Goodness-of-Fit Tests for the Gamma Distribution based on Residual Extropy

Sara Chabbi[#1],Ahmad Zghoul[*2]
Department of Mathematics, The University of Jordan. Amman 11942, Jordan

e-mail: chabbi.sarah30@gmail.com.

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Abstract

In this article, we propose test statistics based on so-called residual extropy to test the conformity of a random sample with a two-parameter gamma distribution. First, we characterize the gamma distribution based on residual extropy. Then we form tests as the integrated deviation (or square deviation) between the sample and population residual extropies. Finally, Monte Carlo simulations will be conducted to calculate critical values and powers of the proposed tests and as well as the classical EDF tests; namely, Kolmogorov-Smirnov (KS), Cramer-von Mises (CvM), and Anderson-Darling (AD) tests. Power comparisons show that the proposed tests outperform the classical tests for a broad spectrum of alternative distributions.

Keywords: -Gamma distributiont; residual extropy; power; goodness-of-fit.

1. Introduction

A random variable X has a two-parameter gamma distribution if its probability density function is of the form

$$f(x;\alpha,\beta)=\frac{\beta^{-\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta},0<x;\alpha>0,\beta>0, \qquad (1)$$

αandβ are, respectively, the shape and scale parameters.

The gamma family of distributions, type III of the Pearson system, has attracted the attention of researchers from both theoretical and applied perspectives. The gamma distribution is also related to the normal distribution; the sum of the squares of independent standard normal variables is distributed gamma. In addition, the gamma distributions have been used to model data from different fields of applications. Besides including the exponential distribution, which is widely used in lifetime analysis, the gamma distribution can make necessary adjustments to account for lifetime cases that the exponential distribution falls short of.

Among other applications, the gamma distribution has been fitted to personal income data [Salem & Mount [1]]; medical data [Robson [2], Belikov [3] and Belikov et al. [4]], and communication systems, Al-Zubi et al. [5]. Furthermore, the gamma distribution has also been extensively applied in hydrological data, including rainfall precipitations and river flows[Markovic [6], Sen and Eljadid [7], Bobee&Ashkar [8], and Aksoy [9]], to mention some. Furthermore, the reproductive property of the gamma distribution (the sum of independent gamma variables with a common scale parameter is also distributed gamma) leads to the appearance of gamma distributions in the theoryof counting processes such as the insurance claims process, Boland [10], and meteorological precipitation process, Kotz and Neumann [11].

An essential aspect of data analysis is the problem of fitting given data to a particular probability distribution model. In this sense, the gamma distribution has received the attention of many researchers. Classical tests such as Kolmogorov-Smirnov, Cramer-von-Mises, and Anderson-Darling tests may be naturally applied to test for the gamma distribution. These tests based on the empirical distribution function (EDF) are distribution-free. However, if one or more of the distribution parameters is unknown and must be estimated from the sample, then the distribution of these tests will depend on the method of estimation used and are not distribution-free anymore.

In addition to the EDF tests, we find some other tests introduced based on different aspects or characterizations of the gamma distribution; Dahiya and Gurland [12] used a generalized minimum χ^2 method to develop a goodness-of-fit test for the gamma and exponential distributions. To discriminate between three families of distributions: lognormal, gamma, and Weibull, Kubler [13] used the likelihood ratio tests. Wodruff et al. [14] modified KS, CvM, and AD tests. Stephen (D'Agostino and Stephens, [15]) used regression and correlation methods to test for gamma distribution. Based on Kullback-Leibler divergence as a measure of the proportionate reduction of uncertainty, Cameron and Windmeijer [16] developed an R-squared goodness-of-fit test for the exponential family regression models, including the gamma distribution. Zhang [17] modified old-nonparametric tests and proposed new ones through a specific parameterization process. Raschke [18] proposed a goodness-of-fit test using the biased transformation. Wilding and Mudholkar [19] utilized a characterization of the gamma distribution that the sample mean and the sample coefficient of variation are independent to produce a new gamma test. Finally, Henze et al. [20] introduced a test based on the empirical Laplace transform.

