Some Operations on Fermatean Fuzzy Soft Matrices

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Abstract

In this paper,we combine the characteristics of Fermatean fuzzy set with the parameterization of soft matrix and construct the Fermatean fuzzy soft matrix. Some operations such as union, intersection, complement, algebraic difference, algebraic division, meet, join, implication are given and proved some properties using these operations.

Keywords: - Intuitionistic fuzzy matrix, Pythagorean fuzzy matrix, Fermatean fuzzy matrix, Soft matrix, Pythagorean fuzzy soft matrix, Fermatean fuzzy soft matrix.

1. Introduction

Zadeh [24] introduced fuzzy set theory and Atanassov [2] generalised the fuzzy set by defining the intuitionistic fuzzy set. Yager and Abbasov [22] introduced Pythagorean membership grades. Since the PFM was initiated, it has been widely applied in different fields, such as investment decision making, service quality of transports, collaborative-based recommended systems, and so on. Although the PFM generalizes the IFM, it cannot describe the following decision information. A panel of experts were invited to give their opinions about the feasibility of an investment plan, and they were divided into two independent groups to make a decision. In [21] Yager and Senapati studied some new operations over Fermatean fuzzy numbers. The concept of an intuitionistc fuzzy matrix (IFM) was introduced by Khan et. al., [8] which is an extension of Thomason's [23] fuzzy matrix. Muthuraji et. al., [13] studied decomposition of intuitionistic fuzzy matrices. In [10] Meenakshi developed some new concepts on fuzzy matrices. Boobalan and Sriram [3] introduced the implication operator $\rightarrow$ and @ operations on intuitionistic fuzzy matrices.

Silambarasan [18,19] defined Fermatean fuzzy matrix and some new operators. In 1999, Molodtsov[11] presented the theory of soft set. In 2001, Maji et al.[9] studied the theory of soft set initiated by Molodtsov [11] and developed several basic notions of soft set theory and presented the idea of fuzzy soft sets. Naim and Serdar [14] introduced the soft matrices which are representations of the Molodtsov's soft sets and successfully applied the soft matrices in decision-making problems. Cagman and Enginoglu [4,5] has introduced the concept of soft matrix theory and applied it to decision making problems. Yong and Chenli [23] introduced fuzzy soft matrices and studied their basic properties. Chetia and Das [6] extended the matrix representation of soft set to fuzzy soft set and intuitionistic fuzzy soft matrix. Guleria and Bajaj [7] introduced Pythagorean fuzzy soft matrices and the various binary operations are analogously proposed for the PFSM and a new decision making algorithm has been proposed. Arikrishnan and Sriram [1] defined modal and implication operators of Pythagorean fuzzy soft matrices and established their algebraic properties. Seenivasan et al [15,16,17] studied performance analysis of two heterogenous server queueing model with intermittenly obtainable server using matrix geometric method. Various researchers have also worked on the concept of soft set and soft matrices which are available in the literature.

Motivated by the work of Murat [12], we defined operators $\boxminus_F, \boxslash_F, @, \cup, \cap$ and considered some properties of Fermatean fuzzy soft Matrix. In this paper, we have developed some new operators for Farmatean fuzzy soft matrices and discussed several properties.

2. Preliminaries

In this section, we recall some basic definitions and concepts related to Fermatean fuzzy set (FFS) which are well known in litrature.

Definition 2.1 [20]: A fuzzy matrix of A of order $m \times n$ is defined as $P = (p_{ij})$, where $p_{ij} \in [0,1]$.

Definition 2.2 [8]: An intuitionistic fuzzy matrix (IFM) is a pair $[\langle \xi_{p_{ij}}, \varsigma_{p_{ij}} \rangle]$ of non negative real numbers $\xi_{p_{ij}}, \varsigma_{p_{ij}} \in [0,1]$ satisfying $0 \leq \xi_{p_{ij}} + \varsigma_{p_{ij}} \leq 1$ for all i, j.

Definition 2.3 [15]: Fermatean fuzzy matrix (FFM) is a pair $P = [\langle \xi_{p_{ij}}, \varsigma_{p_{ij}} \rangle]$ of non negative real numbers $\langle \xi_{p_{ij}}, \varsigma_{p_{ij}} \rangle \in [0,1]$ satisfying the condition $0 \leq \xi_{p_{ij}}^3 + \varsigma_{p_{ij}}^3 \leq 1$ for all i, j. Where

$\xi_{p_{ij}} \in [0,1]$ is called the degree of membership and $\varsigma_{p_{ij}} \in [0,1]$ is called the degree of non-membership.

The generalization in terms of development of concepts from soft sets to Fermatean fuzzy soft sets is available with explanatory examples in literature ([10],[7]). Let $X = \{x_1, x_2, x_3, \dots x_n\}$ be the universe of discourse and $E = \{e_1, e_2, e_3, \dots e_n\}$ be the set of parameters. Consider $P \subseteq E$. The following are the basic notions of soft set, soft matrix and Fermatean fuzzy soft set/matrix which are well known in literature:

Definition 2.4 [11]: The pair (F_P, E) is called soft set over X if and only if $F_P: P \to P(X)$ is the power set of X.

Definition 2.5 [9]: Let $F(X)$ denotes the set of all fuzzy sets of X. P pair (F_P, E) is called a fuzzy soft set over $F(X)$, where F is a mapping given by $F_P: P \to P(F(X))$.

Definition 2.6 [14]: Let (F_P, E) be a soft set over X. Then the subset $X \times E$ is uniquely defined by $R_P = \{(x,e), e \in P, x \in F_P(e)\}$. The Characteristic function of R_P is $\chi_{R_P}: X \times E \to [0,1]$ given by $\chi_{R_P}(x,e) = 1 \; if \; (x,e) \in P$ and $\chi_{R_P}(x,e) = 0 \; if \; (x,e) \notin P$ If $a_{ij} = \chi_{R_P}(x,e)$, then a matrix $[p_{ij}] = [\chi_{R_P}(x,e)]$ is called soft matrix of the soft set (F_P, E) over X of order $m \times n$.

Definition 2.7 [7]: If (F_P, E) be a Pythagorean fuzzy soft set over X, then the subset, $X \times E$ is uniquely defined by $R_P = (x,e), e \in P, x \in F_P(e)$. The R_P can be characterized by its membership function and non-membership function given by $\xi_R: X \times E \to [0,1]$ and $\varsigma_R: X \times E \to [0,1]$, respectively. If $(\xi_{i,j}, \varsigma_{i,j}) = (\xi_{R_P}(x_i, e_j), \varsigma_{R_P}(x_i, ej))$, where $\xi_{R_P}(x_i, e_j)$ is the membership of x_i in the Pythagorean fuzzy set $F(e_j)$ and $\varsigma_{R_P}(x_i, e_j)$ is the non-membership of x_i in the Pythagorean fuzzy set $F(e_j)$, respectively, then we define a matrix given by

$$[M] = [m_{i,j}]_{m\times n} = [(\xi^M_{i,j}, \varsigma^M_{i,j})]_{m\times n} = \begin{bmatrix} (\xi_{11},\varsigma_{11}) & (\xi_{12},\varsigma_{12}) & \cdots (\xi_{1n},\varsigma_{1n}) \\ (\xi_{21},\varsigma_{21}) & (\xi_{21},\varsigma_{21}) & \cdots (\xi_{2n},\varsigma_{2n}) \\ \vdots & \vdots & \vdots \\ (\xi_{m1},\varsigma_{m1}) & (\xi_{m2},\varsigma_{m2}) & \cdots (\xi_{mn},\varsigma_{mn}) \end{bmatrix}$$

which is called **Pythagorean fuzzy soft matrix** of order $m \times n$ over X.

