2. Assume that $\lambda = 0.15$, $\mu = 1$, $\beta = 0.4$, $\bar{\beta} = 0.6$, $\nu = 0.05$ and $R = \begin{pmatrix} 0.4286 & 0.3387 \\ 0.5714 & 0.3259 \end{pmatrix}$

Table 2**:** Probability vector for λ=0.15

Π_j	π_{0j}	π_{1j}	Total
π_0	0.1476	0.0369	0.1845
π_1	0.0843	0.0620	0.1463
π_2	0.0716	0.0488	0.1204
π_3	0.0586	0.0401	0.0987
π_4	0.0480	0.0329	0.0809
π_5	0.0394	0.0270	0.0664
π_6	0.0323	0.0221	0.0544
π_7	0.0265	0.0182	0.0447
π_8	0.0217	0.0149	0.0366
π_9	0.0178	0.0122	0.0300
π_{10}	0.0146	0.0100	0.0246
π_{11}	0.0120	0.0082	0.0202
π_{12}	0.0098	0.0067	0.0165
π_{13}	0.0081	0.0055	0.0136
π_{14}	0.0066	0.0045	0.0111
π_{15}	0.0054	0.0037	0.0091
π_{16}	0.0044	0.0030	0.0074
π_{17}	0.0036	0.0025	0.0061
π_{18}	0.0030	0.0021	0.0051
π_{19}	0.0025	0.0017	0.0042
π_{20}	0.0020	0.0014	0.0034
π_{21}	0.0017	0.0011	0.0028
π_{22}	0.0014	0.0009	0.0023
π_{23}	0.0011	0.0008	0.0019
π_{24}	0.0009	0.0006	0.0015
π_{25}	0.0007	0.0005	0.0012
π_{26}	0.0006	0.0004	0.0010
π_{27}	0.0005	0.0003	0.0008
π_{28}	0.0004	0.0003	0.0007
π_{29}	0.0003	0.0002	0.0005
π_{30}	0.0003	0.0002	0.0005
π_{31}	0.0002	0.0002	0.0004
π_{32}	0.0002	0.0001	0.0003

π_{33}	0.0002	0.0001	0.0003
π_{34}	0.0001	0.0001	0.0002
π_{35}	0.0001	0.0001	0.0002
π_{36}	0.0001	0.0001	0.0002
π_{37}	0.0001	0.0000	0.0001
π_{38}	0.0001	0.0000	0.0001
π_{39}	0.0000	0.0000	0.0000
Total			0.9992

The above table represent the probability vectors which has been obtained from using R matrix in Eq. 6 and Eq. 8, we get Π_0= (0.1476, 0.0369). Using that Π_0 in Eq. 5 all other following vectors are obtained. Overall prob. was validated to be 0.9992 ≈ 1.

Case 3. Assume that λ = 0.2, μ = 1, β = 0.4, $\bar{\beta}$ = 0.6, ν = 0.05 and $R = \begin{pmatrix} 0.4444 & 0.3847 \\ 0.5556 & 0.3781 \end{pmatrix}$

Table 3**:** Probability vector for λ=0.2

Π_j	π_{0j}	π_{1j}	Total
π_0	0.0957	0.0319	0.1276
π_1	0.0603	0.0489	0.1092
π_2	0.0539	0.0417	0.0956
π_3	0.0471	0.0365	0.0836
π_4	0.0412	0.0319	0.0731
π_5	0.0361	0.0279	0.0640
π_6	0.0315	0.0244	0.0559
π_7	0.0276	0.0214	0.0490
π_8	0.0241	0.0187	0.0428
π_9	0.0211	0.0164	0.0375
π_{10}	0.0185	0.0143	0.0328
π_{11}	0.0162	0.0125	0.0287
π_{12}	0.0141	0.0109	0.0250
π_{13}	0.0124	0.0096	0.0220
π_{14}	0.0108	0.0084	0.0192
π_{15}	0.0095	0.0073	0.0168
π_{16}	0.0083	0.0064	0.0147
π_{17}	0.0072	0.0056	0.0128
π_{18}	0.0063	0.0049	0.0112
π_{19}	0.0055	0.0043	0.0098
π_{20}	0.0048	0.0038	0.0086
π_{21}	0.0042	0.0033	0.0075
π_{22}	0.0037	0.0029	0.0066