The extropy is an information measure recently introduced by Lad et al. [21] as the complement dual of Shannon [22]. This article will build goodness of fit tests based on the extropy measure for the gamma distribution. The differential extropy of a continuous random variable (r.v.) X with a probability density function (pdf) f was defined by Lad et al. as

$$J(X) = \frac{-1}{2}\int_{-\infty}^{\infty} f^2(x)dx, \qquad (2)$$

If X~gamma(α,1) with pdf given in (1), then

$$J(X) = \frac{-1}{2}\int_{-\infty}^{\infty} f^2(x;\alpha)dx$$

$$= \frac{-1}{2\Gamma^2(\alpha)}\int_{0}^{\infty} x^{2\alpha-2}e^{-2x}dx$$

$$= -\frac{\Gamma(2\alpha-1)}{4^{\alpha}\Gamma^2(\alpha)}, \alpha > \frac{1}{2}.$$

Residual extropy was defined by Qiu and Jia [23] as a measure of residual uncertainty of a non-negative continuous random variable X as

$$J(X,t)=\frac{-1}{2\bar{F}^2(t)}\int_t^{\infty} f^2(x)dx\,,t>0, \tag{3}$$

where $\bar{F}(t)=1-F(t)$, is the survival of X. $\bar{F}(t;\alpha)=1-F(t;\alpha)$

For the gamma case with scale parameter $\beta=1$, we have

$$\begin{aligned}J(X,t)&=\frac{-1}{2\bar{F}^2(t;\alpha)}\int_t^{\infty} f^2(x;\alpha)dx\\ &=\frac{-1}{2\Gamma^2(\alpha)\bar{F}^2(t)}\int_t^{\infty} x^{2\alpha-2}e^{-2x}dx\\ &=\frac{-\Gamma(2\alpha-1)F(2t;2\alpha-1)}{4^{\alpha}\Gamma^2(\alpha)\bar{F}^2(t;\alpha)},\alpha>\frac{1}{2},t>0.\end{aligned} \tag{4}$$

We will utilize the residual extropy measure to form new tests for the gamma distribution. The remainder of this article is structured as follows. Section 2 will prove a characterization based on the gamma residual extropy given in (4) and then use this characterization to propose new tests. In Section 3, using Monte Carlo simulations, critical values will be calculated and then used to compute the powers of the proposed tests.

Comparisons to the classical EDF tests will be conducted in Section 4. Finally, discussions of the results will be given in Section 5.

2. Characterization and Derivation of Tests

In this section, we will prove a characterization of the gamma distribution based on residual extropy, and then we will utilize this characterization to derive tests for gamma.

Theorem: *LetX be a non-negative random variable with pdf f and cdf F. Then $X \sim gamma(\alpha,1)$ iff*

$$2E_t[Xf(X)+(2\alpha-1)J(X,t)\bar{F}^2(t)]=tf^2(t),t>0, \tag{5}$$

Where

$$E_t[Xf(X)]=\int_t^{\infty} xf^2(x)dx, \tag{6}$$

Proof: Assume $X \sim gamma(\alpha,1)$, then

$$2E_t[X\,f(X)+(2\alpha-1)J(X,t)\bar{F}^2(t)]=2\int_t^{\infty} xf^2(x)\,dx-(2\alpha-1)\int_t^{\infty} f^2(x)\,dx$$

$$= \frac{2}{\Gamma^2(\alpha)}\int_t^\infty x^{2\alpha-1}e^{-2x}\,dx - \frac{2\alpha-1}{\Gamma^2(\alpha)}\int_t^\infty x^{2\alpha-2}e^{-2x}\,dx \tag{7}$$

Integrating the first term of the right-hand side of equation (7) by parts, we get

$$\frac{2}{\Gamma^2(\alpha)}\int_t^\infty x^{2\alpha-1}e^{-2x}\,dx = t^{2\alpha-1}e^{-2t} + \frac{2\alpha-1}{\Gamma^2(\alpha)}\int_t^\infty x^{2\alpha-2}e^{-2x}\,dx$$

