Distance Measure Approaches to Rank Interval-Valued Trapezoidal Intuitionistic Fuzzy Sets

S.N.MurtyKodukulla[#1], V. Sireesha[*2], V Anusha[#3]

Dept. Of Mathematics, GITAM Institute of Science, GITAM (Deemed to be University), Visakhapatnam, 530045

[1] snmurtyk@gmail.com, [2*]vsirisha80@gmail.com, sveerama@gitam.edu, anushavulimiri@gmail.com[3]

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Abstract

Similarity/Distance measures are proven to be potential in evaluating uncertain information. It compares the objects with ambiguous and imprecise feature by measuring the degree of deviation of objects. It is observed that the existing Euclidean distance measure on Interval-Valued Trapezoidal Intuitionistic Fuzzy Sets (IVTrIFSs) is failing to discriminating fuzzy sets in some cases. This paper proposes, a modified Euclidean distance measure by redefining the terms of non-memberships in the existing ED. Further a new distance measure-Jaccard distance is proposed by using the modified Euclidean distance. The desirable properties of the measure have been proven. Numerical examples are provided to demonstrate the applicability of the distance measure. Comparative study is done. The results show that the proposed distance measures effectively ranking IVTrIFSs and the ranking is close to human intuition.

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Introduction

Making decisions involves growing amounts of uncertainty and vagueness [1]. Intuitionistic fuzzy sets (IFSs) defined by Atanassov [2] are one among widely used sets to model the uncertainity. Numerous generalisations to IFSs have been proposed since their inception. These generalisations include the interval-valued intuitionistic fuzzy sets (IVIFSs), the triangular intuitionistic fuzzy sets (TIFSs), the trapezoidal intuitionistic fuzzy sets (TrIFSs), and the interval-valued trapezoidal intuitionistic fuzzy sets (IVTrIFSs). Wan [1] proposed the IVTrIFSs, by combining IFSs, IVFSs, and trapezoidal fuzzy sets. IVTrIFSs uses intervals rather than crisp numbers to define membership and non-membership values of trapezoidal numbers and hence can describe uncertainty stronger than that of the TIFN and TrIFN [3]. Therefore, these sets were utilised for adaptable and reliable expert decision assistance.

Ranking of fuzzy numbers plays a fundamental role in fuzzy decision making. Since their inception, researchers have used concepts like center of gravity, areas, score and accuracy, similarity measure etc. to rank fuzzy numbers and their extensions (IFNs, IVIFNs, TrIFNs, TIFNs). However, relatively few ranking methods have been developed on IVTrIFSs.Wan [1] proposed a ranking approach for IVTrIFSs based on score function and accuracy functions. Wu and Liu [18] proposed a ranking method allowing experts' risk attitude by developing the new score and expected accuracy functions for IVITrIFSs. Dong and Wan [4] suggested the expectation and expectant score of IVTrIFSs and thereby proposed a new ranking method. They also defined some generalized aggregation operators to solve IVITrFSs decision making problems. Maoying and Jing [5] proposed some averaging aggregation operators for IVTrIFS MCDM problems. Sireesha and Himabindu [6] defined a raking method based score and accuracy functions and used it in decision making methods.

The similarity/distance measure is a significant tool for analysing confusing data. The fuzzy similarity/distance metric [7] illustrates the similarity (difference) between fuzzy sets. It can be applied to determine how closely related the fuzzy sets are to one another. The measure's capacity to differentiate between the sets is stronger the more data it has [8]. As a result, since the invention of fuzzy sets, researchers have been interested in studying distance measures on fuzzy sets [8, 9, 10, 11, 12, 13, 14, 15, 16–19]. Although some distance measures (Hamming and Euclidean) of IVTrIFSs have been proposed in the literature [1, 20], there is no attempt of using the Jaccard index to define distance measure on IVTrIFSs. The Jaccard index similarity measure which is a class of fuzzy preference relation ranking method has also been proposed in ranking fuzzy numbers [21]. It is a statistic that is used to assess the similarity and diversity of sets [22]. Also it has been noted that the existing Euclidean distance measure has limitation in ranking some IVTrIFs. Therefore, this study aims to propose a modified Euclidean distance measure and Jaccard distance measure for IVTrIFSs. The purposeof defining the modified Euclidean distance is illustrated using numerical examples then it is employed in defining Jaccard distance.

The paper is constructed as follows: In section 2, concepts of IVTrIFSs and distance measures are discussed. In section3, the counter cases of existing Euclidean distance measure are mentioned and the modified Euclidean distancefor IVTrIFSs is presented. In section 4, some operations on IVTrIFSs are defined and then the Jaccard distance for IVTrIFSs is proposed and numerical examples are given. Conclusion is presented in the final section.

Preliminaries

In this section, the definition of IVTrIFS, the arithmetic operations and distance measures on IVTrIFSs from literature are reviewed.

Definition 1: Interval-valued trapezoidal intuitionistic fuzzy set [1]

Fuzzy sets are expressed in the universal set of real numbers. Let $\tilde{\alpha} = ([\mathrm{a},\mathrm{b},\mathrm{c},\mathrm{d}];[m_{\tilde{\alpha}}^{L}, m_{\tilde{\alpha}}^{U}];[n_{\tilde{\alpha}}^{L}, n_{\tilde{\alpha}}^{U}])$ is an IVTrIFS, its membership and non-membership functions are defined as follows:

$$m_{\tilde{\alpha}}^{U}(\mathrm{x}) = \begin{cases} \frac{x-a}{b-a} m_{\tilde{\alpha}}^{U} & \text{for a} \leq \mathrm{x} < b \\ m_{\tilde{\alpha}}^{U} & \text{for b} \leq \mathrm{x} \leq \mathrm{c} \\ \frac{d-x}{d-c} m_{\tilde{\alpha}}^{U} & \text{for c} < x \leq d \\ 0 & \text{otherwise.} \end{cases}$$

$$m_{\tilde{\alpha}}^{L}(\mathrm{x}) = \begin{cases} \frac{x-a}{b-a} m_{\tilde{\alpha}}^{L} & \text{for a} \leq \mathrm{x} \leq \mathrm{b} \\ m_{\tilde{\alpha}}^{L} & \text{for b} \leq \mathrm{x} \leq \mathrm{c} \\ \frac{d-x}{d-c} m_{\tilde{\alpha}}^{L} & \text{for c} < x \leq d \\ 0 & \text{otherwise.} \end{cases}$$

Its non-membership function is given by

$$n_{\tilde{\alpha}}^{U}(\mathrm{x}) = \begin{cases} \frac{b-x\ +n_{\tilde{\alpha}}^{U}\ (\mathrm{x-a})}{b-a} & \text{for a} \leq \mathrm{x} < b \\ n_{\tilde{\alpha}}^{U} & \text{for b} \leq \mathrm{x} \leq \mathrm{c} \\ \frac{x-c\ +n_{\tilde{\alpha}}^{U}\ (\mathrm{d-x})}{d-c} & \text{for c} < x \leq d \\ 0 & \text{otherwise.} \end{cases}$$

$$n^L_{\tilde{\alpha}}(x) = \begin{cases} \dfrac{b-x+n^L_{\tilde{\alpha}}(x-a)}{b-a} & \text{for } a \le x < b \\ n^L_{\tilde{\alpha}} & \text{for } b \le x \le c \\ \dfrac{x-c+n^L_{\tilde{\alpha}}(d-x)}{d-c} & \text{for } c < x \le d \\ 0 & \text{otherwise.} \end{cases}$$

Where $0 \le m^L_{\tilde{\alpha}} \le m^U_{\tilde{\alpha}} \le 1$, $0 \le n^L_{\tilde{\alpha}} \le n^U_{\tilde{\alpha}} \le 1$, $0 \le m^U_{\tilde{\alpha}} + n^U_{\tilde{\alpha}} \le 1$ and $0 \le m^L_{\tilde{\alpha}} + n^L_{\tilde{\alpha}} \le 1$

$a, b, c, d \in R$ …… (2.1)

then $\tilde{\alpha} = ([a, b, c, d]; [m^L_{\tilde{\alpha}}, m^U_{\tilde{\alpha}}]; [n^L_{\tilde{\alpha}}, n^U_{\tilde{\alpha}}])$ is called an IVTrIFS.