3. Fermatean fuzzy soft matrices and various operations

Since matrices play an important role in many computational techniques, handling dimensionality feature of various problems of engineering, medical sciences, social sciences, etc., it motivates to extend the concept of Fermatean fuzzy soft set into Fermatean fuzzy soft matrices. In this section, we propose the concept of Fermatean fuzzy soft matrix with various operations over it. Further, we proved some theorems and corollaries are proved by using Fermatean fuzzy soft matrix.

Definition 3.1: If (F_P, E) be a Fermatean fuzzy soft set over X, then the subset, $X \times E$ is uniquely defined by $R_P = (x, e), e \in P, x \in F_P(e)$. The R_P can be characterized by its membership function and non-membership function given by $\xi_R: X \times E \to [0,1]$ and $\varsigma_R: X \times E \to [0,1]$, respectively.

If $(\xi_{i,j}, \varsigma_{i,j}) = (\xi_{R_P}(x_i, e_j), \varsigma_{R_P}(x_i, ej))$, where $\xi_{R_P}(x_i, e_j)$ is the membership of x_i in the Fermatean fuzzy set $F(e_j)$ and $\varsigma_{R_P}(x_i, e_j)$ is the non-membership of x_i in the Fermatean fuzzy set $F(e_j)$, respectively, then we define a matrix given by

$$[M] = [m_{i,j}]_{m\times n} = [(\xi^M_{i,j}, \varsigma^M_{i,j})]_{m\times n} = \begin{bmatrix} (\xi_{11},\varsigma_{11}) & (\xi_{12},\varsigma_{12}) & \cdots (\xi_{1n},\varsigma_{1n}) \\ (\xi_{21},\varsigma_{21}) & (\xi_{21},\varsigma_{21}) & \cdots (\xi_{2n},\varsigma_{2n}) \\ \vdots & \vdots & \vdots \\ (\xi_{m1},\varsigma_{m1}) & (\xi_{m2},\varsigma_{m2}) & \cdots (\xi_{mn},\varsigma_{mn}) \end{bmatrix}$$

is called **Fermatean fuzzy soft matrix** of order $m \times n$ over X. For better understanding let us consider $X = \{x_1, x_2, x_3\}$ as a universal set and $E = \{e_1, e_2, e_3, e_4\}$ as a set of parameter. If $P = \{e_1, e_2, e_3\} \subseteq E$ and

$F_P(e_1) = \{(x_1, 0.7, 0.6), (x_2, 0.8, 0.5), (x_3, 0.9, 0.3)\}$

$F_P(e_2) = \{(x_1, 0.8, 0.6), (x_2, 0.7, 0.8), (x_3, 0.8, 0.7)\}$

$F_P(e_3) = \{(x_1, 0.9,0.4), (x_2, 0.5,0.9), (x_3, 0.6,0.7)\}$ Then (F_P, E) is the parameterized family of $F_P(e_1), F_P(e_2), F_P(e_3)$ over X. Hence the Fermatean fuzzy soft matrix is

$$M = [\xi_{ij}, \varsigma_{ij}]_{m\times n} = \begin{bmatrix} (0.7,0.6) & (0.8,0.6) & (0.9,0.4) \\ (0.8,0.5) & (0.7,0.8) & (0.5,0.9) \\ (0.9,0.3) & (0.8,0.7) & (0.6,0.7) \end{bmatrix}$$

Definition 3.2: Let $F_{m\times n}$ denote the family of all FFSMs for all i,j and $P, Q \in F_{m\times n}$ be given as

$P = [\xi_{p_{ij}}, \varsigma_{p_{ij}}]$ and $Q = [\xi_{q_{ij}}, \varsigma_{q_{ij}}]$ then

(i)$P \vee_F Q = [\max\{\xi_{p_{ij}}, \xi_{q_{ij}}\}, \min\{\varsigma_{p_{ij}}, \varsigma_{q_{ij}}\}]$

(ii)$P \wedge_F Q = [\min\{\xi_{p_{ij}}, \xi_{q_{ij}}\}, \max\{\varsigma_{p_{ij}}, \varsigma_{q_{ij}}\}]$

(iii) $P^C = [\langle \varsigma_{p_{ij}}, \xi_{p_{ij}} \rangle]$

(iv)$P \boxminus_F Q = \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}} \right\rangle\right]$

(v)$P \boxslash_F Q = \left[\left\langle \frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 - \varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}} \right\rangle\right]$

(vi)$P@Q = \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}{2}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}{2}} \right\rangle\right]$

(vii)$P\#Q = \left[\left\langle \frac{\sqrt[3]{2}\xi_{p_{ij}}\xi_{q_{ij}}}{\sqrt[3]{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}}, \frac{\sqrt[3]{2}\varsigma_{p_{ij}}\varsigma_{q_{ij}}}{\sqrt[3]{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}} \right\rangle\right]$

(viii)$P\$Q = \left[\left\langle \sqrt[3]{\xi_{p_{ij}}\xi_{q_{ij}}}, \sqrt[3]{\varsigma_{p_{ij}}\varsigma_{q_{ij}}} \right\rangle\right]$

(ix)$P \to Q = [\max\{\varsigma_{p_{ij}}, \xi_{q_{ij}}\}, \min\{\xi_{p_{ij}}, \varsigma_{q_{ij}}\}]$

Theorem 3.3 For $P, Q \in F_{m\times n}$, we have

(i)$(P^C \to Q) \boxminus_F (P \to Q^C)^C = (P \boxminus_F Q)$

(ii)$(P^C \to Q) \boxslash_F (P \to Q^C)^C = (P \boxslash_F Q)$

(iii)$(P \to Q)^C \boxminus_F (Q \to P) = (P \boxminus_F Q^C)$

(iv)$(P \to Q)^C \boxslash_F (Q \to P) = (P \boxslash_F Q^C)$

Proof. (i) Since

$$\begin{aligned}(P^C \to Q) \underset{F}{\boxminus} (P \to Q^C)^C &= \left[\sqrt[3]{\frac{\max\left\{\xi^3_{p_{ij}}, \varsigma^3_{q_{ij}}\right\} - \min\left\{\xi^3_{p_{ij}}, \xi^3_{q_{ij}}\right\}}{1-\min\left\{\xi^3_{p_{ij}}, \xi^3_{q_{ij}}\right\}}}, \frac{\min\left\{\varsigma_{p_{ij}}, \varsigma_{q_{ij}}\right\}}{\max\left\{\varsigma_{p_{ij}}, \varsigma_{q_{ij}}\right\}}\right] \\ &= \left[\left\langle \sqrt[3]{\frac{\xi^3_{p_{ij}} - \xi^3_{q_{ij}}}{1-\xi^3_{q_{ij}}}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}\right\rangle\right] = \left(P \underset{F}{\boxminus} Q\right)\end{aligned}$$