Replacing this in equation (5), we get

$$2E_t[X\,f(X)] + (2\alpha-1)J(X,t)\bar{F}^2(t)] = \frac{t^{2\alpha-1}e^{-2t}}{\Gamma^2(\alpha)} = tf^2(t).$$

Now, assume $F(t)$ is absolutely continuous, and equation (5) is satisfied. From equation (3), we have

$$2\bar{F}^2(t)J(X,t)] = -\int_t^\infty f^2(x)dx,$$

so equation (5) can be written as

$$2\int_t^\infty xf^2(x)dx - (2\alpha-1)\int_t^\infty f^2(x)dx = tf^2(t) \tag{8}$$

Differentiation of both sides of equation (8) gives,

$$2tf^2(t) + (2\alpha-1)f^2(t) = 2tf(t)f'(t) + f^2(t) \tag{9}$$

which, when simplified, reduces to

$$tf^2(t) + (\alpha-1)f^2(t) = tf(t)f'(t) \tag{10}$$

Dividing both sides of equation (10) by $tf^2(t)$, it simplified to

$$\frac{(\alpha-1)}{t} - 1 = \frac{f'(t)}{f(t)} \tag{11}$$

Solving equation (11) for t, we get

$f(t) = ct^{\alpha-1}e^{-t}$.

For $f(t)$ to be a pdf, $c = 1/\Gamma(\alpha)$.

3. Proposed Tests

In this section, we construct test statistics based on the above characterization. Let $X_1,\ldots,X_n$ be a random sample from an absolutely continuous distribution with density function f and distribution function F, then the empirical counterpart of the left-hand side of equation (5), say $\Psi_n(\mathbf{X},\mathrm{t})$, is

$$\Psi_n(\mathbf{X},t) \equiv \frac{1}{n}\sum_{j=1}^{n}\psi(X_j)I_{X_j>t}$$

where

$$\psi(x) = 2xf(\mathrm{x}) - (2\alpha - 1)f^2(x)$$

By the laws of large numbers, $\Psi_n(\mathbf{X},t)$ converges almost surely to its mean,

$$\Psi_n(\mathbf{X},\mathrm{t}) \xrightarrow{n\to\infty} E\left[\Psi_n(\mathbf{X},\mathrm{t})\right] = tf^2(t).$$

Thus $\Psi_n(\mathbf{X},\mathrm{t})$ approximates $tf^2(t)$. Measures of deviation of $\Psi_n(\mathbf{X},\mathrm{t})$ from its limit, $tf^2(t)$, will be employed to form our test statistics. Examples of such deviance measures are:

$$D_{1,n}(\mathbf{X},t) = \int_0^\infty (\Psi_n(\mathbf{X},\mathrm{t}) - tf^2(t))\omega_1(t)dt \qquad (12)$$

and

$$D_{2,n}(\mathbf{X},\mathrm{t}) = \int_0^\infty (\Psi_n(\mathbf{X},\mathrm{t}) - tf^2(t))^2\omega_2(t)dt \qquad (13)$$

where $\omega_j(t)$, $j = 1,2$, are properly chosen weight functions such that $D_{1,n}(\mathbf{X},\mathrm{t})$ and $D_{2,n}(\mathbf{X},\mathrm{t})$ exist and have conveniently computationally closed forms.