Definition 2: Euclidean Distance measures for IVTrIFSs[1]

Let $\widetilde{\alpha_1} = ([a_1, b_1, c_1, d_1]; [m^L_{\widetilde{\alpha_1}}, m^U_{\widetilde{\alpha_1}}]; [n^L_{\widetilde{\alpha_1}}, n^U_{\widetilde{\alpha_1}}])$

and $\widetilde{\alpha_2} = ([a_2, b_2, c_2, d_2]; [m^L_{\widetilde{\alpha_2}}, m^U_{\widetilde{\alpha_2}}]; [n^L_{\widetilde{\alpha_2}}, n^U_{\widetilde{\alpha_2}}])$ be two IVTrIFSs.

The Euclidean distance between $\widetilde{\alpha_1}$ and $\widetilde{\alpha_2}$ is defined as:

$$d_E(\widetilde{\alpha_1}, \widetilde{\alpha_2}) = \frac{1}{2\sqrt{2}}\left(\sqrt{\begin{array}{c} \left((m^L_{\widetilde{\alpha_1}} - n^U_{\widetilde{\alpha_1}})a_1 - (m^L_{\widetilde{\alpha_2}} - n^U_{\widetilde{\alpha_2}})a_2\right)^2 + \left((m^U_{\widetilde{\alpha_1}} - n^L_{\widetilde{\alpha_1}})a_1 - (m^U_{\widetilde{\alpha_2}} - n^L_{\widetilde{\alpha_2}})a_2\right)^2 + \\ \left((m^L_{\widetilde{\alpha_1}} - n^U_{\widetilde{\alpha_1}})b_1 - (m^L_{\widetilde{\alpha_2}} - n^U_{\widetilde{\alpha_2}})b_2\right)^2 + \left((m^U_{\widetilde{\alpha_1}} - n^L_{\widetilde{\alpha_1}})b_1 - (m^U_{\widetilde{\alpha_2}} - n^L_{\widetilde{\alpha_2}})b_2\right)^2 \\ + \left((m^L_{\widetilde{\alpha_1}} - n^U_{\widetilde{\alpha_1}})c_1 - (m^L_{\widetilde{\alpha_2}} - n^U_{\widetilde{\alpha_2}})c_2\right)^2 + \left((m^U_{\widetilde{\alpha_1}} - n^L_{\widetilde{\alpha_1}})c_1 - (m^U_{\widetilde{\alpha_2}} - n^L_{\widetilde{\alpha_2}})c_2\right)^2 \\ + \left((m^L_{\widetilde{\alpha_1}} - n^U_{\widetilde{\alpha_1}})d_1 - (m^L_{\widetilde{\alpha_2}} - n^U_{\widetilde{\alpha_2}})d_2\right)^2 + \left((m^U_{\widetilde{\alpha_1}} - n^L_{\widetilde{\alpha_1}})d_1 - (m^U_{\widetilde{\alpha_2}} - n^L_{\widetilde{\alpha_2}})d_2\right)^2 \end{array}}\right)$$

…… (2.2)

For ranking of IVTrIFSs, the Euclidean distance/Hamming distance is calculated from origin and the set with greater distance is given higher ranking.

Definition 3: Ranking method of SP Wan [17] and Sireesha&Himabindu[6]

S P Wan andSireesha&Himabindu developed ranking method for IVTrIFSs based on score and accuracy functions.

For any interval-valued trapezoidal intuitionistic fuzzy set

$$\tilde{\alpha} = ([\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}]; [m_{\tilde{\alpha}}^{L}, m_{\tilde{\alpha}}^{U}]; [n_{\tilde{\alpha}}^{L}, n_{\tilde{\alpha}}^{U}])$$

S P Wan [24] defined the score and accuracy functions as:

Score function: $S(\tilde{\alpha}) = \left[\frac{a+b+c+d}{4}\right] S_X(\tilde{\alpha})$

Accuracy function: $H(\tilde{\alpha}) = \left[\frac{a+b+c+d}{4}\right] H_X(\tilde{\alpha})$

Sireesha and Himabindu [12] defined the Value index and Ambiguity index as below:

Value index:$V(\tilde{\alpha}) = \left[\frac{a+2b+2c+d}{12}\right] [1 + S_X(\tilde{\alpha}) - H_X(\tilde{\alpha})]$ and

Ambiguity index:$A(\tilde{\alpha}) = \left[\frac{(d-a)-2(b-c)}{12}\right] [1 + S_X(\tilde{\alpha}) - H_X(\tilde{\alpha})]$

Here $S_X(\tilde{\alpha}) = \left[\frac{m_{\tilde{\alpha}}^{L} + m_{\tilde{\alpha}}{}^{U} - n_{\tilde{\alpha}}^{L} - n_{\tilde{\alpha}}^{U}}{2}\right]$ is the score function

$H_X(\tilde{\alpha}) = \left[\frac{m_{\tilde{\alpha}}^{L} + m_{\tilde{\alpha}}{}^{U} + n_{\tilde{\alpha}}^{L} + n_{\tilde{\alpha}}^{U}}{2}\right]$is the accuracy function.

For raking IVTrIFSs, the set with high score is preferred. If value indices are equal, then compare with ambiguity indices. The set with high ambiguity index is preferred the most. If both value index and ambiguity index are same then they are said to be equal.

Definition 4: Jaccard distance [24]

Jaccard distance is a measure of dissimilarity between two sets, given by

$$d_J(A_1, A_2) = 1 - J_{SI}(A_1, A_2)$$

where $J_{SI}(A_1, A_2) = \frac{|A_1 \cap A_2|}{|A_1 \cup A_2|}$ is the Jaccard similarity index.

Proposed Modified Euclidean distance on IVTrIFSs

In this part, we first present some tested contexts of IVTrIFSs where the existing ED [1] fails in ranking, and then we propose a modified ED that overcomes the drawback of the existing ED.

Example 1: Consider the IVTrIFSs

$\widetilde{\alpha_1} = ([a_1, b_1, c_1, d_1]; [m^L_{\widetilde{\alpha_1}}, m^U_{\widetilde{\alpha_1}}]; [n^L_{\widetilde{\alpha_1}}, n^U_{\widetilde{\alpha_1}}])$= ([0.29, 0.42, 0.54, 0.69]; [0.4, 0.5], [0.2, 0.4]),

$\widetilde{\alpha_2} = ([a_2, b_2, c_2, d_2]; [m^L_{\widetilde{\alpha_2}}, m^U_{\widetilde{\alpha_2}}]; [n^L_{\widetilde{\alpha_2}}, n^U_{\widetilde{\alpha_2}}])$= ([0.25, 0.41, 0.52, 0.66]; [0.2, 0.4], [0.3, 0.5])

and $\widetilde{\alpha_0} = ([a_0, b_0, c_0, d_0]; [m^L_{\widetilde{\alpha_0}}, m^U_{\widetilde{\alpha_0}}]; [n^L_{\widetilde{\alpha_0}}, n^U_{\widetilde{\alpha_0}}]) = ([0, 0, 0, 0]; [0, 0], [0, 0])$