(ii) Since

$$\begin{aligned}(P^C \to Q) \underset{F}{\boxslash} (P \to Q^C)^C &= \left[\frac{\max\left\{\xi_{p_{ij}}, \xi_{q_{ij}}\right\}}{\min\left\{\xi_{p_{ij}}, \xi_{q_{ij}}\right\}}, \sqrt[3]{\frac{\min\left\{\varsigma^3_{p_{ij}}, \varsigma^3_{q_{ij}}\right\} - \max\left\{\varsigma^3_{p_{ij}}, \varsigma^3_{q_{ij}}\right\}}{1-\max\left\{\varsigma^3_{p_{ij}}, \varsigma^3_{q_{ij}}\right\}}}\right] \\ &= \left[\left\langle \frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}, \sqrt[3]{\frac{\varsigma^3_{p_{ij}} - \varsigma^3_{q_{ij}}}{1-\varsigma^3_{q_{ij}}}}\right\rangle\right] = \left(P \underset{F}{\boxslash} Q\right)\end{aligned}$$

(iii) Since

$$\begin{aligned}(P \to Q)^C \underset{F}{\boxminus} (Q \to P) &= \left[\sqrt[3]{\frac{\min\left\{\xi^3_{p_{ij}}, \varsigma^3_{q_{ij}}\right\} - \max\left\{\xi^3_{p_{ij}}, \varsigma^3_{q_{ij}}\right\}}{1-\max\left\{\xi^3_{p_{ij}}, \varsigma^3_{q_{ij}}\right\}}}, \frac{\max\left\{\varsigma_{p_{ij}}, \xi_{q_{ij}}\right\}}{\min\left\{\varsigma_{p_{ij}}, \xi_{q_{ij}}\right\}}\right] \\ &= \left[\left\langle \sqrt[3]{\frac{\xi^3_{p_{ij}} - \varsigma^3_{q_{ij}}}{1-\varsigma^3_{q_{ij}}}}, \frac{\varsigma_{p_{ij}}}{\xi_{q_{ij}}}\right\rangle\right] = \left(P \underset{F}{\boxminus} Q^C\right)\end{aligned}$$

(iv) Since

$$
\begin{aligned}
(P \rightarrow Q)^C \underset{F}{\boxslash} (Q \rightarrow P) &= \left[\frac{\min\left\{\xi_{p_{ij}}, \varsigma_{q_{ij}}\right\}}{\max\left\{\xi_{p_{ij}}, \xi_{q_{ij}}\right\}}, \sqrt[3]{\frac{\max\left\{\varsigma^3_{p_{ij}}, \xi^3_{q_{ij}}\right\} - \min\left\{\varsigma^3_{p_{ij}}, \xi^3_{q_{ij}}\right\}}{1 - \min\left\{\varsigma^3_{p_{ij}}, \xi^3_{q_{ij}}\right\}}} \right] \\
&= \left[\left\langle \frac{\xi_{p_{ij}}}{\varsigma_{q_{ij}}}, \sqrt[3]{\frac{\varsigma^3_{p_{ij}} - \xi^3_{q_{ij}}}{1 - \xi^3_{q_{ij}}}} \right\rangle\right] = \left(P \underset{F}{\boxslash} Q^C\right)
\end{aligned}
$$

Theorem 3.4 For $P, Q \in F_{m \times n}$, we have

(i)$((P \boxminus_F Q) \rightarrow (P@Q)^C)^C = ((P@Q) \rightarrow (P \boxminus_F Q)^C)^C = (P \boxminus_F Q)$

(ii)$((P \boxminus_F Q)^C \rightarrow (P@Q)) = ((P@Q)^C \rightarrow (P \boxminus_F Q)) = (P@Q)$

(iii)$((P \boxslash_F Q) \rightarrow (P@Q)^C)^C = ((P@Q) \rightarrow (P \boxslash_F Q)^C)^C = (P@Q)$

(iv)$((P \boxslash_F Q)^C \rightarrow (P@Q)) = ((P@Q)^C \rightarrow (P \boxslash_F Q)) = (P \boxslash_F Q)$

(v)$((P \boxminus_F Q) \rightarrow (P\#Q)^C)^C = ((P\#Q) \rightarrow (P \boxminus_F Q)^C)^C = (P \boxminus_F Q)$

(vi)$((P \boxminus_F Q)^C \rightarrow (P\#Q)) = ((P\#Q)^C \rightarrow (P \boxminus_F Q)) = (P\#Q)$

(vii)$((P \boxslash_F Q) \rightarrow (P\#Q)^C)^C = ((P\#Q) \rightarrow (P \boxslash_F Q)^C)^C = (P\#Q)$

(viii)$((P \boxslash_F Q)^C \rightarrow (P\#Q)) = ((P\#Q)^C \rightarrow (P \boxslash_F Q)) = (P \boxslash_F Q)$

(ix)$((P \boxminus_F Q) \rightarrow (P\$Q)^C)^C = ((P\$Q) \rightarrow (P \boxminus_F Q)^C)^C = (P \boxminus_F Q)$

(x)$((P \boxminus_F Q)^C \rightarrow (P\$Q)) = ((P\$Q)^C \rightarrow (P \boxminus_F Q)) = (P\$Q)$

(xi)$((P \boxslash_F Q) \rightarrow (P\$Q)^C)^C = ((P\$Q) \rightarrow (P \boxslash_F Q)^C)^C = (P\$Q)$

(xii)$((P \boxslash_F Q)^C \rightarrow (P\$Q)) = ((P\$Q)^C \rightarrow (P \boxslash_F Q)) = (P \boxslash_F Q)$

(xiii)$((P \boxminus_F Q) \rightarrow (P \boxslash_F Q)^C)^C = ((P \boxslash_F Q) \rightarrow (P \boxminus_F Q)^C)^C = (P \boxminus_F Q)$

(xiv)$((P \boxslash_F Q)^C \rightarrow (P \boxminus_F Q)) = ((P \boxminus_F Q)^C \rightarrow (P \boxslash_F Q)) = (P \boxslash_F Q)$

Proof. (i) Since

$$
\begin{aligned}
((P \underset{F}{\boxminus} Q) \to (P@Q)^C)^C &= \left[\min\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \sqrt[3]{\frac{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}{2}}\right\}, \max\left\{\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}{2}}\right\}\right] \\
&= \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}\right\rangle\right] = \left(P \underset{F}{\boxminus} Q\right)
\end{aligned}
$$

and

$$
\begin{aligned}
((P@Q) \to (P \underset{F}{\boxminus} Q)^C)^C &= \left[\min\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}{2}}, \sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}\right\}, \max\left\{\sqrt[3]{\frac{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}{2}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}\right\}\right] \\
&= \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}\right\rangle\right] = \left(P \underset{F}{\boxminus} Q\right)
\end{aligned}
$$

(ii) Since

$$
\begin{aligned}
((P \underset{F}{\boxminus} Q)^C \to (P@Q) &= \left[\max\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \sqrt[3]{\frac{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}{2}}\right\}, \min\left\{\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}{2}}\right\}\right] \\
&= \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}{2}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}{2}}\right\rangle\right] = P@Q
\end{aligned}
$$

and

$$
\begin{aligned}
(P@Q)^C \to (P \underset{F}{\boxminus} Q) &= \left[\max\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}{2}}, \sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}\right\}, \min\left\{\sqrt[3]{\frac{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}{2}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}\right\}\right] \\
&= \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}{2}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}{2}}\right\rangle\right] = P@Q
\end{aligned}
$$