When $\omega_1(t) = \omega_2(t) = \exp(-at), a > 0$, we have

$$\begin{aligned}
D_{1,n}(\mathbf{X},t) &= \int_0^\infty (\Psi_n(\mathbf{X},\mathrm{t}) - tf^2(t))\omega_1(t)dt \\
&= \int_0^\infty \frac{1}{n}\sum_{j=1}^{n}[\psi(X_j)\, I_{X_j>t} - tf^2(t)]e^{-at}dt \\
&= \int_0^\infty \left[\frac{1}{n}\sum_{j=1}^{n}\left(2X_j f(X_j) - (2\alpha-1)f^2(X_j)\right) I_{X_j>t} - tf^2(t))\right] e^{-at}dt \\
&= \frac{1}{n}\sum_{j=1}^{n}\left[2X_j f(X_j) - (2\alpha-1)f^2(X_j)\right]\int_0^{X_j} e^{-at}dt - \int_0^\infty tf^2(t)e^{-at}dt \\
&= \frac{1}{n}\sum_{j=1}^{n}\left[2X_j\, \mathrm{f}(X_j) - (2\alpha-1)f^2(X_j)\right]\left(\frac{1-e^{-aX_j}}{a}\right) - \frac{1}{\Gamma^2(\alpha)}\int_0^\infty t^{2\alpha-1}e^{-(a+2)t}dt
\end{aligned}$$

$$=\frac{1}{an}\sum_{j=1}^{n}\left[2X_j\, \mathrm{f}\left(X_j\right)-(2\alpha-1)f^2\left(X_j\right)\right](1-e^{-aX_j})-\frac{\Gamma(2\alpha)}{(a+2)^{2\alpha}\Gamma^2(\alpha)}. \qquad (14)$$

To derive a computational form for $D_{2,n}$, we have

$$D_{2,n}=\int_0^\infty [\Psi_n(\mathbf{X},t)-tf^2(t)]^2e^{-at}\,dt$$

$$=\underbrace{\int_0^\infty \Psi_n^2(\mathbf{X},t)e^{-at}\,dt}_{\mathrm{I}_1}-\underbrace{2\int_0^\infty tf^2(t)\Psi_n(\mathbf{X},t)e^{-at}\,dt}_{\mathrm{I}_2}+\underbrace{\int_0^\infty t^2f^4(t)e^{-at}\,dt}_{\mathrm{I}_3}.$$

$$\mathrm{I}_1=\int_0^\infty \Psi_n^2(\mathbf{X},t)e^{-at}\,dt$$

$$=\int_0^\infty\left(\frac{1}{n}\sum_{j=1}^{n}\psi(X_j)\,I_{X_j>t}\right)^2e^{-at}\,dt$$

$$=\frac{1}{n^2}\sum_{i,j=1}^{n}\psi(X_i)\psi(X_j)\int_0^\infty e^{-at}I_{X_j\wedge X_j>t}\,dt$$

$$=\frac{1}{n^2}\sum_{i,j=1}^{n}\psi(X_i)\psi(X_j)\int_0^\infty e^{-at}I_{X_j\wedge X_j>t}\,dt$$

$$=\frac{1}{an^2}\sum_{i,j=1}^{n}\psi(X_i)\psi(X_j)\,(1-e^{-a(X_j\wedge X_j)}). \qquad (15)$$

$$\mathrm{I}_2=2\int_0^\infty tf^2(t)\Psi_n(\mathbf{X},t)e^{-at}\,dt$$

$$=\frac{2}{n}\sum_{i,j=1}^{n}\psi(X_j)\int_0^\infty tf^2(t)e^{-at}\,I_{X_j>t}dt$$

$$=\frac{2\Gamma(2\alpha)}{\Gamma^2(\alpha)}\frac{1}{n}\sum_{i,j=1}^{n}\psi(X_j)\int_0^{X_j}\frac{t^{2\alpha-1}e^{-(a+2)t}}{\Gamma(2\alpha)}dt$$

$$=\frac{2\Gamma(2\alpha)}{(a+2)^{2\alpha}\Gamma^2(\alpha)}\frac{1}{n}\sum_{i,j=1}^{n}\psi\left(X_j\right)G((2+a)X_j,2\alpha), \qquad (16)$$

where$G(x,2\alpha)$ is the *cdf* of the gamma$(2\alpha,1)$ random variable.

$$
\begin{aligned}
\mathrm{I}_3 &= \int_0^\infty t^2 f^4(t) e^{-at}\, dt \\
&= \frac{1}{\Gamma^4(\alpha)} \int_0^\infty t^{4\alpha-2} e^{-(a+4)t}\, dt \\
&= \frac{\Gamma(4\alpha-1)}{(a+4)^{(4\alpha-1)}\Gamma^4(\alpha)}.
\end{aligned}
\tag{17}
$$