π_{23}	0.0032	0.0025	0.0057
π_{24}	0.0028	0.0022	0.0050
π_{25}	0.0025	0.0019	0.0044
π_{26}	0.0022	0.0017	0.0039
π_{27}	0.0019	0.0015	0.0034
π_{28}	0.0017	0.0013	0.0030
π_{29}	0.0015	0.0011	0.0026
π_{30}	0.0013	0.0010	0.0023
π_{31}	0.0011	0.0009	0.0020
π_{32}	0.0010	0.0008	0.0018
π_{33}	0.0009	0.0007	0.0016
π_{34}	0.0007	0.0006	0.0013
π_{35}	0.0007	0.0005	0.0012
π_{36}	0.0006	0.0004	0.0010
π_{37}	0.0005	0.0004	0.0009
π_{38}	0.0004	0.0003	0.0007
π_{39}	0.0004	0.0003	0.0007
π_{40}	0.0003	0.0003	0.0006
π_{41}	0.0003	0.0002	0.0005
π_{42}	0.0003	0.0002	0.0005
π_{43}	0.0002	0.0002	0.0004
π_{44}	0.0002	0.0002	0.0004
π_{45}	0.0002	0.0001	0.0003
π_{46}	0.0001	0.0001	0.0002
π_{47}	0.0001	0.0001	0.0002
π_{48}	0.0001	0.0001	0.0002
π_{49}	0.0001	0.0001	0.0002
π_{50}	0.0001	0.0001	0.0002
π_{51}	0.0001	0.0001	0.0002
π_{52}	0.0001	0.0001	0.0002
π_{53}	0.0001	0.0000	0.0001
π_{54}	0.0001	0.0000	0.0001
π_{55}	0.0000	0.0000	0.0000
Total			0.9998

The above table represent the probability vectors which has been obtained from using R matrix in Eq. 6 and Eq. 8, we get Π_0= (0.0957, 0.0319). Using that Π_0 in Eq. 5 all other following vectors are obtained. Overall prob. was validated to be 0.9998 ≈ 1.

Case 4. Assume that $\lambda = 0.25$, $\mu = 1$, $\beta = 0.4$, $\bar{\beta} = 0.6$, $\nu = 0.05$ and $R = \begin{pmatrix} 0.4444 & 0.4206 \\ 0.5333 & 0.4203 \end{pmatrix}$

Table 4**:** Probability vector for λ=0.25

Π_j	π_{0j}	π_{1j}	Total
π_0	0.0897	0.0374	0.1271
π_1	0.0606	0.0486	0.1092
π_2	0.0540	0.0417	0.0957
π_3	0.0472	0.0365	0.0837
π_4	0.0413	0.0320	0.0733
π_5	0.0361	0.0280	0.0641
π_6	0.0316	0.0245	0.0561
π_7	0.0276	0.0214	0.0490
π_8	0.0242	0.0187	0.0420
π_9	0.0211	0.0164	0.0375
π_{10}	0.0185	0.0143	0.0328
π_{11}	0.0162	0.0125	0.0287
π_{12}	0.0141	0.0110	0.0251
π_{13}	0.0124	0.0096	0.0220
π_{14}	0.0108	0.0084	0.0192
π_{15}	0.0095	0.0073	0.0168
π_{16}	0.0083	0.0064	0.0147
π_{17}	0.0072	0.0056	0.0128
π_{18}	0.0063	0.0049	0.0112
π_{19}	0.0055	0.0043	0.0098
π_{20}	0.0049	0.0038	0.0087
π_{21}	0.0042	0.0033	0.0075
π_{22}	0.0037	0.0029	0.0066
π_{23}	0.0032	0.0025	0.0057
π_{24}	0.0028	0.0022	0.0050
π_{25}	0.0025	0.0019	0.0044
π_{26}	0.0022	0.0017	0.0039
π_{27}	0.0019	0.0015	0.0034
π_{28}	0.0017	0.0013	0.0030
π_{29}	0.0015	0.0011	0.0026
π_{30}	0.0013	0.0010	0.0023
π_{31}	0.0011	0.0009	0.0020
π_{32}	0.0010	0.0008	0.0018
π_{33}	0.0009	0.0007	0.0016
π_{34}	0.0007	0.0006	0.0013
π_{35}	0.0007	0.0005	0.0012
π_{36}	0.0006	0.0004	0.0010
π_{37}	0.0005	0.0004	0.0009

π_{38}	0.0004	0.0003	0.0007
π_{39}	0.0004	0.0003	0.0007
π_{40}	0.0003	0.0003	0.0006
π_{41}	0.0003	0.0002	0.0005
π_{42}	0.0003	0.0002	0.0005
π_{43}	0.0002	0.0002	0.0004
π_{44}	0.0002	0.0002	0.0004
π_{45}	0.0002	0.0001	0.0003
π_{46}	0.0001	0.0001	0.0002
π_{47}	0.0001	0.0001	0.0002
π_{48}	0.0001	0.0001	0.0002
π_{49}	0.0001	0.0001	0.0002
π_{50}	0.0001	0.0001	0.0002
π_{51}	0.0001	0.0001	0.0002
π_{52}	0.0001	0.0001	0.0002
π_{53}	0.0001	0.0000	0.0001
π_{54}	0.0001	0.0000	0.0001
π_{55}	0.0000	0.0000	0.0000
Total			0.9994

The above table represent the probability vectors which has been obtained from using R matrix in Eq. 6 and Eq. 8, we get Π_0= (0.0897, 0.0374). Using that Π_0 in Eq. 5 all other following vectors are obtained. Overall prob. was validated to be 0.9994 ≈ 1.