(iii) Since

$$
\begin{aligned}
((P \underset{F}{\boxslash} Q) \to (P@Q)^C)^C &= \left[\min\left\{\frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}, \sqrt[3]{\frac{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}{2}}\right\}, \max\left\{\sqrt[3]{\frac{\varsigma_{p_{ij}}^3 - \varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}{2}}\right\}\right] \\
&= \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}{2}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}{2}}\right\rangle\right] = P@Q
\end{aligned}
$$

and

$$
\begin{aligned}
\left((P@Q) \to (P \underset{F}{\boxslash} Q)^C\right)^C &= \left[\min\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3+\xi_{q_{ij}}^3}{2}}, \frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}\right\}, \max\left\{\sqrt[3]{\frac{\varsigma_{p_{ij}}^3+\varsigma_{q_{ij}}^3}{2}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3-\varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}\right\}\right] \\
&= \left[\left\langle\sqrt[3]{\frac{\xi_{p_{ij}}^3+\xi_{q_{ij}}^3}{2}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3+\varsigma_{q_{ij}}^3}{2}}\right\rangle\right] = P@Q
\end{aligned}
$$

(iv) Since

$$
\begin{aligned}
\left((P \underset{F}{\boxslash} Q)^C \to (P@Q)\right) &= \left[\max\left\{\frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}, \sqrt[3]{\frac{\xi_{p_{ij}}^3+\xi_{q_{ij}}^3}{2}}\right\}, \min\left\{\sqrt[3]{\frac{\varsigma_{p_{ij}}^3-\varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3+\varsigma_{q_{ij}}^3}{2}}\right\}\right] \\
&= \left[\left\langle\frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3-\varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}\right\rangle\right] = \left(P \underset{F}{\boxslash} Q\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\left((P@Q)^C \to (P \underset{F}{\boxslash} Q)\right) &= \left[\max\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3+\xi_{q_{ij}}^3}{2}}, \frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}\right\}, \min\left\{\sqrt[3]{\frac{\varsigma_{p_{ij}}^3+\varsigma_{q_{ij}}^3}{2}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3-\varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}\right\}\right] \\
&= \left[\left\langle\frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3-\varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}\right\rangle\right] = \left(P \underset{F}{\boxslash} Q\right)
\end{aligned}
$$

(v) Since

$$
\begin{aligned}
\left((P \underset{F}{\boxminus} Q) \to (P\#Q)^C\right)^C &= \left[\min\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3-\xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \frac{\sqrt[3]{2}\,\xi_{p_{ij}}\xi_{q_{ij}}}{\sqrt[3]{\xi_{p_{ij}}^3+\xi_{q_{ij}}^3}}\right\}, \max\left\{\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}, \frac{\sqrt[3]{2}\,\varsigma_{p_{ij}}\varsigma_{q_{ij}}}{\sqrt[3]{\varsigma_{p_{ij}}^3+\varsigma_{q_{ij}}^3}}\right\}\right] \\
&= \left[\left\langle\sqrt[3]{\frac{\xi_{p_{ij}}^3-\xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}\right\rangle\right] = \left(P \underset{F}{\boxminus} Q\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\left((P\#Q) \to (P \underset{F}{\boxminus} Q)^C\right)^C &= \left[\min\left\{\frac{\sqrt[3]{2}\,\xi_{p_{ij}}\xi_{q_{ij}}}{\sqrt[3]{\xi_{p_{ij}}^3+\xi_{q_{ij}}^3}}, \sqrt[3]{\frac{\xi_{p_{ij}}^3-\xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}\right\}, \max\left\{\frac{\sqrt[3]{2}\,\varsigma_{p_{ij}}\varsigma_{q_{ij}}}{\sqrt[3]{\varsigma_{p_{ij}}^3+\varsigma_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}\right\}\right] \\
&= \left[\left\langle\sqrt[3]{\frac{\xi_{p_{ij}}^3-\xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}\right\rangle\right] = \left(P \underset{F}{\boxminus} Q\right)
\end{aligned}
$$

(vi) Since

$$\begin{aligned}((P \underset{F}{\boxminus} Q)^C \to (P\#Q)) &= \left[\max\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3-\xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \frac{\sqrt[3]{2}\xi_{p_{ij}}\xi_{q_{ij}}}{\sqrt[3]{\xi_{p_{ij}}^3+\xi_{q_{ij}}^3}}\right\}, \min\left\{\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}, \frac{\sqrt[3]{2}\varsigma_{p_{ij}}\varsigma_{q_{ij}}}{\sqrt[3]{\varsigma_{p_{ij}}^3+\varsigma_{q_{ij}}^3}}\right\}\right] \\ &= \left[\left\langle \frac{\sqrt[3]{2}\xi_{p_{ij}}\xi_{q_{ij}}}{\sqrt[3]{\xi_{p_{ij}}^3+\xi_{q_{ij}}^3}}, \frac{\sqrt[3]{2}\varsigma_{p_{ij}}\varsigma_{q_{ij}}}{\sqrt[3]{\varsigma_{p_{ij}}^3+\varsigma_{q_{ij}}^3}}\right\rangle\right] = P\#Q\end{aligned}$$

and

$$\begin{aligned}((P\#Q)^C \to (P \underset{F}{\boxminus} Q)) &= \left[\max\left\{\frac{\sqrt[3]{2}\xi_{p_{ij}}\xi_{q_{ij}}}{\sqrt[3]{\xi_{p_{ij}}^3+\xi_{q_{ij}}^3}}, \sqrt[3]{\frac{\xi_{p_{ij}}^3-\xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}\right\}, \min\left\{\frac{\sqrt[3]{2}\varsigma_{p_{ij}}\varsigma_{q_{ij}}}{\sqrt[3]{\varsigma_{p_{ij}}^3+\varsigma_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}\right\}\right] \\ &= \left[\left\langle \frac{\sqrt[3]{2}\xi_{p_{ij}}\xi_{q_{ij}}}{\sqrt[3]{\xi_{p_{ij}}^3+\xi_{q_{ij}}^3}}, \frac{\sqrt[3]{2}\varsigma_{p_{ij}}\varsigma_{q_{ij}}}{\sqrt[3]{\varsigma_{p_{ij}}^3+\varsigma_{q_{ij}}^3}}\right\rangle\right] = P\#Q\end{aligned}$$