Combining (15)-(17), we obtain the following computation form for $D_{2,n}(\mathbf{X},\mathrm{t})$

$$
\begin{aligned}
D_{2,n}(\mathbf{X},\mathrm{t}) &= \frac{1}{an^2} \sum_{i,j=1}^{n} \psi(X_i)\psi(X_j)\left(1 - e^{-a(X_j \wedge X_j)}\right) \\
&- \frac{2\Gamma(2\alpha)}{(a+2)^{2\alpha}\Gamma^2(\alpha)} \frac{1}{n} \sum_{i,j=1}^{n} \psi(X_j)\, G(X_j, 2\alpha) + \frac{\Gamma(4\alpha-1)}{(a+4)^{(4\alpha-1)}\Gamma^4(\alpha)}
\end{aligned}
\tag{18}
$$

In the above analysis, we have assumed, without loss of generality, that $\beta = 1$; otherwise, we divide X_j by β to rescale. If α and β are unknown, which is the practical case, then α and β must be estimated from the sample. In this case, α is replaced by $\hat{\alpha}$, and X_j is replaced by $X_j/\hat{\beta}, j = 1, \dots, n$, where $\hat{\alpha}$and$\hat{\beta}$ are estimates of α and β. Studies have shown that the power of the tests can vary depending on the estimation method used. This paper assumes that the parameters α and β are unknown, and we will use the maximum likelihood method to estimate both parameters.

4. Simulation of critical values and powers

Monte Carlo simulation is commonly used to handle non-analytically tractable statistical issues. In particular, the method generates random samples from a given distribution and utilizes that sample to evaluate some measurements of interest. We employ this procedure to calculate specified quantiles for the introduced tests and then calculate the powers of the tests.

The simulation proceeds as follows: We generate a random sample of specified size from gamma$(\alpha, \beta)$for a given value of the shape parameter α and, without loss of generality, for$\beta = 1$. We compute the MLEs for α and β, then calculate the underlined tests' values based on this sample. To find a simulated percentile of a given test, we repeat this process 10,000 times, then sort the computed test values in increasing order. The $100(1-\alpha)^{\text{th}}$ percentile, $0 < \alpha < 1$, is the $10000 \times (1-\alpha)^{\text{th}}$ ordered value. Our power computations will be for the nominal value $\alpha = 0.05$.

Critical values for the proposed tests D1 and D2 and the three EDF-based tests; Kolmogorov-Smirnov (KS), Cramer-von Mises (CM), and Anderson-Darling tests, are simulated and displayed in Table 1. The simulation is carried out for samples of sizes n = 10, 20, and 50 and shape parameters of values $\alpha = 0.5$ to$5.0$with increments of length 0.5.

Table 1.Critical values for KS, AD, CvM, D1 and D2 tests based on nominal value 0.05.