The ED [9] of $\widetilde{\alpha_1}$ and $\widetilde{\alpha_2}$from $\widetilde{\alpha_0}$

$$d_E(\widetilde{\alpha_1}, \widetilde{\alpha_0})$$

$$= \left(\sqrt{ \frac{1}{8} \begin{pmatrix} \left((m^L_{\widetilde{\alpha_1}} - n^U_{\widetilde{\alpha_1}}) a_1 - (m^L_{\widetilde{\alpha_0}} - n^U_{\widetilde{\alpha_0}}) a_0 \right)^2 + \left((m^U_{\widetilde{\alpha_1}} - n^L_{\widetilde{\alpha_1}}) a_1 - (m^U_{\widetilde{\alpha_0}} - n^L_{\widetilde{\alpha_0}}) a_0 \right)^2 + \\ \left((m^L_{\widetilde{\alpha_1}} - n^U_{\widetilde{\alpha_1}}) b_1 - (m^L_{\widetilde{\alpha_0}} - n^U_{\widetilde{\alpha_0}}) b_0 \right)^2 + \left((m^U_{\widetilde{\alpha_1}} - n^L_{\widetilde{\alpha_1}}) b_1 - (m^U_{\widetilde{\alpha_0}} - n^L_{\widetilde{\alpha_0}}) b_0 \right)^2 \\ + \left((m^L_{\widetilde{\alpha_1}} - n^U_{\widetilde{\alpha_1}}) c_1 - (m^L_{\widetilde{\alpha_0}} - n^U_{\widetilde{\alpha_0}}) c_0 \right)^2 + \left((m^U_{\widetilde{\alpha_1}} - n^L_{\widetilde{\alpha_1}}) c_1 - (m^U_{\widetilde{\alpha_0}} - n^L_{\widetilde{\alpha_0}}) c_0 \right)^2 \\ + \left((m^L_{\widetilde{\alpha_1}} - n^U_{\widetilde{\alpha_1}}) d_1 - (m^L_{\widetilde{\alpha_0}} - n^U_{\widetilde{\alpha_0}}) d_0 \right)^2 + \left((m^U_{\widetilde{\alpha_1}} - n^L_{\widetilde{\alpha_1}}) d_1 - (m^U_{\widetilde{\alpha_0}} - n^L_{\widetilde{\alpha_0}}) d_0 \right)^2 \end{pmatrix} } \right)$$

$$= \sqrt{\frac{1}{8}(0.093)} = \sqrt{0.0116} = 0.1076$$

$$d_E(\widetilde{\alpha_2}, \widetilde{\alpha_0})$$

$$= \left(\sqrt{ \frac{1}{8} \begin{pmatrix} \left((m^L_{\widetilde{\alpha_2}} - n^U_{\widetilde{\alpha_2}}) a_2 - (m^L_{\widetilde{\alpha_0}} - n^U_{\widetilde{\alpha_0}}) a_0 \right)^2 + \left((m^U_{\widetilde{\alpha_2}} - n^L_{\widetilde{\alpha_2}}) a_2 - (m^U_{\widetilde{\alpha_0}} - n^L_{\widetilde{\alpha_0}}) a_0 \right)^2 + \\ \left((m^L_{\widetilde{\alpha_2}} - n^U_{\widetilde{\alpha_2}}) b_2 - (m^L_{\widetilde{\alpha_0}} - n^U_{\widetilde{\alpha_0}}) b_0 \right)^2 + \left((m^U_{\widetilde{\alpha_2}} - n^L_{\widetilde{\alpha_2}}) b_2 - (m^U_{\widetilde{\alpha_0}} - n^L_{\widetilde{\alpha_0}}) b_0 \right)^2 \\ + \left((m^L_{\widetilde{\alpha_2}} - n^U_{\widetilde{\alpha_2}}) c_2 - (m^L_{\widetilde{\alpha_0}} - n^U_{\widetilde{\alpha_0}}) c_0 \right)^2 + \left((m^U_{\widetilde{\alpha_2}} - n^L_{\widetilde{\alpha_2}}) c_2 - (m^U_{\widetilde{\alpha_0}} - n^L_{\widetilde{\alpha_0}}) c_0 \right)^2 \\ + \left((m^L_{\widetilde{\alpha_2}} - n^U_{\widetilde{\alpha_2}}) d_2 - (m^L_{\widetilde{\alpha_0}} - n^U_{\widetilde{\alpha_0}}) d_0 \right)^2 + \left((m^U_{\widetilde{\alpha_2}} - n^L_{\widetilde{\alpha_2}}) d_2 - (m^U_{\widetilde{\alpha_0}} - n^L_{\widetilde{\alpha_0}}) d_0 \right)^2 \end{pmatrix} } \right)$$

$$= \sqrt{\frac{1}{8}(0.094)} = \sqrt{0.0117} = 0.1082$$

i.e., $d_E(\widetilde{\alpha_2}, \widetilde{\alpha_0}) > d_E(\widetilde{\alpha_1}, \widetilde{\alpha_0})$ but it is evident from the sets that $d(\widetilde{\alpha_2}, \widetilde{\alpha_0}) < d(\widetilde{\alpha_1}, \widetilde{\alpha_0})$.

Example 2: Consider the IVTrIFSs

$\widetilde{\alpha_1}$ = ([0.35, 0.45, 0.53, 0.64]; [0.4, 0.6], [0.3, 0.4]) and $\widetilde{\alpha_2}$ = ([0.31, 0.44, 0.53, 0.64]; [0.3, 0.4], [0.5, 0.6])

The ED [9] of $\widetilde{\alpha_1}$ and $\widetilde{\alpha_2}$from $\widetilde{\alpha_0}$

$d_E(\widetilde{\alpha_1},\widetilde{\alpha_0})=\sqrt{\frac{1}{8}(0.091)} = \sqrt{0.0114} = 0.1069$

$$d_E(\widetilde{\alpha_2},\widetilde{\alpha_0}) = \sqrt{\frac{1}{8}(0.094)} = \sqrt{0.0117} = 0.1106$$

i.e., $d_E(\widetilde{\alpha_2},\widetilde{\alpha_0}) > d_E(\widetilde{\alpha_1},\widetilde{\alpha_0})$ but it is evident from the sets$d(\widetilde{\alpha_2},\widetilde{\alpha_0}) < d(\widetilde{\alpha_1},\widetilde{\alpha_0})$.

Definition 5: Modified Euclidean distance for IVTrIFSs

Let $\widetilde{\alpha_1}$ and $\widetilde{\alpha_2}$ be two IVTrIFSs then the Modified ED between$\widetilde{\alpha_1}$ and $\widetilde{\alpha_2}$ is denoted by $d_e(\widetilde{\alpha_1},\widetilde{\alpha_2})$and is defined as follows:

$$d_e(\widetilde{\alpha_1},\widetilde{\alpha_2}) = \frac{1}{2\sqrt{2}}\left(\sqrt{d^m(\widetilde{\alpha_1},\widetilde{\alpha_2}) + d^n(\widetilde{\alpha_1},\widetilde{\alpha_2})}\right)$$

$$d^m(\widetilde{\alpha_1},\widetilde{\alpha_2}) = \left(m^L_{\widetilde{\alpha_1}} * a_1 - m^L_{\widetilde{\alpha_2}} * a_2\right)^2 + \left(m^L_{\widetilde{\alpha_1}} * b_1 - m^L_{\widetilde{\alpha_2}} * b_2\right)^2 + \left(m^U_{\widetilde{\alpha_1}} * c_1 - m^U_{\widetilde{\alpha_2}} * c_2\right)^2 + \left(m^U_{\widetilde{\alpha_1}} * d_1 - m^U_{\widetilde{\alpha_2}} * d_2\right)^2$$

$$d^n(\widetilde{\alpha_1},\widetilde{\alpha_2}) = \left(\left(1 - n^L_{\widetilde{\alpha_1}}\right) * a_1 - \left(1 - n^L_{\widetilde{\alpha_2}}\right) * a_2\right)^2 + \left(\left(1 - n^L_{\widetilde{\alpha_1}}\right) * b_1 - \left(1 - n^L_{\widetilde{\alpha_2}}\right) * b_2\right)^2 + \left(\left(1 - n^U_{\widetilde{\alpha_1}}\right) * c_1 - \left(1 - n^U_{\widetilde{\alpha_2}}\right) * c_2\right)^2 + \left(\left(1 - n^U_{\widetilde{\alpha_1}}\right) * d_1 - \left(1 - n^U_{\widetilde{\alpha_2}}\right) * d_2\right)^2$$

Proposition 3.1: The proposed d_e on IVTrIFSs satisfies the following axioms:

(A1) $0 \leq d_e(\widetilde{\alpha_1},\widetilde{\alpha_2}) \leq 1, \forall\, \widetilde{\alpha_1},\widetilde{\alpha_2} \in \mathcal{C}(X)$, where $\mathcal{C}(X)$is class of subsets of X.