Performance Analysis

Performance analysis are found from steady-state probabilities, and it's given below

Prob. mass function of server idle:

$$\text{PMSI}=\sum_{j=0}^{\infty} \pi_{0,j} \quad (11)$$

Prob. mass function of server busy:

$$\text{PMSB}=\sum_{j=0}^{\infty} \pi_{1,j} \quad (12)$$

Prob. mass function that the orbit has no customer:

$$\text{PNCO}=\sum_{i=0}^{1} \pi_{i,0} \quad (13)$$

Mean number of customers in the orbit:

$$\text{MNCO}=\sum_{j=0}^{\infty} \sum_{i=0}^{1} j\pi_{i,0} \quad (14)$$

Table 5. Performance analysis.

λ	0.1	0.15	0.2	0.25
PMSI	0.6927	0.6288	0.5859	0.5808
PMSB	0.3067	0.3704	0.4139	0.4195
PNCO	0.2961	0.1845	0.1276	0.1271
MNCO	2.4615	4.5137	6.8774	6.9451

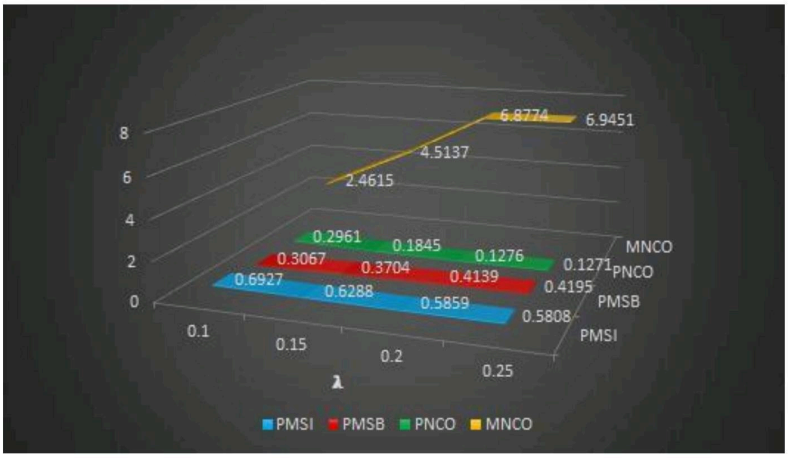

Fig. 3 Performance Analysis with the effect of λ

By varying arrival rate **λ** we derived probability vectors. Using the given equations, From Eq. 11 to Eq. 14, we get some performance analysis. It shows that if arrival rate increases, then Prob. mass function of server idle and Prob. mass function that the orbit has no customer are decreases. And if arrival rate increases, then Prob. mass function of server busy and mean number of customers in the orbit are increases. These are clearly shown in Fig. 3.

Summary

In this queueing model, we have analysed single server retrial queue with feedback. We found the probability vectors using some particular parameter by utilizing MGA. Such as Prob. mass function of server idle, Prob. mass function of server busy, Prob. mass function that the orbit has no customer and mean number of customers in the orbit.

References

[1] Aissani and J. R. Artalejo, On the single server retrial queue subject to breakdown, Queueing Sys., 30, 309-321, (1998).

[2] J. R. Artalejo and M. J. Lopez-Herrero, The single server retrial queue with finite population: A BSDE approach, (2012).

[3] G. Ayyappan, A. Muthu Ganapathi Subramanian and Gopal Sekar, M/M/1 Retrial queueing System with Pre-emptive priority service, International Journal of Computer Applications, Vol.2, 2010, no.6, 0975-8887, (2010).

[4] Chang Fu-Min, Liu Tzu-Hsin and Ke Jau-Chuan, On an unreliable server retrial queue with customer feedback and impatience, Appl.Math. modeling., vol.55, pp.171-182(2018).

[5] B.D. Choi and Y. Chang, Single server retrial queues with priority calls, Math.Comput. Model.,30 (3-4), pp. 7-32, (1999).

[6] B.D. Choi, Y. C. Kim and Y. W. Lee, The M/M/c retrial queue with Geometric loss and Feedback, Computers Math. Applic. vol.36, pp.41-52, (1998).

[7] B. D. Choi and V. G. Kulkarni, Feedback retrial queueing system, Stochastic Model Relat. Fields pp. 93-105, (1992).

[8] G. I. Falin, A survey of retrial queues, Queueing Syst.,7, pp. 127-168, (1990).

[9] M. F. Neuts, Matrix-Geometric solutions in stochastic models, vol.2of johns Hopkins's series in the mathematical series, Johns Hopkins University Press, Baltimore, USA, (1981).

[10] M. Seenivasan and R. Abinaya, Markovian queueing model with single working vacation and feedback, J. Computational Analysis & Applications, vol. 30, No. 2, pp. 210-221(2021).

[11] M. Seenivasan, G. Rajarajan, and M. Indumathi, Retrial queueing model with two heterogeneous servers using the Matrix Geometric Method, Mater SciEng, Vol.1070, (2021).

[12] R. Ramesh and S. Kumaraguru, Performance measures of retrial queueuing model with fuzzy parameters using robust ranking technique, international journal of advanced and innovative research, vol 3, No. 4, pp. 62-66, (2014).