n	α	*KS*	*AD*	*CvM*	*D1*	*D2* 2.5%	*D2* 97.5%
		95%					
10	0.5	0.280	0.781	0.139	-0.054	0.052	0.011
10	1.0	0.273	0.740	0.130	-0.085	0.084	0.009
10	1.5	0.270	0.736	0.126	-0.113	0.115	0.014
10	2.0	0.271	0.737	0.127	-0.112	0.115	0.014
10	2.5	0.269	0.736	0.126	-0.119	0.120	0.015
10	3	0.269	0.742	0.127	-0.121	0.127	0.016
10	3.5	0.267	0.723	0.125	-0.123	0.125	0.016
10	4	0.267	0.726	0.123	-0.126	0.128	0.016
10	4.5	0.267	0.722	0.123	-0.127	0.131	0.017
10	5.0	0.268	0.723	0.124	-0.127	0.129	0.016
20	0.5	0.205	0.803	0.144	-0.031	0.032	0.003
20	1.0	0.197	0.754	0.131	-0.054	0.053	0.004
20	1.5	0.197	0.761	0.130	-0.068	0.068	0.005
20	2.0	0.196	0.750	0.128	-0.076	0.076	0.006
20	2.5	0.195	0.746	0.127	-0.082	0.082	0.007
20	3	0.194	0.750	0.127	-0.084	0.085	0.007
20	3.5	0.194	0.756	0.128	-0.087	0.090	0.008
20	4	0.195	0.740	0.126	-0.089	0.089	0.008
20	4.5	0.194	0.746	0.125	-0.090	0.090	0.008
20	5.0	0.195	0.746	0.127	-0.091	0.091	0.008
50	0.5	0.132	0.811	0.144	-0.018	0.018	0.0008
50	1.0	0.129	0.771	0.133	-0.031	0.031	0.001
50	1.5	0.127	0.766	0.131	-0.039	0.040	0.002
50	2.0	0.127	0.769	0.130	-0.047	0.046	0.002
50	2.5	0.124	0.749	0.127	-0.051	0.050	0.002
50	3	0.126	0.740	0.125	-0.055	0.052	0.003
50	3.5	0.126	0.763	0.128	-0.055	0.054	0.003
50	4	0.126	0.751	0.127	-0.055	0.057	0.003
50	4.5	0.125	0.752	0.128	-0.057	0.056	0.003
50	5.0	0.125	0.748	0.126	-0.056	0.057	0.003

We notice little variations withα in the critical values of K, CM, AD, and D2 tests; however, D1 has more variations, yet small ones. All tests, except CM, show a noticeable effect of the sample size on the critical values.

5. Simulated Powers

To assess the performance of the proposed tests compared to other gamma tests, we will use Monte Carlo simulation to compute the powers of these tests against a set of alternatives. First, we will apply these tests to test the gamma distribution against three competitive families of distributions; the gamma family, G(α,β), the lognormal family, LN(α,β), the Weibull family, W(α,β), and the half-normal family, HN(α). The first parameter (α) in the two-parameter families is the shape parameter and the second parameter (β) is the scale parameter. We will simulate gamma samples with the shape parameters α = 0.5,1, 1.5, 2.5, and 4 and all with the scale parameter, without loss of generality, $\beta = 1$. Before evaluating the underlined tests based on the simulated sample, we will use the maximum likelihood method to estimate the gamma parameters. Then, we will apply the tests under the null hypothesis that the given sample comes from a gamma population with unknown parameters. The powers will be computed for a nominal value of 0.05 and samples of sizes 10, 20, and 50. The power computation procedure follows the following steps: First, simulate a random sample of size n, say $x_1, \dots, x_n$, from G(α, β) with a specific value of α and with $\beta = 1$. Second, use the MLE method to estimate each of α and β as if they are unknown. Third, evaluate each underlined test at the sample values and the estimated parameters. Fourth, reject the null hypothesis if the computed value of a test lies in the critical region of the corresponding test given in Table1; otherwise, the hypothesis is not rejected. Then, repeat these four steps 10,000 times to calculate the approximate power as the proportion of rejections.

Table2 displays the simulated powers when testing gamma distributions against gamma alternatives with different shape parameters. Notice that the diagonal blocks represent testing gamma(α, β) against itself. We see that EDF as well as the proposed tests recover the nominal value $\alpha = 0.05$.

Table2. Power of the underlined tests when tested against gamma alternatives at the nominal value of 0.05 and n=20.

Test	*Alt.* *Null*	G(0.5,1)	G(1,1)	G(1.5,1)	G(2.5,1)	G(4,1)
K	G(0.5,1)	0.055	0.033	0.039	0.028	0.030
CM		0.055	0.030	0.028	0.024	0.032
AD		0.052	0.036	0.032	0.034	0.041
D1		0.049	0.225	0.336	0.432	0.496
D2		0.055	0.093	0.160	0.218	0.262
K	G(1,1)	0.071	0.047	0.053	0.055	0.043
CM		0.070	0.050	0.045	0.051	0.041
AD		0.068	0.049	0.046	0.061	0.048
D1		0.003	0.056	0.107	0.203	0.261
D2		0.028	0.054	0.094	0.157	0.189
K	G(1.5,1)	0.062	0.050	0.049	0.036	0.037
CM		0.056	0.045	0.053	0.039	0.038
AD		0.049	0.042	0.051	0.041	0.037
D1		0.000	0.009	0.046	0.095	0.119
D2		0.018	0.013	0.050	0.080	0.086
K	G(2.5,1)	0.070	0.051	0.056	0.051	0.037
CM		0.066	0.054	0.046	0.049	0.040
AD		0.061	0.055	0.046	0.052	0.045
D1		0.000	0.003	0.021	0.052	0.061
D2		0.015	0.006	0.025	0.051	0.052