(A2) $d_e(\widetilde{\alpha_1},\widetilde{\alpha_2}) = 0$, if $\widetilde{\alpha_1} = \widetilde{\alpha_2}$

(A3) $d_e(\widetilde{\alpha_1},\widetilde{\alpha_2}) = d_e(\widetilde{\alpha_2},\widetilde{\alpha_1})$

(A4) $d_e(\widetilde{\alpha_1}, \widetilde{\alpha_3}) \leq d_e(\widetilde{\alpha_1}, \widetilde{\alpha_2}) + d_e(\widetilde{\alpha_2}, \widetilde{\alpha_3})$.

Proof: Let $\widetilde{\alpha_1} = ([a_1, b_1, c_1, d_1]; [m^L_{\widetilde{\alpha_1}}, m^U_{\widetilde{\alpha_1}}]; [n^L_{\widetilde{\alpha_1}}, n^U_{\widetilde{\alpha_1}}])\widetilde{\alpha_2} = ([a_2, b_2, c_2, d_2]; [m^L_{\widetilde{\alpha_2}}, m^U_{\widetilde{\alpha_2}}]; [n^L_{\widetilde{\alpha_2}}, n^U_{\widetilde{\alpha_2}}])$and $\widetilde{\alpha_3} = ([a_3, b_3, c_3, d_3]; [m^L_{\widetilde{\alpha_3}}, m^U_{\widetilde{\alpha_3}}]; [n^L_{\widetilde{\alpha_3}}, n^U_{\widetilde{\alpha_3}}])$ be three IVITrFSs.

(A1)$0 \leq d_e(\widetilde{\alpha_1}, \widetilde{\alpha_2}) \leq 1$, can be observed by equation (2.1).

(A2) To prove $d_e(\widetilde{\alpha_1}, \widetilde{\alpha_2}) = 0$, if $\widetilde{\alpha_1} = \widetilde{\alpha_2}$

Suppose $\widetilde{\alpha_1} = \widetilde{\alpha_2}$
Then
$m^L_{\widetilde{\alpha_1}} * a_1 = m^L_{\widetilde{\alpha_2}} * a_2, m^L_{\widetilde{\alpha_1}} * b_1 = m^L_{\widetilde{\alpha_2}} * b_2, m^U_{\widetilde{\alpha_1}} * c_1 = m^U_{\widetilde{\alpha_2}} * c_2$ and $m^U_{\widetilde{\alpha_1}} * d_1 = m^U_{\widetilde{\alpha_2}} * d_2$
So, $d^m(\widetilde{\alpha_1}, \widetilde{\alpha_2}) = 0$

And $\left(1 - n^L_{\widetilde{\alpha_1}}\right) * a_1 = \left(1 - n^L_{\widetilde{\alpha_2}}\right) * a_2$, $\left(1 - n^L_{\widetilde{\alpha_1}}\right) * b_1 = \left(1 - n^L_{\widetilde{\alpha_2}}\right) * b_2$, $\left(1 - n^U_{\widetilde{\alpha_1}}\right) * c_1 = \left(1 - n^U_{\widetilde{\alpha_2}}\right) * c_2$ and $\left(1 - n^U_{\widetilde{\alpha_1}}\right) * d_1 = \left(1 - n^U_{\widetilde{\alpha_2}}\right) * d_2$

So, $d^n(\widetilde{\alpha_1}, \widetilde{\alpha_2}) = 0$

$$\Rightarrow d_e(\widetilde{\alpha_1}, \widetilde{\alpha_2}) = \frac{1}{2\sqrt{2}}\left(\sqrt{d^m(\widetilde{\alpha_1}, \widetilde{\alpha_2}) + d^n(\widetilde{\alpha_1}, \widetilde{\alpha_2})}\right) = 0$$

Thus, $d_e(\widetilde{\alpha_1}, \widetilde{\alpha_2}) = 0$, if $\widetilde{\alpha_1} = \widetilde{\alpha_2}$

(A3)$d_e(\widetilde{\alpha_1}, \widetilde{\alpha_2}) = d_e(\widetilde{\alpha_2}, \widetilde{\alpha_1})$ can be observed easily.

(A4) To prove $d_e(\widetilde{\alpha_1}, \widetilde{\alpha_3}) \leq d_e(\widetilde{\alpha_1}, \widetilde{\alpha_2}) + d_e(\widetilde{\alpha_2}, \widetilde{\alpha_3})$

The above inequality is equivalent to

$$d^m(\widetilde{\alpha_1}, \widetilde{\alpha_3}) + d^n(\widetilde{\alpha_1}, \widetilde{\alpha_3}) \leq d^m(\widetilde{\alpha_1}, \widetilde{\alpha_2}) + d^n(\widetilde{\alpha_1}, \widetilde{\alpha_2}) + d^m(\widetilde{\alpha_2}, \widetilde{\alpha_3}) + d^n(\widetilde{\alpha_2}, \widetilde{\alpha_3})$$

Consider $d^m(\widetilde{\alpha_1}, \widetilde{\alpha_2}) + d^n(\widetilde{\alpha_1}, \widetilde{\alpha_2}) + d^m(\widetilde{\alpha_2}, \widetilde{\alpha_3}) + d^n(\widetilde{\alpha_2}, \widetilde{\alpha_3}) =$

$$\left(m^L_{\widetilde{\alpha_1}} * a_1 - m^L_{\widetilde{\alpha_2}} * a_2\right)^2 + \left(m^L_{\widetilde{\alpha_1}} * b_1 - m^L_{\widetilde{\alpha_2}} * b_2\right)^2 + \left(m^U_{\widetilde{\alpha_1}} * c_1 - m^U_{\widetilde{\alpha_2}} * c_2\right)^2 + \left(m^U_{\widetilde{\alpha_1}} * d_1 - m^U_{\widetilde{\alpha_2}} * d_2\right)^2$$
$$+ \left(\left(1 - n^L_{\widetilde{\alpha_1}}\right) * a_1 - \left(1 - n^L_{\widetilde{\alpha_2}}\right) * a_2\right)^2 + \left(\left(1 - n^L_{\widetilde{\alpha_1}}\right) * b_1 - \left(1 - n^L_{\widetilde{\alpha_2}}\right) * b_2\right)^2$$
$$+ \left(\left(1 - n^U_{\widetilde{\alpha_1}}\right) * c_1 - \left(1 - n^U_{\widetilde{\alpha_2}}\right) * c_2\right)^2 + \left(\left(1 - n^U_{\widetilde{\alpha_1}}\right) * d_1 - \left(1 - n^U_{\widetilde{\alpha_2}}\right) * d_2\right)^2$$
$$+ \left(m^L_{\widetilde{\alpha_2}} * a_2 - m^L_{\widetilde{\alpha_3}} * a_3\right)^2 + \left(m^L_{\widetilde{\alpha_2}} * b_2 - m^L_{\widetilde{\alpha_3}} * b_3\right)^2 + \left(m^U_{\widetilde{\alpha_2}} * c_2 - m^U_{\widetilde{\alpha_3}} * c_3\right)^2$$
$$+ \left(m^U_{\widetilde{\alpha_2}} * d_2 - m^U_{\widetilde{\alpha_3}} * d_3\right)^2 + \left(\left(1 - n^L_{\widetilde{\alpha_2}}\right) * a_2 - \left(1 - n^L_{\widetilde{\alpha_3}}\right) * a_3\right)^2$$
$$+ \left(\left(1 - n^L_{\widetilde{\alpha_2}}\right) * b_2 - \left(1 - n^L_{\widetilde{\alpha_3}}\right) * b_3\right)^2 + \left(\left(1 - n^U_{\widetilde{\alpha_2}}\right) * c_2 - \left(1 - n^U_{\widetilde{\alpha_3}}\right) * c_3\right)^2$$
$$+ \left(\left(1 - n^U_{\widetilde{\alpha_2}}\right) * d_2 - \left(1 - n^U_{\widetilde{\alpha_3}}\right) * d_3\right)^2$$