(vii) Since

$$\begin{aligned}((P \underset{F}{\boxslash} Q) \to (P\#Q)^C)^C &= \left[\min\left\{\frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}, \frac{\sqrt[3]{2}\xi_{p_{ij}}\xi_{q_{ij}}}{\sqrt[3]{\xi_{p_{ij}}^3+\xi_{q_{ij}}^3}}\right\}, \max\left\{\sqrt[3]{\frac{\varsigma_{p_{ij}}^3-\varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}, \frac{\sqrt[3]{2}\varsigma_{p_{ij}}\varsigma_{q_{ij}}}{\sqrt[3]{\varsigma_{p_{ij}}^3+\varsigma_{q_{ij}}^3}}\right\}\right] \\ &= \left[\left\langle \frac{\sqrt[3]{2}\xi_{p_{ij}}\xi_{q_{ij}}}{\sqrt[3]{\xi_{p_{ij}}^3+\xi_{q_{ij}}^3}}, \frac{\sqrt[3]{2}\varsigma_{p_{ij}}\varsigma_{q_{ij}}}{\sqrt[3]{\varsigma_{p_{ij}}^3+\varsigma_{q_{ij}}^3}}\right\rangle\right] = P\#Q\end{aligned}$$

and

$$\begin{aligned}((P\#Q) \to (P \underset{F}{\boxslash} Q)^C)^C &= \left[\min\left\{\frac{\sqrt[3]{2}\xi_{p_{ij}}\xi_{q_{ij}}}{\sqrt[3]{\xi_{p_{ij}}^3+\xi_{q_{ij}}^3}}, \frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}\right\}, \max\left\{\frac{\sqrt[3]{2}\varsigma_{p_{ij}}\varsigma_{q_{ij}}}{\sqrt[3]{\varsigma_{p_{ij}}^3+\varsigma_{q_{ij}}^3}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3-\varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}\right\}\right] \\ &= \left[\left\langle \frac{\sqrt[3]{2}\xi_{p_{ij}}\xi_{q_{ij}}}{\sqrt[3]{\xi_{p_{ij}}^3+\xi_{q_{ij}}^3}}, \frac{\sqrt[3]{2}\varsigma_{p_{ij}}\varsigma_{q_{ij}}}{\sqrt[3]{\varsigma_{p_{ij}}^3+\varsigma_{q_{ij}}^3}}\right\rangle\right] = P\#Q\end{aligned}$$

(viii) Since

$$
\begin{aligned}
((P \underset{F}{\boxslash} Q)^C \to (P\#Q)) &= \left[\max\left\{\frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}, \frac{\sqrt[3]{2}\xi_{p_{ij}}\xi_{q_{ij}}}{\sqrt[3]{\xi_{p_{ij}}^3+\xi_{q_{ij}}^3}}\right\}, \min\left\{\sqrt[3]{\frac{\varsigma_{p_{ij}}^3-\varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}, \frac{\sqrt[3]{2}\varsigma_{p_{ij}}\varsigma_{q_{ij}}}{\sqrt[3]{\varsigma_{p_{ij}}^3+\varsigma_{q_{ij}}^3}}\right\}\right] \\
&= \left[\left\langle\frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3-\varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}\right\rangle\right] = \left(P \underset{F}{\boxslash} Q\right)
\end{aligned}
$$

and

$$
\begin{aligned}
((P\#Q)^C \to (P \underset{F}{\boxslash} Q)) &= \left[\max\left\{\frac{\sqrt[3]{2}\xi_{p_{ij}}\xi_{q_{ij}}}{\sqrt[3]{\xi_{p_{ij}}^3+\xi_{q_{ij}}^3}}, \frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}\right\}, \min\left\{\frac{\sqrt[3]{2}\varsigma_{p_{ij}}\varsigma_{q_{ij}}}{\sqrt[3]{\varsigma_{p_{ij}}^3+\varsigma_{q_{ij}}^3}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3-\varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}\right\}\right] \\
&= \left[\left\langle\frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3-\varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}\right\rangle\right] = \left(P \underset{F}{\boxslash} Q\right)
\end{aligned}
$$

(ix) Since

$$
\begin{aligned}
((P \underset{F}{\boxminus} Q) \to (P\$Q)^C)^C &= \left[\min\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3-\xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \sqrt[3]{\xi_{p_{ij}}\xi_{q_{ij}}}\right\}, \max\left\{\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}, \sqrt[3]{\varsigma_{p_{ij}}\varsigma_{q_{ij}}}\right\}\right] \\
&= \left[\left\langle\sqrt[3]{\frac{\xi_{p_{ij}}^3-\xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}\right\rangle\right] = \left(P \underset{F}{\boxminus} Q\right)
\end{aligned}
$$

and

$$
\begin{aligned}
((P\$Q) \to (P \underset{F}{\boxminus} Q)^C)^C &= \left[\min\left\{\sqrt[3]{\xi_{p_{ij}}\xi_{q_{ij}}}, \sqrt[3]{\frac{\xi_{p_{ij}}^3-\xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}\right\}, \max\left\{\sqrt[3]{\varsigma_{p_{ij}}\varsigma_{q_{ij}}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}\right\}\right] \\
&= \left[\left\langle\sqrt[3]{\frac{\xi_{p_{ij}}^3-\xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}\right\rangle\right] = \left(P \underset{F}{\boxminus} Q\right)
\end{aligned}
$$

(x) Since

$$
\begin{aligned}
((P \underset{F}{\boxminus} Q)^C \to (P\$Q)) &= \left[\max\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3-\xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \sqrt[3]{\xi_{p_{ij}}\xi_{q_{ij}}}\right\}, \min\left\{\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}, \sqrt[3]{\varsigma_{p_{ij}}\varsigma_{q_{ij}}}\right\}\right] \\
&= \left[\left\langle\sqrt[3]{\xi_{p_{ij}}\xi_{q_{ij}}}, \sqrt[3]{\varsigma_{p_{ij}}\varsigma_{q_{ij}}}\right\rangle\right] = (P\$Q)
\end{aligned}
$$

and

$$\begin{aligned}((P\$Q)^C \rightarrow (P \underset{F}{\boxminus} Q)) &= \left[\max\left\{\sqrt[3]{\xi_{p_{ij}}\xi_{q_{ij}}}, \sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}\right\}, \min\left\{\sqrt[3]{\varsigma_{p_{ij}}\varsigma_{q_{ij}}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}\right\}\right] \\ &= \left[\left\langle \sqrt[3]{\xi_{p_{ij}}\xi_{q_{ij}}}, \sqrt[3]{\varsigma_{p_{ij}}\varsigma_{q_{ij}}} \right\rangle\right] = (P\$Q)\end{aligned}$$

(xi) Since

$$\begin{aligned}((P \underset{F}{\boxslash} Q) \rightarrow (P\$Q)^C)^C &= \left[\min\left\{\frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}, \sqrt[3]{\xi_{p_{ij}}\xi_{q_{ij}}}\right\}, \max\left\{\sqrt[3]{\frac{\varsigma_{p_{ij}}^3 - \varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}, \sqrt[3]{\varsigma_{p_{ij}}\varsigma_{q_{ij}}}\right\}\right] \\ &= \left[\left\langle \sqrt[3]{\xi_{p_{ij}}\xi_{q_{ij}}}, \sqrt[3]{\varsigma_{p_{ij}}\varsigma_{q_{ij}}} \right\rangle\right] = (P\$Q)\end{aligned}$$

and

$$\begin{aligned}((P\$Q) \rightarrow (P \underset{F}{\boxslash} Q)^C)^C &= \left[\min\left\{\sqrt[3]{\xi_{p_{ij}}\xi_{q_{ij}}}, \frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}\right\}, \max\left\{\sqrt[3]{\varsigma_{p_{ij}}\varsigma_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 - \varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}\right\}\right] \\ &= \left[\left\langle \sqrt[3]{\xi_{p_{ij}}\xi_{q_{ij}}}, \sqrt[3]{\varsigma_{p_{ij}}\varsigma_{q_{ij}}} \right\rangle\right] = (P\$Q)\end{aligned}$$