K	G(4,1)	0.063	0.062	0.054	0.043	0.037
CM		0.072	0.060	0.060	0.046	0.035
AD		0.070	0.062	0.060	0.052	0.053
D1		0.000	0.004	0.016	0.030	0.049
D2		0.014	0.006	0.021	0.030	0.046

We can notice from Table3 that all considered EDF-based tests (K, CM, and AD) result in poor power when gamma distribution with a specific shape parameter is tested against another gamma with a different shape. Because as we noticed earlier in Section2, the EDF-based test quantiles, particularly the tests' critical values, are not noticeably affected by the value of the gamma shape parameter. Except for G[0.5,1], D1 and D2 tests also show low power values.

The power values of the tests when testing gamma against lognormal family members are displayed in Table3.

Table 3. The underlined tests' power when testing gamma against lognormal alternatives at the nominal value of 0.05 and n=20.

Test	*Alt.* *Null*	LN(0,0.5)	LN(0,1)	LN(0,1.5)	LN(0,2)
K	G(0.5,1)	0.051	0.154	0.290	0.443
CM		0.048	0.178	0.351	0.505
AD		0.070	0.207	0.388	0.524
D1		0.589	0.594	0.421	0.144
D2		0.346	0.347	0.201	0.318
K	G(1,1)	0.079	0.194	0.366	0.488
CM		0.080	0.218	0.417	0.556
AD		0.099	0.236	0.416	0.562
D1		0.347	0.217	0.037	0.001
D2		0.268	0.225	0.095	0.240
K	G(1.5,1)	0.073	0.175	0.342	0.464
CM		0.081	0.197	0.395	0.547
AD		0.091	0.208	0.391	0.538
D1		0.219	0.064	0.006	0.000
D2		0.168	0.075	0.030	0.146
K	G(2.5,1)	0.078	0.208	0.370	0.481
CM		0.079	0.225	0.415	0.554
AD		0.087	0.239	0.421	0.547
D1		0.131	0.027	0.001	0.000
D2		0.111	0.046	0.013	0.137
K	G(4,1)	0.066	0.192	0.354	0.480
CM		0.083	0.246	0.436	0.570
AD		0.094	0.255	0.426	0.567
D1		0.098	0.013	0.000	0.000

D2		0.095	0.036	0.015	0.140

The following can be mainly noticed in Table3:

1) When testing against LN (0.5,1), D1 and D2 tests perform much better than K, CM, and AD tests. In fact, K, CM, and AD show poor power against LN (0,0.5). We also notice that the power of D1 and D2 declines when the gamma distribution shape parameter increases.
2) Opposite to D1 and D2, the performance of K, CM, and AD improves with the rise of the LN shape parameter.
3) Compared to D1 and D2, the power of K, CM, and AD is higher when testing each of G(1.5,1), G(2.5,1), and G(4,1) against LN(0,1), LN(0,1.5), and LN(0,2.5).

Table 4 shows the simulated powers of the underlined tests when testing gamma against members of the Weibull family.

Table 4. The underlined tests' power when testing gamma against Weibull alternatives at the nominal value of 0.05 and n=20.