$$= \Big[\left(m^L_{\widetilde{\alpha_1}} * a_1\right)^2 + \left(m^L_{\widetilde{\alpha_2}} * a_2\right)^2 - 2 * \left(m^L_{\widetilde{\alpha_1}} * a_1 * m^L_{\widetilde{\alpha_2}} * a_2\right) + \left(m^L_{\widetilde{\alpha_1}} * b_1\right)^2 + \left(m^L_{\widetilde{\alpha_2}} * b_2\right)^2 - 2$$
$$* \left(m^L_{\widetilde{\alpha_1}} * b_1 * m^L_{\widetilde{\alpha_2}} * b_2\right) + \left(m^U_{\widetilde{\alpha_1}} * c_1\right)^2 + \left(m^U_{\widetilde{\alpha_2}} * c_2\right)^2 - 2 * \left(m^U_{\widetilde{\alpha_1}} * c_1 * m^U_{\widetilde{\alpha_2}} * c_2\right) + \left(m^U_{\widetilde{\alpha_1}} * d_1\right)^2$$
$$+ \left(m^U_{\widetilde{\alpha_2}} * d_2\right)^2 - 2 * \left(m^U_{\widetilde{\alpha_1}} * d_1 * m^U_{\widetilde{\alpha_2}} * d_2\right) + \left(\left(1 - n^L_{\widetilde{\alpha_1}}\right) * a_1\right)^2 + \left(\left(1 - n^L_{\widetilde{\alpha_2}}\right) * a_2\right)^2 - 2$$
$$* \left(\left(1 - n^L_{\widetilde{\alpha_1}}\right) * a_1 * \left(1 - n^L_{\widetilde{\alpha_2}}\right) * a_2\right) + \left(\left(1 - n^L_{\widetilde{\alpha_1}}\right) * b_1\right)^2 + \left(\left(1 - n^L_{\widetilde{\alpha_2}}\right) * b_2\right)^2 - 2$$
$$* \left(\left(1 - n^L_{\widetilde{\alpha_1}}\right) * b_1 * \left(1 - n^L_{\widetilde{\alpha_2}}\right) * b_2\right) + \left(\left(1 - n^U_{\widetilde{\alpha_1}}\right) * c_1\right)^2 + \left(\left(1 - n^U_{\widetilde{\alpha_2}}\right) * c_2\right)^2 - 2$$
$$* \left(\left(1 - n^U_{\widetilde{\alpha_1}}\right) * c_1 * \left(1 - n^U_{\widetilde{\alpha_2}}\right) * c_2\right) + \left(\left(1 - n^U_{\widetilde{\alpha_1}}\right) * d_1\right)^2 + \left(\left(1 - n^U_{\widetilde{\alpha_2}}\right) * d_2\right)^2 - 2 * \ \left(1 - n^U_{\widetilde{\alpha_1}}\right) * d_1$$
$$* \ \left(1 - n^U_{\widetilde{\alpha_2}}\right) * d_2\Big]$$
$$+ \Big[\left(m^L_{\widetilde{\alpha_2}} * a_2\right)^2 + \left(m^L_{\widetilde{\alpha_3}} * a_3\right)^2 - 2 * \left(m^L_{\widetilde{\alpha_2}} * a_2 * m^L_{\widetilde{\alpha_3}} * a_3\right) + \left(m^L_{\widetilde{\alpha_2}} * b_2\right)^2 + \left(m^L_{\widetilde{\alpha_3}} * b_3\right)^2 - 2$$
$$* \left(m^L_{\widetilde{\alpha_3}} * b_3 * m^L_{\widetilde{\alpha_2}} * b_2\right) + \left(m^U_{\widetilde{\alpha_2}} * c_2\right)^2 + \left(m^U_{\widetilde{\alpha_3}} * c_3\right)^2 - 2 * \left(m^U_{\widetilde{\alpha_3}} * c_3 * m^U_{\widetilde{\alpha_2}} * c_2\right) + \left(m^U_{\widetilde{\alpha_3}} * d_3\right)^2$$
$$+ \left(m^U_{\widetilde{\alpha_2}} * d_2\right)^2 - 2 * \left(m^U_{\widetilde{\alpha_3}} * d_3 * m^U_{\widetilde{\alpha_2}} * d_2\right) + \left(\left(1 - n^L_{\widetilde{\alpha_3}}\right) * a_3\right)^2 + \left(\left(1 - n^L_{\widetilde{\alpha_2}}\right) * a_2\right)^2 - 2$$
$$* \left(\left(1 - n^L_{\widetilde{\alpha_3}}\right) * a_3 * \left(1 - n^L_{\widetilde{\alpha_2}}\right) * a_2\right) + \left(\left(1 - n^L_{\widetilde{\alpha_3}}\right) * b_3\right)^2 + \left(\left(1 - n^L_{\widetilde{\alpha_2}}\right) * b_2\right)^2 - 2$$
$$* \left(\left(1 - n^L_{\widetilde{\alpha_3}}\right) * b_3 * \left(1 - n^L_{\widetilde{\alpha_2}}\right) * b_2\right) + \left(\left(1 - n^U_{\widetilde{\alpha_3}}\right) * c_3\right)^2 + \left(\left(1 - n^U_{\widetilde{\alpha_2}}\right) * c_2\right)^2 - 2$$
$$* \left(\left(1 - n^U_{\widetilde{\alpha_3}}\right) * c_3 * \left(1 - n^U_{\widetilde{\alpha_2}}\right) * c_2\right) + \left(\left(1 - n^U_{\widetilde{\alpha_3}}\right) * d_3\right)^2 + \left(\left(1 - n^U_{\widetilde{\alpha_2}}\right) * d_2\right)^2 - 2 * \ \left(1 - n^U_{\widetilde{\alpha_3}}\right) * d_3$$
$$* \ \left(1 - n^U_{\widetilde{\alpha_2}}\right) * d_2\Big]$$

$$\geq \Big[\left(m^{L}_{\widetilde{\alpha_1}} * a_1\right)^2 + \left(m^{L}_{\widetilde{\alpha_3}} * a_3\right)^2 - 2 * \left(m^{L}_{\widetilde{\alpha_1}} * a_1 * m^{L}_{\widetilde{\alpha_3}} * a_3\right) + \left(m^{L}_{\widetilde{\alpha_1}} * b_1\right)^2 + \left(m^{L}_{\widetilde{\alpha_3}} * b_3\right)^2 - 2 * \left(m^{L}_{\widetilde{\alpha_1}} * b_1 * m^{L}_{\widetilde{\alpha_3}} * b_3\right) + \left(m^{U}_{\widetilde{\alpha_1}} * c_1\right)^2 + \left(m^{U}_{\widetilde{\alpha_3}} * c_3\right)^2 - 2 * \left(m^{U}_{\widetilde{\alpha_1}} * c_1 * m^{U}_{\widetilde{\alpha_3}} * c_3\right) + \left(m^{U}_{\widetilde{\alpha_1}} * d_1\right)^2 + \left(m^{U}_{\widetilde{\alpha_3}} * d_3\right)^2 - 2 * \left(m^{U}_{\widetilde{\alpha_1}} * d_1 * m^{U}_{\widetilde{\alpha_3}} * d_3\right) + \left(\left(1 - n^{L}_{\widetilde{\alpha_1}}\right) * a_1\right)^2 + \left(\left(1 - n^{L}_{\widetilde{\alpha_3}}\right) * a_3\right)^2 - 2 * \left(\left(1 - n^{L}_{\widetilde{\alpha_1}}\right) * a_1 * \left(1 - n^{L}_{\widetilde{\alpha_3}}\right) * a_3\right) + \left(\left(1 - n^{L}_{\widetilde{\alpha_1}}\right) * b_1\right)^2 + \left(\left(1 - n^{L}_{\widetilde{\alpha_3}}\right) * b_3\right)^2 - 2 * \left(\left(1 - n^{L}_{\widetilde{\alpha_1}}\right) * b_1 * \left(1 - n^{L}_{\widetilde{\alpha_3}}\right) * b_3\right) + \left(\left(1 - n^{U}_{\widetilde{\alpha_1}}\right) * c_1\right)^2 + \left(\left(1 - n^{U}_{\widetilde{\alpha_3}}\right) * c_3\right)^2 - 2 * \left(\left(1 - n^{U}_{\widetilde{\alpha_1}}\right) * c_1 * \left(1 - n^{U}_{\widetilde{\alpha_3}}\right) * c_3\right) + \left(\left(1 - n^{U}_{\widetilde{\alpha_1}}\right) * d_1\right)^2 + \left(\left(1 - n^{U}_{\widetilde{\alpha_3}}\right) * d_3\right)^2 - 2 * \left(1 - n^{U}_{\widetilde{\alpha_1}}\right) * d_1 * \left(1 - n^{U}_{\widetilde{\alpha_3}}\right) * d_3\Big]$$