(xii) Since

$$\begin{aligned}((P \underset{F}{\boxslash} Q)^C \rightarrow (P\$Q)) &= \left[\max\left\{\frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}, \sqrt[3]{\xi_{p_{ij}}\xi_{q_{ij}}}\right\}, \min\left\{\sqrt[3]{\frac{\varsigma_{p_{ij}}^3 - \varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}, \sqrt[3]{\varsigma_{p_{ij}}\varsigma_{q_{ij}}}\right\}\right] \\ &= \left[\left\langle \frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 - \varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}} \right\rangle\right] = \left(P \underset{F}{\boxslash} Q\right)\end{aligned}$$

and

$$\begin{aligned}((P\$Q)^C \rightarrow (P \underset{F}{\boxslash} Q)) &= \left[\max\left\{\sqrt[3]{\xi_{p_{ij}}\xi_{q_{ij}}}, \frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}\right\}, \min\left\{\sqrt[3]{\varsigma_{p_{ij}}\varsigma_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 - \varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}\right\}\right] \\ &= \left[\left\langle \frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 - \varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}} \right\rangle\right] = \left(P \underset{F}{\boxslash} Q\right)\end{aligned}$$

(xiii) Since

$$\begin{aligned}((P \underset{F}{\boxminus} Q) \to (P \underset{F}{\boxslash} Q)^C)^C &= \left[\min\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3-\xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}},\frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}\right\},\max\left\{\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}},\sqrt[3]{\frac{\varsigma_{p_{ij}}^3-\varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}\right\}\right] \\ &= \left[\left\langle\sqrt[3]{\frac{\xi_{p_{ij}}^3-\xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}},\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}\right\rangle\right] = \left(P \underset{F}{\boxminus} Q\right)\end{aligned}$$

and

$$\begin{aligned}((P \underset{F}{\boxslash} Q) \to (P \underset{F}{\boxminus} Q)^C)^C &= \left[\min\left\{\frac{\xi_{p_{ij}}}{\xi_{q_{ij}}},\sqrt[3]{\frac{\xi_{p_{ij}}^3-\xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}\right\},\max\left\{\sqrt[3]{\frac{\varsigma_{p_{ij}}^3-\varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}},\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}\right\}\right] \\ &= \left[\left\langle\sqrt[3]{\frac{\xi_{p_{ij}}^3-\xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}},\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}\right\rangle\right] = \left(P \underset{F}{\boxminus} Q\right)\end{aligned}$$

(xiv) Since

$$\begin{aligned}((P \underset{F}{\boxslash} Q)^C \to (P \underset{F}{\boxminus} Q)) &= \left[\max\left\{\frac{\xi_{p_{ij}}}{\xi_{q_{ij}}},\sqrt[3]{\frac{\xi_{p_{ij}}^3-\xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}\right\},\min\left\{\sqrt[3]{\frac{\varsigma_{p_{ij}}^3-\varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}},\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}\right\}\right] \\ &= \left[\left\langle\frac{\xi_{p_{ij}}}{\xi_{q_{ij}}},\sqrt[3]{\frac{\varsigma_{p_{ij}}^3-\varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}\right\rangle\right] = \left(P \underset{F}{\boxslash} Q\right)\end{aligned}$$

and

$$\begin{aligned}((P \underset{F}{\boxminus} Q)^C \to (P \underset{F}{\boxslash} Q)) &= \left[\max\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3-\xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}},\frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}\right\},\min\left\{\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}},\sqrt[3]{\frac{\varsigma_{p_{ij}}^3-\varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}\right\}\right] \\ &= \left[\left\langle\frac{\xi_{p_{ij}}}{\xi_{q_{ij}}},\sqrt[3]{\frac{\varsigma_{p_{ij}}^3-\varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}\right\rangle\right] = \left(P \underset{F}{\boxslash} Q\right)\end{aligned}$$

Corollary 3.5 For $P, Q \in F_{m\times n}$, we have

$$\begin{gathered}((P \underset{F}{\boxminus} Q) \to (P @ Q)^C)^C = ((P @ Q) \to (P \underset{F}{\boxminus} Q)^C)^C = \\ ((P \underset{F}{\boxminus} Q) \to (P \# Q)^C)^C = ((P \# Q) \to (P \underset{F}{\boxminus} Q)^C)^C = \\ ((P \underset{F}{\boxminus} Q) \to (P \$ Q)^C)^C = ((P \$ Q) \to (P \underset{F}{\boxminus} Q)^C)^C = \\ ((P \underset{F}{\boxminus} Q) \to (P \underset{F}{\boxslash} Q)^C)^C = ((P \underset{F}{\boxslash} Q) \to (P \underset{F}{\boxminus} Q)^C)^C = (P \underset{F}{\boxminus} Q)\end{gathered}$$

Corollary 3.6 For $P, Q \in F_{m\times n}$, we have

$$((P \underset{F}{\boxtimes} Q)^C \rightarrow (P @ Q)) = ((P @ Q)^C \rightarrow (P \underset{F}{\boxtimes} Q)) =$$
$$((P \underset{F}{\boxtimes} Q)^C \rightarrow (P \# Q)) = ((P \# Q)^C \rightarrow (P \underset{F}{\boxtimes} Q)) =$$
$$((P \underset{F}{\boxtimes} Q)^C \rightarrow (P \$ Q)) = ((P \$ Q)^C \rightarrow (P \underset{F}{\boxtimes} Q)) =$$
$$((P \underset{F}{\boxtimes} Q)^C \rightarrow (P \underset{F}{\boxminus} Q)) = ((P \underset{F}{\boxminus} Q)^C \rightarrow (P \underset{F}{\boxtimes} Q)) = (P \underset{F}{\boxtimes} Q)$$

Theorem 3.7 For $P, Q \in F_{m \times n}$ we have

$$\{[(P^C \rightarrow Q) \underset{F}{\boxminus} (P \rightarrow Q^C)^C] \cap [(P^C \rightarrow Q) \underset{F}{\boxtimes} (P \rightarrow Q^C)^C]\} \cup \{[(P^C \rightarrow Q) \underset{F}{\boxminus} (P \rightarrow Q^C)^C] \cup \left[(P^C \rightarrow Q) \underset{F}{\boxtimes} (P \rightarrow Q^C)^C\right]\right\} = P \underset{F}{\boxtimes} Q$$

Proof. Since $[(P^C \rightarrow Q) \boxminus_F (P \rightarrow Q^C)^C] = \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1 - \xi_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}} \right\rangle\right]$

and $[(P^C \rightarrow Q) \boxtimes_F (P \rightarrow Q^C)^C] = \left[\left\langle \frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 - \varsigma_{q_{ij}}^3}{1 - \varsigma_{q_{ij}}^3}} \right\rangle\right]$

Now

$$\{[(P^C \rightarrow Q) \underset{F}{\boxminus} (P \rightarrow Q^C)^C] \cap \left[(P^C \rightarrow Q) \underset{F}{\boxtimes} (P \rightarrow Q^C)^C\right]\right\}$$
$$= \left[\min\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1 - \xi_{q_{ij}}^3}}, \frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}\right\}, \max\left\{\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 - \varsigma_{q_{ij}}^3}{1 - \varsigma_{q_{ij}}^3}}\right\}\right]$$
$$= \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1 - \xi_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}} \right\rangle\right]$$

and

$$\{[(P^C \rightarrow Q) \underset{F}{\boxminus} (P \rightarrow Q^C)^C] \cup [(P^C \rightarrow Q) \underset{F}{\boxtimes} (P \rightarrow Q^C)^C]\}$$
$$= \left[\max\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1 - \xi_{q_{ij}}^3}}, \frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}\right\}, \min\left\{\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 - \varsigma_{q_{ij}}^3}{1 - \varsigma_{q_{ij}}^3}}\right\}\right]$$
$$= \left[\left\langle \frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 - \varsigma_{q_{ij}}^3}{1 - \varsigma_{q_{ij}}^3}} \right\rangle\right]$$