Test	*Alt.* / *Null*	Wei(0,0.5)	Wei(0,1)	Wei(0,1.5)	Wei(0,2)
K	G(0.5,1)	0.111	0.042	0.034	0.047
CM		0.125	0.038	0.041	0.051
AD		0.135	0.042	0.058	0.069
D1		0.026	0.221	0.419	0.601
D2		0.254	0.098	0.231	0.370
K	G(1,1)	0.148	0.059	0.061	0.080
CM		0.169	0.058	0.057	0.098
AD		0.156	0.062	0.061	0.115
D1		0.001	0.046	0.182	0.376
D2		0.213	0.048	0.148	0.309
K	G(1.5,1)	0.144	0.052	0.050	0.069
CM		0.162	0.048	0.053	0.084
AD		0.149	0.049	0.057	0.096
D1		0.000	0.008	0.079	0.249
D2		0.148	0.011	0.076	0.203
K	G(2.5,1)	0.150	0.060	0.058	0.088
CM		0.170	0.067	0.053	0.092
AD		0.149	0.061	0.053	0.098
D1		0.000	0.004	0.030	0.164
D2		0.128	0.007	0.036	0.142
K	G(4,1)	0.147	0.057	0.056	0.076
CM		0.175	0.067	0.069	0.107
AD		0.165	0.064	0.074	0.117
D1		0.000	0.001	0.028	0.135

D2		0.122	0.004	0.039	0.133

We can conclude the following from Table 4:

1) The three EDF tests have low powers when testing gamma against almost all considered Weibull alternatives regardless of the gamma shape.
2) The proposed D1 and D2 tests and the EDF tests show low performance when testing gamma against the considered Weibull alternatives, except the case G(0.5,1), for which D1 and D2 show reasonable power values, especially for Wei(1.5,1) and Wei(2,1) alternatives.

The powers of the considered tests against Halfnormal family members are displayed in Table 5.

Table 5. The underlined tests power when testing gamma against Halfnormal alternatives at the nominal value of 0.05 and n=20.

Test	*Alt.* / *Null*	HN(0.5)	HN (1)	HN (1.5)	HN (5)
K	G(0.5,1)	0.0635	0.0620	0.0735	0.0655
CM		0.0610	0.0615	0.0760	0.0780
AD		0.0780	0.0780	0.0985	0.0965
D1		0.4195	0.4120	0.4020	0.4140
D2		0.2335	0.2425	0.2420	0.2395
K	G(1,1)	0.0955	0.0940	0.0825	0.0860
CM		0.0980	0.1125	0.0940	0.0950
AD		0.1115	0.1190	0.1045	0.1040
D1		0.1670	0.1800	0.1500	0.1680
D2		0.1525	0.1680	0.1380	0.1595
K	G(1.5,1)	0.0860	0.0715	0.0845	0.0910
CM		0.0915	0.0855	0.0935	0.0910
AD		0.0935	0.0875	0.0945	0.0960
D1		0.0615	0.0570	0.0630	0.0655
D2		0.0665	0.0685	0.0655	0.0720
K	G(2.5,1)	0.0980	0.0855	0.1020	0.0930
CM		0.1045	0.0870	0.1070	0.0955
AD		0.1075	0.0860	0.1105	0.0990
D1		0.0305	0.0260	0.0320	0.0275
D2		0.0465	0.0385	0.0450	0.0400
K	G(4,1)	0.0940	0.0970	0.0790	0.0840
CM		0.1140	0.1125	0.1000	0.0955
AD		0.1175	0.1210	0.1095	0.1085
D1		0.0160	0.0145	0.0125	0.0150
D2		0.0340	0.0275	0.0280	0.0280

Table5 shows that all the considered tests have low power against Halfnormal alternatives except for G(0.5,1) when D1 and D2 have reasonable power values.

6. **Conclusions**

The proposed test shows higher power than EDF tests (K, CM, and AD) when testing against lognormal distributions with inrise shape parameters. Moreover, when testing against family of gamma distribution alternatives with shape parameter increases, we notice that the power of D1 and D2 declines. We also notice that the three EDF tests have low powers when testing gamma against almost all considered Weibull alternatives regardless of the gamma shape. Finally, all considered tests have lowpower against Halfnormal alternatives except for G(0.5,1) when D1 and D2 have reasonable power values.

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