$$= d^m(\widetilde{\alpha_1}, \widetilde{\alpha_3}) + d^n(\widetilde{\alpha_1}, \widetilde{\alpha_3})$$

$$d^m(\widetilde{\alpha_1}, \widetilde{\alpha_2}) + d^n(\widetilde{\alpha_1}, \widetilde{\alpha_2}) + d^m(\widetilde{\alpha_2}, \widetilde{\alpha_3}) + d^n(\widetilde{\alpha_2}, \widetilde{\alpha_3}) \geq d^m(\widetilde{\alpha_1}, \widetilde{\alpha_3}) + d^n(\widetilde{\alpha_1}, \widetilde{\alpha_3})$$

Thus, $d_e(\widetilde{\alpha_1}, \widetilde{\alpha_3}) \leq d_e(\widetilde{\alpha_1}, \widetilde{\alpha_2}) + d_e(\widetilde{\alpha_2}, \widetilde{\alpha_3})$.

Hence, the metric properties are proved for d_e.

Numerical examples

In this section, we demonstrate how the suggested Modified ED overcomes the drawback of existing ED [1] by using the Examples 1 and 2 discussed in section 3.

For the IVTrIFSs mentioned in Example 1, the modified ED is obtained as:

$$d_e(\widetilde{\alpha_1}, \widetilde{\alpha_0}) = \left(\sqrt{\frac{1}{8}\begin{pmatrix} \left(m^{L}_{\widetilde{\alpha_1}} * a_1 - m^{L}_{\widetilde{\alpha_0}} * a_0\right)^2 + \left(m^{L}_{\widetilde{\alpha_1}} * b_1 - m^{L}_{\widetilde{\alpha_0}} * b_0\right)^2 + \\ \left(m^{U}_{\widetilde{\alpha_1}} * c_1 - m^{U}_{\widetilde{\alpha_0}} * c_0\right)^2 + \left(m^{U}_{\widetilde{\alpha_1}} * d_1 - m^{U}_{\widetilde{\alpha_0}} * d_0\right)^2 + \\ \left(\left(1 - n^{L}_{\widetilde{\alpha_1}}\right) * a_1 - \left(1 - n^{L}_{\widetilde{\alpha_0}}\right) * a_0\right)^2 + \left(\left(1 - n^{L}_{\widetilde{\alpha_1}}\right) * b_1 - \left(1 - n^{L}_{\widetilde{\alpha_0}}\right) * b_0\right)^2 \\ + \left(\left(1 - n^{U}_{\widetilde{\alpha_1}}\right) * c_1 - \left(1 - n^{U}_{\widetilde{\alpha_0}}\right) * c_0\right)^2 + \left(\left(1 - n^{U}_{\widetilde{\alpha_1}}\right) * d_1 - \left(1 - n^{U}_{\widetilde{\alpha_0}}\right) * d_0\right)^2 \end{pmatrix}}\right)$$

$$= \sqrt{\frac{1}{8}(0.6767)} = 0.2908$$

$$d_e(\widetilde{\alpha_2}, \widetilde{\alpha_0})$$
$$= \left(\sqrt{\frac{1}{8}\begin{pmatrix} \left(m^L_{\widetilde{\alpha_2}} * a_2 - m^L_{\widetilde{\alpha_0}} * a_0\right)^2 + \left(m^L_{\widetilde{\alpha_2}} * b_2 - m^L_{\widetilde{\alpha_0}} * b_0\right)^2 + \\ \left(m^U_{\widetilde{\alpha_2}} * c_2 - m^U_{\widetilde{\alpha_0}} * c_0\right)^2 + \left(m^U_{\widetilde{\alpha_2}} * d_2 - m^U_{\widetilde{\alpha_0}} * d_0\right)^2 + \\ \left(\left(1 - n^L_{\widetilde{\alpha_2}}\right) * a_2 - \left(1 - n^L_{\widetilde{\alpha_0}}\right) * a_0\right)^2 + \left(\left(1 - n^L_{\widetilde{\alpha_2}}\right) * b_2 - \left(1 - n^L_{\widetilde{\alpha_0}}\right) * b_0\right)^2 \\ + \left(\left(1 - n^U_{\widetilde{\alpha_2}}\right) * c_2 - \left(1 - n^U_{\widetilde{\alpha_0}}\right) * c_0\right)^2 + \left(\left(1 - n^U_{\widetilde{\alpha_2}}\right) * d_2 - \left(1 - n^U_{\widetilde{\alpha_0}}\right) * d_0\right)^2 \end{pmatrix}}\right)$$

$$= \sqrt{\frac{1}{8}(0.4117)} = 0.2268$$

i.e., $d_e(\widetilde{\alpha_2}, \widetilde{\alpha_0}) < d_e(\widetilde{\alpha_1}, \widetilde{\alpha_0})$ which agrees with human intuition.

Similarly, for Example 2, we get

$$d_e(\widetilde{\alpha_1}, \widetilde{\alpha_0}) = \sqrt{\frac{1}{8}(0.7084)} = 0.2976$$

$$d_e(\widetilde{\alpha_2}, \widetilde{\alpha_0}) = \sqrt{\frac{1}{8}(0.3194)} = 0.1998$$

i.e., $d_e(\widetilde{\alpha_2}, \widetilde{\alpha_0}) < d_e(\widetilde{\alpha_1}, \widetilde{\alpha_0})$ agrees with human intuition.

From the tested examples it is observed that the proposed modified Euclidean distance is overcoming the limitation of existing ED [1] and effectively ranking.

Proposed Jaccard distance and ranking on IVTrIFSs

In this section, the Jaccard distance (JD) between two IVTrIFSs is proposed by using modified Euclidean distance and exemplified through a numerical example. The ranking methodology of IVTrIFSs is presented using proposed Jaccard distance. The procedure is demonstrated through illustrative example.

Definition 6: Set operations on IVTrIFSs

Let $\widetilde{\alpha_1} = ([a_1, b_1, c_1, d_1]; [m^L_{\widetilde{\alpha_1}}, m^U_{\widetilde{\alpha_1}}]; [n^L_{\widetilde{\alpha_1}}, n^U_{\widetilde{\alpha_1}}])$

and $\widetilde{\alpha_2} = ([a_2, b_2, c_2, d_2]; [m^L_{\widetilde{\alpha_2}}, m^U_{\widetilde{\alpha_2}}]; [n^L_{\widetilde{\alpha_2}}, n^U_{\widetilde{\alpha_2}}])$ be two IVTrIFSs, then

$$\widetilde{\alpha_1} \cup \widetilde{\alpha_2} = \begin{pmatrix} [\max(a_1, a_2), \max(b_1, b_2), \max(c_1, c_2), \max(d_1, d_2)]; \\ [\max(m^L_{\widetilde{\alpha_1}}, m^L_{\widetilde{\alpha_2}}), \max(m^U_{\widetilde{\alpha_1}}, m^U_{\widetilde{\alpha_2}}), \min(n^L_{\widetilde{\alpha_1}}, n^L_{\widetilde{\alpha_2}}), \min(n^U_{\widetilde{\alpha_1}}, n^U_{\widetilde{\alpha_2}})] \end{pmatrix} \quad \dots\dots (4.1)$$

$$\widetilde{\alpha_1} \cap \widetilde{\alpha_2} = \begin{pmatrix} [\min(a_1, a_2), \min(b_1, b_2), \min(c_1, c_2), \min(d_1, d_2)]; \\ [\min(m^L_{\widetilde{\alpha_1}}, m^L_{\widetilde{\alpha_2}}), \min(m^U_{\widetilde{\alpha_1}}, m^U_{\widetilde{\alpha_2}}), \max(n^L_{\widetilde{\alpha_1}}, n^L_{\widetilde{\alpha_2}}), \max(n^U_{\widetilde{\alpha_1}}, n^U_{\widetilde{\alpha_2}})] \end{pmatrix} \quad \dots\dots (4.2)$$