Hence

$$\{[(P^C \to Q) \underset{F}{\boxminus} (P \to Q^C)^C] \cap [(P^C \to Q) \underset{F}{\boxslash} (P \to Q^C)^C]\} \cup \{[(P^C \to Q) \underset{F}{\boxminus} (P \to Q^C)^C] \cup \left[(P^C \to Q) \underset{F}{\boxslash} (P \to Q^C)^C\right]\}$$

$$= \left[\max\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}\right\}, \min\left\{\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 - \varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}\right\}\right]$$

$$= \left[\left\langle \frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 - \varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}} \right\rangle\right]$$

$$= P \underset{F}{\boxslash} Q$$

Theorem 3.8 For $P, Q \in F_{m\times n}$ we have

$$\{[(P^C \to Q) \underset{F}{\boxminus} (P \to Q^C)^C] \cap [(P^C \to Q) \underset{F}{\boxslash} (P \to Q^C)^C]\} \cap \{[(P^C \to Q) \underset{F}{\boxminus} (P \to Q^C)^C] \cup \left[(P^C \to Q) \underset{F}{\boxslash} (P \to Q^C)^C\right]\} = P \underset{F}{\boxminus} Q$$

Proof. Since $[(P^C \to Q) \boxminus_F (P \to Q^C)^C] = \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}} \right\rangle\right]$

and $[(P^C \to Q) \boxslash_F (P \to Q^C)^C] = \left[\left\langle \frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 - \varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}} \right\rangle\right]$

Now

$$\{[(P^C \to Q) \underset{F}{\boxminus} (P \to Q^C)^C] \cap [(P^C \to Q) \underset{F}{\boxslash} (P \to Q^C)^C]\}$$

$$= \left[\min\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}\right\}, \max\left\{\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 - \varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}\right\}\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}} \right\rangle\right]$$

and

$$\{[(P^C \to Q) \underset{F}{\boxminus} (P \to Q^C)^C] \cup [(P^C \to Q) \underset{F}{\boxslash} (P \to Q^C)^C]\}$$

$$= \left[\max\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}\right\}, \min\left\{\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 - \varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}\right\}\right]$$

$$= \left[\left\langle \frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 - \varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}} \right\rangle\right]$$

Hence

$$\{[(P^C \to Q) \underset{F}{\boxminus} (P \to Q^C)^C] \cap [(P^C \to Q) \underset{F}{\boxtimes} (P \to Q^C)^C]\} \cap \{[(P^C \to Q) \underset{F}{\boxminus} (P \to Q^C)^C] \cup \left[(P^C \to Q) \underset{F}{\boxtimes} (P \to Q^C)^C\right]\}$$

$$= \left[\min\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1 - \xi_{q_{ij}}^3}}, \frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}\right\}, \max\left\{\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 - \varsigma_{q_{ij}}^3}{1 - \varsigma_{q_{ij}}^3}}\right\}\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1 - \xi_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}} \right\rangle\right] = P \underset{F}{\boxminus} Q$$

Theorem 3.9 For $P, Q \in F_{m\times n}$ we have

$$\{[(P \underset{F}{\boxminus} Q) \to (P@Q)^C]^C \cup [(P \underset{F}{\boxtimes} Q) \to (P@Q)^C]^C\} \cup \{[(P \underset{F}{\boxminus} Q) \to (P@Q)^C]^C \cup [\left(P \underset{F}{\boxtimes} Q\right) \to (P@Q)^C]^C\} = P@Q$$

Proof. Since $[(P \boxminus_F Q) \to (P@Q)^C]^C = \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}} \right\rangle\right]$

and $[(P \boxtimes_F Q) \to (P@Q)^C]^C = \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}{2}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}{2}} \right\rangle\right]$

Now

$$\{[(P \underset{F}{\boxminus} Q) \to (P@Q)^C]^C \cup [(P \underset{F}{\boxtimes} Q) \to (P@Q)^C]^C\}$$

$$= \left[\max\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1 - \xi_{q_{ij}}^3}}, \sqrt[3]{\frac{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}{2}}\right\}, \min\left\{\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}{2}}\right\}\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}{2}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}{2}} \right\rangle\right]$$

and

$$\{[(P \underset{F}{\boxminus} Q) \to (P@Q)^C]^C \cap [(P \underset{F}{\boxtimes} Q) \to (P@Q)^C]^C\}$$

$$= \left[\min\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1 - \xi_{q_{ij}}^3}}, \sqrt[3]{\frac{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}{2}}\right\}, \max\left\{\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}{2}}\right\}\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1 - \xi_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}} \right\rangle\right]$$

Hence

$$\{[(P \underset{F}{\boxminus} Q) \to (P@Q)^C]^C \cup [(P \underset{F}{\boxslash} Q) \to (P@Q)^C]^C\} \cup \{[(P \underset{F}{\boxminus} Q) \to (P@Q)^C]^C \cup [\left(P \underset{F}{\boxslash} Q\right) \to (P@Q)^C]^C\}$$

$$= \left[\max\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1 - \xi_{q_{ij}}^3}}, \sqrt[3]{\frac{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}{2}}\right\}, \min\left\{\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}{2}}\right\}\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}{2}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}{2}} \right\rangle\right] = P@Q$$

Theorem 3.10 For $P, Q \in F_{m \times n}$ we have

$$\{[(P \underset{F}{\boxminus} Q) \to (P@Q)^C]^C \cup [(P \underset{F}{\boxslash} Q) \to (P@Q)^C]^C\} \cap \{[(P \underset{F}{\boxminus} Q) \to (P@Q)^C]^C \cap [\left(P \underset{F}{\boxslash} Q\right) \to (P@Q)^C]^C\} = P \underset{F}{\boxminus} Q$$

Proof. Since $[(P \boxminus_F Q) \to (P@Q)^C]^C = \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1 - \xi_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}} \right\rangle\right]$

and $[(P \boxslash_F Q) \to (P@Q)^C]^C = \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}{2}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}{2}} \right\rangle\right]$

Now

$$\{[(P \underset{F}{\boxminus} Q) \to (P@Q)^C]^C \cup [(P \underset{F}{\boxslash} Q) \to (P@Q)^C]^C\}$$

$$= \left[\max\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1 - \xi_{q_{ij}}^3}}, \sqrt[3]{\frac{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}{2}}\right\}, \min\left\{\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}{2}}\right\}\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}{2}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}{2}} \right\rangle\right]$$

and

$$\{[(P \underset{F}{\boxminus} Q) \to (P@Q)^C]^C \cap [(P \underset{F}{\boxslash} Q) \to (P@Q)^C]^C\}$$

$$= \left[\min\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1 - \xi_{q_{ij}}^3}}, \sqrt[3]{\frac{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}{2}}\right\}, \max\left\{\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}{2}}\right\}\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1 - \xi_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}} \right\rangle\right]$$