Definition 7: Jaccard Distance on IVTrIFSs

If $\widetilde{\alpha_1}$ and $\widetilde{\alpha_2}$ are two IVTrIFSs then the Jaccard distance between $\widetilde{\alpha_1}$ and $\widetilde{\alpha_2}$ is denoted by $d_{JD}(\widetilde{\alpha_1}, \widetilde{\alpha_2})$ is defined as follows:

$$d_{JD}(\widetilde{\alpha_1}, \widetilde{\alpha_2}) = 1 - \frac{|\widetilde{\alpha_1} \cap \widetilde{\alpha_2}|}{|\widetilde{\alpha_1} \cup \widetilde{\alpha_2}|} = 1 - \frac{d_e(\widetilde{\alpha_1} \cap \widetilde{\alpha_2}, \widetilde{\alpha_0})}{d_e(\widetilde{\alpha_1} \cup \widetilde{\alpha_2}, \widetilde{\alpha_0})}$$

Where $\widetilde{\alpha_0} = ([0,0,0,0]; [0,0], [0,0])$,

$\widetilde{\alpha_1} \cup \widetilde{\alpha_2}$ follows from equation (4.1)

$\widetilde{\alpha_1} \cap \widetilde{\alpha_2}$ follows from equation (4.2)

Proposition 4.1: The proposed JD on IVTrIFSs satisfies the following axioms:

(A1) $0 \leq d_{JD}(\widetilde{\alpha_1}, \widetilde{\alpha_2}) \leq 1, \forall \widetilde{\alpha_1}, \widetilde{\alpha_2} \in \mathcal{C}(X)$, where $\mathcal{C}(X)$is class of subsets of X.

(A2) $d_{JD}(\widetilde{\alpha_1}, \widetilde{\alpha_2}) = 0$, if $\widetilde{\alpha_1} = \widetilde{\alpha_2}$

(A3) $d_{JD}(\widetilde{\alpha_1}, \widetilde{\alpha_2}) = d_{JD}(\widetilde{\alpha_2}, \widetilde{\alpha_1})$

(A4) If $d_{JD}(\widetilde{\alpha_1}, \widetilde{\alpha_2}) = 0, d_{JD}(\widetilde{\alpha_2}, \widetilde{\alpha_3}) = 0$ for $\widetilde{\alpha_3} \in \mathcal{C}(X)$then$d_{JD}(\widetilde{\alpha_1}, \widetilde{\alpha_3}) = 0$.

Proof: Proofs are verified (similar to proposition 3.1).

Example 3: Consider two IVTrIFSs

$\widetilde{\alpha_1}$ = ([0.4, 0.5, 0.6, 0.7]; [0.4, 0.5], [0.3, 0.4]) and $\widetilde{\alpha_2}$ = ([0.5, 0.6, 0.7, 0.9]; [0.2, 0.4], [0.1, 0.2])

The Jaccard distance between $\widetilde{\alpha_1}$ and $\widetilde{\alpha_2}$is $d_{JD}(\widetilde{\alpha_1}, \widetilde{\alpha_2}) = 1 - \frac{|\widetilde{\alpha_1} \cap \widetilde{\alpha_2}|}{|\widetilde{\alpha_1} \cup \widetilde{\alpha_2}|}$

From equation (4.1), we have

$$\widetilde{\alpha_1} \cup \widetilde{\alpha_2} = ([0.5, 0.6, 0.7, 0.9];\ [0.4, 0.5], [0.1, 0.2])$$

From equation (4.2), we have

$$\widetilde{\alpha_1} \cap \widetilde{\alpha_2} = ([0.4, 0.5, 0.6, 0.7];\ [0.2, 0.4], [0.3, 0.4])$$

then $|\widetilde{\alpha_1} \cap \widetilde{\alpha_2}| = d_e(\widetilde{\alpha_1} \cap \widetilde{\alpha_2}, \widetilde{\alpha_0}) = \sqrt{\frac{1}{8}(0.875)} = \sqrt{0.109} = 0.3331$

and $|\widetilde{\alpha_1} \cup \widetilde{\alpha_2}| = d_e(\widetilde{\alpha_1} \cup \widetilde{\alpha_2}, \widetilde{\alpha_0}) = \sqrt{\frac{1}{8}(2.308)} = \sqrt{0.288} = 0.537$

therefore, $d_{JD}(\widetilde{\alpha_1}, \widetilde{\alpha_2}) = 1 - \frac{0.3331}{0.537} = 1 - 0.616 = 0.384$

Thus, $d_{JD}(\widetilde{\alpha_1}, \widetilde{\alpha_2}) = 0.384$.

To assess the efficacy of the modified ED and the existing ED when used in other methods, we calculated the JD using existing ED [16] as well as modified ED for Example 1 discussed in section 3.

Let $\widetilde{\alpha_1}$ = ([0.29, 0.42, 0.54, 0.69]; [0.4, 0.5], [0.2, 0.4]), $\widetilde{\alpha_2}$ = ([0.25, 0.41, 0.52, 0.66]; [0.2, 0.4], [0.3, 0.5]) be two IVITrFSs.

JD with existing ED [16]:

$$d_{JD}\,(\widetilde{\alpha_1},\widetilde{\alpha_2}) = 1 - \frac{d_E(\widetilde{\alpha_1} \cap \widetilde{\alpha_2},\widetilde{\alpha_0})}{d_E(\widetilde{\alpha_1} \cup \widetilde{\alpha_2},\widetilde{\alpha_0})}$$

$$= 1 - \frac{d_E\big(([0.25, 0.41, 0.52, 0.66];\ [0.2, 0.4], [0.3, 0.5]), ([0,0,0,0]; [0,0], [0,0])\big)}{d_E\big(([0.29, 0.42, 0.54, 0.69];\ [0.4, 0.5], [0.2, 0.4])), ([0,0,0,0]; [0,0], [0,0])\big)}$$

$$= 1 - \frac{0.1076}{0.1082} = 1 - 1.006 = -0.01$$

therefore, $d_{JD}\,(\widetilde{\alpha_1},\widetilde{\alpha_2}) = -0.01$, which is absurd as the distance measure cannot be negative.

Whereas, JD with Modified ED is obtained as:

$$d_{JD}\,(\widetilde{\alpha_1},\widetilde{\alpha_2}) = 1 - \frac{d_e(\widetilde{\alpha_1} \cap \widetilde{\alpha_2},\widetilde{\alpha_0})}{d_e(\widetilde{\alpha_1} \cup \widetilde{\alpha_2},\widetilde{\alpha_0})}$$

$$= 1 - \frac{d_e\big(([0.25, 0.41, 0.52, 0.66];\ [0.2, 0.4], [0.3, 0.5]), ([0,0,0,0]; [0,0], [0,0])\big)}{d_e\big(([0.29, 0.42, 0.54, 0.69];\ [0.4, 0.5], [0.2, 0.4])), ([0,0,0,0]; [0,0], [0,0])\big)}$$

$$= 1 - \frac{0.353}{0.283} = 1 - 0.802 = 0.198$$

therefore, $d_{JD}\,(\widetilde{\alpha_1},\widetilde{\alpha_2}) = 0.198$

This demonstrates that the existing ED[1] have significant limitations when used in other approaches.