Hence

$$\{[(P \underset{F}{\boxminus} Q) \rightarrow (P@Q)^C]^C \cup [(P \underset{F}{\boxtimes} Q) \rightarrow (P@Q)^C]^C\} \cap \{[(P \underset{F}{\boxminus} Q) \rightarrow (P@Q)^C]^C \cap [\left(P \underset{F}{\boxtimes} Q\right) \rightarrow (P@Q)^C]^C\}$$

$$= \left[\min\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \sqrt[3]{\frac{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}{2}}\right\}, \max\left\{\frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}{2}}\right\}\right]$$

$$= \left[\left\langle\sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}\right\rangle\right] = P \underset{F}{\boxminus} Q$$

Theorem 3.11 For $P, Q \in F_{m\times n}$ we have

$$[(P \underset{F}{\boxminus} Q)^C \rightarrow (P@Q)] \cup [(P \underset{F}{\boxminus} Q) \rightarrow (P@Q)^C]^C = P@Q$$

Proof. Since $[(P \boxminus_F Q)^C \rightarrow (P@Q)] = \left[\left\langle\sqrt[3]{\frac{\xi_{p_{ij}}^3+\xi_{q_{ij}}^3}{2}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3+\varsigma_{q_{ij}}^3}{2}}\right\rangle\right]$

and $[(P \boxminus_F Q) \rightarrow (P@Q)^C]^C = \left[\left\langle\sqrt[3]{\frac{\xi_{p_{ij}}^3-\xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}\right\rangle\right]$

Hence

$$[(P \underset{F}{\boxminus} Q)^C \rightarrow (P@Q)] \cup [(P \underset{F}{\boxminus} Q) \rightarrow (P@Q)^C]^C$$

$$= \left[\max\left\{\sqrt[3]{\frac{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}{2}}, \sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}\right\}, \min\left\{\sqrt[3]{\frac{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}{2}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}\right\}\right]$$

$$= \left[\left\langle\sqrt[3]{\frac{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}{2}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}{2}}\right\rangle\right]$$

$$= P@Q$$

Theorem 3.12 For $P, Q \in F_{m\times n}$ we have $[(P \boxminus_F Q)^C \rightarrow (P\#Q)] \cup [(P\#Q) \rightarrow (P \boxminus_F Q)^C] = P\#Q$

Proof. Since $[(P \boxminus_F Q)^C \rightarrow (P\#Q)] = \left[\left\langle\frac{\sqrt[3]{2}\xi_{p_{ij}}\xi_{q_{ij}}}{\sqrt[3]{\xi_{p_{ij}}^3+\xi_{q_{ij}}^3}}, \frac{\sqrt[3]{2}\varsigma_{p_{ij}}\varsigma_{q_{ij}}}{\sqrt[3]{\varsigma_{p_{ij}}^3+\varsigma_{q_{ij}}^3}}\right\rangle\right]$

and $[(P \boxminus_F Q) \rightarrow (P@Q)^C]^C = \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}} \right\rangle\right]$

Hence

$[(P \underset{F}{\boxminus} Q)^C \rightarrow (P\#Q)] \cup [(P\#Q) \rightarrow (P \underset{F}{\boxminus} Q)^C]$

$$= \left[\max\left\{\frac{\sqrt[3]{2}\xi_{p_{ij}}\xi_{q_{ij}}}{\sqrt[3]{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}}, \sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}\right\}, \min\left\{\frac{\sqrt[3]{2}\varsigma_{p_{ij}}\varsigma_{q_{ij}}}{\sqrt[3]{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}\right\}\right]$$

$$= \left[\left\langle \frac{\sqrt[3]{2}\xi_{p_{ij}}\xi_{q_{ij}}}{\sqrt[3]{\xi_{p_{ij}}^3 + \xi_{q_{ij}}^3}}, \frac{\sqrt[3]{2}\varsigma_{p_{ij}}\varsigma_{q_{ij}}}{\sqrt[3]{\varsigma_{p_{ij}}^3 + \varsigma_{q_{ij}}^3}} \right\rangle\right]$$

$$= P\#Q$$

Theorem 3.13 For $P, Q \in F_{m\times n}$ we have $[(P \boxminus_F Q)^C \rightarrow (P\$Q)] \cup [(P\$Q) \rightarrow (P \boxminus_F Q)^C] = P\$Q$

Proof. Since $[(P \boxminus_F Q)^C \rightarrow (P\$Q)] = \left[\left\langle \sqrt[3]{\xi_{p_{ij}}\xi_{q_{ij}}}, \sqrt[3]{\varsigma_{p_{ij}}\varsigma_{q_{ij}}} \right\rangle\right]$

and $[(P \boxminus_F Q) \rightarrow (P@Q)^C]^C = \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}} \right\rangle\right]$

Hence

$[(P \underset{F}{\boxminus} Q)^C \rightarrow (P\$Q)] \cup [(P\$Q) \rightarrow (P \underset{F}{\boxminus} Q)^C]$

$$= \left[\max\left\{\sqrt[3]{\xi_{p_{ij}}\xi_{q_{ij}}}, \sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}\right\}, \min\left\{\sqrt[3]{\varsigma_{p_{ij}}\varsigma_{q_{ij}}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}\right\}\right]$$

$$= \left[\left\langle \sqrt[3]{\xi_{p_{ij}}\xi_{q_{ij}}}, \sqrt[3]{\varsigma_{p_{ij}}\varsigma_{q_{ij}}} \right\rangle\right]$$

Theorem 3.14 For $P, Q \in F_{m\times n}$ we have $[(P \boxslash_F Q)^C \rightarrow (P \boxminus_F Q)] \cup [(P \boxminus_F Q) \rightarrow (P \boxslash_F Q)^C] = P \boxslash_F Q$

Proof. Since $[(P \boxslash_F Q)^C \rightarrow (P \boxminus_F Q)] = \left[\left\langle \frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 - \varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}} \right\rangle\right]$

and $[(P \boxminus_F Q) \rightarrow (P@Q)^C]^C = \left[\left\langle \sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}} \right\rangle\right]$

Hence

$$[(P \underset{F}{\boxslash} Q)^C \rightarrow (P \underset{F}{\boxminus} Q)] \cup [(P \underset{F}{\boxminus} Q) \rightarrow (P \underset{F}{\boxslash} Q)^C]$$

$$= \left[\max\left\{\frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}, \sqrt[3]{\frac{\xi_{p_{ij}}^3 - \xi_{q_{ij}}^3}{1-\xi_{q_{ij}}^3}}\right\}, \min\left\{\sqrt[3]{\frac{\varsigma_{p_{ij}}^3 - \varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}}, \frac{\varsigma_{p_{ij}}}{\varsigma_{q_{ij}}}\right\}\right]$$

$$= \left[\left\langle \frac{\xi_{p_{ij}}}{\xi_{q_{ij}}}, \sqrt[3]{\frac{\varsigma_{p_{ij}}^3 - \varsigma_{q_{ij}}^3}{1-\varsigma_{q_{ij}}^3}} \right\rangle\right]$$

$$= P \underset{F}{\boxslash} Q$$

4. Conclusion

In this paper, we defined some new operators $[(P\$Q), (P\#Q), (P \rightarrow Q)]$ of Fermatean fuzzy soft matrices and investigated their algebraic properties. Finally, we proved several of these properties, particularly those involving the operator $P \rightarrow Q$ for Fermatean fuzzy soft implication with other operators.

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