Definition 8: Ranking of IVTrIFSs

For any two IVTrIFSs$\widetilde{\alpha_1}$and $\widetilde{\alpha_2}$ the ranking is given as follows:

(i) If $d_{JD}\,(\widetilde{\alpha_1},\widetilde{\alpha}_I) < d_{JD}\,(\widetilde{\alpha_2},\widetilde{\alpha}_I)$ then $\widetilde{\alpha_1} > \widetilde{\alpha_2}$
(ii) If $d_{JD}\,(\widetilde{\alpha_1},\widetilde{\alpha}_I) > d_{JD}\,(\widetilde{\alpha_2},\widetilde{\alpha}_I)$ then $\widetilde{\alpha_1} < \widetilde{\alpha_2}$
(iii)If $d_{JD}\,(\widetilde{\alpha_1},\widetilde{\alpha}_I) = d_{JD}\,(\widetilde{\alpha_2},\widetilde{\alpha}_I)$ then $\widetilde{\alpha_1} = \widetilde{\alpha_2}$

Where $\widetilde{\alpha}_I = ([\max(a_1, a_2)\,, \max(b_1, b_2)\,, \max(c_1, c_2)\,, \max(d_1, d_2)];$

$[\max(m^L_{\widetilde{\alpha_1}}, m^L_{\widetilde{\alpha_2}}), \max(m^U_{\widetilde{\alpha_1}}, m^U_{\widetilde{\alpha_2}}), \min(n^L_{\widetilde{\alpha_1}}, n^L_{\widetilde{\alpha_2}}), \min(n^U_{\widetilde{\alpha_1}}, n^U_{\widetilde{\alpha_2}})])$ is the positive Ideal IVTrIFS for $\widetilde{\alpha_1}$and $\widetilde{\alpha_2}$.

Example 4: Consider two IVTrIFSs

$\widetilde{\alpha_1}$ = ([0.3, 0.5, 0.6, 0.7]; [0.3, 0.5], [0.1, 0.4]) and $\widetilde{\alpha_2}$ = ([0.4, 0.6, 0.7, 0.8]; [0.4, 0.7], [0.1, 0.2])

The ideal IVTrIFS for $\widetilde{\alpha_1}$and $\widetilde{\alpha_2}$is$\widetilde{\alpha}_I = ([0.4, 0.6, 0.7, 0.8]; [0.4,0.7]; [0.1,0.2])$

Then

$$d_{JD}(\widetilde{\alpha_1}, \widetilde{\alpha}_I) = 1 - \frac{|\widetilde{\alpha_1} \cap \widetilde{\alpha}_I|}{|\widetilde{\alpha_1} \cup \widetilde{\alpha}_I|} = 1 - \frac{d_e(\widetilde{\alpha_1} \cap \widetilde{\alpha}_I, \widetilde{\alpha_0})}{d_e(\widetilde{\alpha_1} \cup \widetilde{\alpha}_I, \widetilde{\alpha_0})}$$

$$= 1 - \frac{d_e\big(([0.3,0.5, 0.6, 0.7]; [0.3,0.5], [0.1,0.4]), ([0,0,0,0]; [0,0], [0,0])\big)}{d_e\big(([0.4, 0.6, 0.7, 0.8]; [0.4,0.7]; [0.1,0.2]), ([0,0,0,0]; [0,0], [0,0])\big)}$$

$$= 1 - \frac{0.495}{0.393} = 1 - 0.795 = 0.205$$

$$d_{JD}(\widetilde{\alpha_1}, \widetilde{\alpha}_I) = 0.205$$

and

$$d_{JD}(\widetilde{\alpha_2}, \widetilde{\alpha}_I) = 1 - \frac{|\widetilde{\alpha_2} \cap \widetilde{\alpha}_I|}{|\widetilde{\alpha_2} \cup \widetilde{\alpha}_I|} = 1 - \frac{d_e(\widetilde{\alpha_2} \cap \widetilde{\alpha}_I, \widetilde{\alpha_0})}{d_e(\widetilde{\alpha_2} \cup \widetilde{\alpha}_I, \widetilde{\alpha_0})}$$

$$= 1 - \frac{d_e\big(([0.4, 0.6, 0.7, 0.8]; [0.4,0.7], [0.1,0.2]), ([0,0,0,0]; [0,0], [0,0])\big)}{d_e\big(([0.4, 0.6, 0.7, 0.8]; [0.4,0.7]; [0.1,0.4]), ([0,0,0,0]; [0,0], [0,0])\big)}$$

$$= 1 - \frac{0.533}{0.495} = 1 - 0.928 = 0.072$$

$$d_{JD}(\widetilde{\alpha_2}, \widetilde{\alpha}_I) = 0.072$$

Therefore, $d_{JD}(\widetilde{\alpha_1}, \widetilde{\alpha}_I) > d_{JD}(\widetilde{\alpha_2}, \widetilde{\alpha}_I)$

Thus, $\widetilde{\alpha_1} < \widetilde{\alpha_2}$.

Comparison study

In this section, the proposed methods are compared with the existing methods based on score and accuracy functions namely Wu & Liu [23] ,Sireesha&Himabindu [6] and also with the existing Euclidean distance [1] for the Examples discussed in above sections. The obtained results are listed in Table 1.

Table 1: Comparative Study

Example	Wu&Liu [18]	Sireesha&Himabindu [12]	Euclidean Distance [16]	Modified Euclidean distance	Jaccard Distance
Ex: 1	$\widetilde{\alpha_1} > \widetilde{\alpha_2}$	$\widetilde{\alpha_1} > \widetilde{\alpha_2}$	$\widetilde{\alpha_2} > \widetilde{\alpha_1}$	$\boldsymbol{\widetilde{\alpha_1} > \widetilde{\alpha_2}}$	$\boldsymbol{\widetilde{\alpha_1} > \widetilde{\alpha_2}}$
Ex: 2	$\widetilde{\alpha_1} > \widetilde{\alpha_2}$	$\widetilde{\alpha_1} > \widetilde{\alpha_2}$	$\widetilde{\alpha_2} > \widetilde{\alpha_1}$	$\boldsymbol{\widetilde{\alpha_1} > \widetilde{\alpha_2}}$	$\boldsymbol{\widetilde{\alpha_1} > \widetilde{\alpha_2}}$
Ex: 3	$\widetilde{\alpha_2} > \widetilde{\alpha_1}$	$\widetilde{\alpha_2} > \widetilde{\alpha_1}$	$\widetilde{\alpha_2} > \widetilde{\alpha_1}$	$\boldsymbol{\widetilde{\alpha_2} > \widetilde{\alpha_1}}$	$\boldsymbol{\widetilde{\alpha_2} > \widetilde{\alpha_1}}$
Ex: 4	$\widetilde{\alpha_2} > \widetilde{\alpha_1}$	$\widetilde{\alpha_2} > \widetilde{\alpha_1}$	$\widetilde{\alpha_2} > \widetilde{\alpha_1}$	$\boldsymbol{\widetilde{\alpha_2} > \widetilde{\alpha_1}}$	$\boldsymbol{\widetilde{\alpha_2} > \widetilde{\alpha_1}}$

The comparison analysis shows that, in contrast to the existing Euclidean distance approach, which fails in the first two examples, the proposed methods agree with the ranking methods based on score and accuracy functions provided by [23&6]. This means that the improved ED and JD approaches that have been developed are efficient distance-based methods for ranking IVTrIFSs.

Conclusion

Due to the ambiguity and complexity that exist in real life, it is difficult to assess the corresponding characteristics of a problem with precision and certainty. IVTrIFSs are one of the generalizations of IFSs that have been shown to be effective in modelling such data. The similarity/distance measure is an important tool for analyzing fuzzy data. As a result, the focus of this paper was on developing effective distance-based measures to rank IVTrIFSs. While researching existing distance measures, it was found that the existing Euclidean distance measure fails to rank the IVTrIFSs in some cases. As a result, a modified Ed distance is proposed in this paper, as well as a new distance measure-Jaccard (JD) proposed by using this modified Euclidean distance (ED). The deserved properties of distance measure are verified for both proposed methods. The limitations of existing method and efficacy of the proposed methods are discussed by taking numerical examples. Further, a comparative analysis is done with other existing ranking methods developed based on score and accuracy functions. The comparison reveals that the proposed methods are on par with score and accuracy based methods. Thereby, these methods can be applicable in solving the decision-making problems such as medical diagnosis and pattern recognition etc.

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