The Generalized Hilfer-Type Integro-Differential Equation with Boundary Condition

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Abstract

This manuscript deals with the existence and uniqueness results of a class of boundary value problems for a generalized Hilfer-type integrodifferential equation. We apply Schauder's and Banach's fixed point theorems to obtain our main results. Also, we establish the stability results of the given problem by applying some mathematical methods. In the end, we give two concrete examples illustrating our main results.

Keywords.ς-Hilfer fractional integrodifferential equation; Boundary conditions; Fixed point theorem; Ulam-Hyers stability.

1 Introduction:

The idea of fractional differential equations (FDEs) is majorly very important because of its nonlocal property. This is the main reason that FDEs have always been looked to be the most reasonable ones to survey different facts and figures in applications of numerous emerging research fields such as biological sciences, economics, polarization, physics, engineering, and traffic modeling. As a result, we can call the recent decade the age of fractional calculus (FC) as this theory is drawing more and more notice from well-renowned mathematicians, for more details, you can see the series of books and research papers [1–8]. Nevertheless, FDEs of fractional order have extensively been deliberated by many investigators. Very briefly, interesting subjects in this scope are the investigation of some qualitative properties of solutions e.g., uniqueness, existence, and stability, through fixed point techniques and many single kinds and numerical solutions for different types of FDEs using diverse classes of FDs have been established (see [9–15]). In the last decades, some investigators presented definitions of FC, involving definitions of Riemann-Liouville (RL), Caputo, Erdelyi-Kober and Hadamard.

We will concentrate our attention on the more general problem so-called here the Hilfer FD (HFD) of order σ_1 and a type $\sigma_2 \in [0, 1]$, (see [16]), where applications of the aforesaid area have been presented (see [17, 18]). Sousa and de Oliveira in [19] introduce a new type of the HFD with respect to another function ψ .

Bashir.et al. [20], they studied the following boundary value problem (BVP) for a nonlinear fractional integrodifferential equation with integral boundary conditions

$$\begin{cases}\left({}^{c}\mathrm{D}^{\sigma_2}\right)\vartheta(\nu)=\varpi(\nu,\vartheta(\nu),\Im\vartheta(\nu)),0<\nu<1,1<\sigma_2\leq 2,\\ \alpha\vartheta(0)+\sigma_2\vartheta'(0)=\int_0^1 q_1(\vartheta(s))ds,,\ \alpha\vartheta(1)+\sigma_2\vartheta'(1)=\int_0^1 q_2(\vartheta(s))ds,\end{cases}\quad(1.1)$$

where ${}^{c}\mathrm{D}^{\sigma_2}$ is the Caputo FD a multivalued map, ϖ: $[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$for $\lambda:[0,1]\times[0,1]\to[0,\infty)$, and

$$\Im\vartheta(\nu)=\int_0^{\tau}\lambda(\tau,s)(\vartheta(s))ds,$$

$q_1, q_2 : \mathbb{R}\to\mathbb{R}and\alpha> 0$, $\sigma_2 \geq 0$, are real numbers.

Newly, the study of various specific properties of solutions to different FDEs including generalized FDs has become the basic theme of applied mathematics surveys. Many studies in connection with the existence and stability of solutions through different kinds of FPTs were formulated, we refer the researcher to some studied work [21–23]. Also, in [24, 25], the authors study some problems of Nonlocal fractional BVPs with ς-HFDs.

In the present manuscript, We will consider the class of BVPs for a generalized Hilfer-type integrodifferential equation:

$$\begin{cases}{}^{H}\mathrm{D}_{a^+}^{\sigma_1,\sigma_2;\varsigma}\vartheta(\nu)=\varpi(\nu,\vartheta(\nu),\Im_{a^+}^{\sigma_1;\varsigma}\vartheta(\nu)),\nu\in\mathbf{J}=[a,b],0<\sigma_1<1,\\ A\vartheta(a)+B\vartheta(b)=C,\ A,B,C\in\mathbb{R},\ 0\leq\sigma_2\leq 1,\end{cases}\quad(1.2)$$

where ${}^{H}\mathrm{D}_{a^+}^{\sigma_1,\sigma_2;\varsigma}$ is the ς-HFD of order σ_1 and type σ_2, and $\Im_{a^+}^{\sigma_1;\varsigma}$ is ς-RL fractional integral of order σ_1.

The Study here in this manuscript is new and adds to the literature, especially in the field of ς-Hilfer kind with nonlinear problems.

In generic, our newly results keep useful for different values of the function ς and a lot of corresponding problems, for example (For $\varsigma(\nu)=\log\nu$, we get Hilfer-Hadamard type problem), (for $\varsigma(v) = v^{\mu}$,$\mu> 0$, we get Hilfer-Katugampola type problem), (for $\varsigma(v) = v$, and $\sigma_2 = 1$, we get Caputo-type problem), and (for $\varsigma(v) = v$, and $\sigma_2 = 0$, we get RL-type problem).

This manuscript is marshaled as follows: Sect. 2, is devoted to some needful definitions and results which are related to our study. The main results related to linear problems correspond to the proposed problems (1.2) are addressed in Sect. 3 and 4. this work is strengthened by providing examples and a short conclusion

2 Preliminaries

In this segment, some necessary definitions, lemmas, properties, and important estimations needed onward for our analysis are given bellow.

Let $\mathrm{L}(\mathbf{J},\mathbb{R})$ and $C(\mathbf{J},\mathbb{R})$ are the Lebesgueintegrable functions and Banach space from **J** into $\mathbb{R}$ with the norms

$$\|\vartheta\|_{\infty} = \sup\{|\vartheta| : \nu \in \mathbf{J}\},$$

and

$$\|\vartheta\|_{\mathrm{L}} = \int_a^b |\vartheta(\nu)| d\nu,$$

respectively.

Definition 2.1 [4] Let $\sigma_1 > 0$and $\vartheta \in \mathrm{L}^1(\mathbf{J},\mathbb{R})$. The$\varsigma$ −RL fractional integral of order σ_1 defined by

$$\mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\vartheta(\nu) = \frac{1}{\Gamma(\sigma_1)}\int_a^{\nu} \varsigma'(\tau)(\varsigma(\nu)-\varsigma(\tau))^{\sigma_1-1}\vartheta(\tau)d\tau.$$

Definition 2.2 [19] Let $n-1 < \sigma_1 < n$, $0 \le \sigma_2 \le 1$. The ς −HFD of order σ_1 and type σ_2 is given by

$${}_H\mathsf{D}_{a^+}^{\sigma_1,\sigma_2;\varsigma}\vartheta(\nu) = \mathfrak{I}^{\sigma_2(n-\sigma_1);\varsigma}\left(\frac{1}{\varsigma'(\nu)}\frac{d}{d\nu}\right)^n \mathfrak{I}^{(1-\sigma_2)(n-\sigma_1);\varsigma}\vartheta(\nu),$$

where$\nu > a$.

Lemma 2.1 [4, 19] Let σ_1, η, and $\delta > 0$. Then

(1) $\mathfrak{I}^{\sigma_1;\varsigma}\mathfrak{I}^{\eta;\varsigma}\vartheta(\nu) = \mathfrak{I}^{\sigma_1+\eta;\varsigma}\vartheta(\nu)$.

(2) $\mathfrak{I}^{\sigma_1;\varsigma}(\varsigma(\nu)-\varsigma(a))^{\delta-1} = \dfrac{\Gamma(\delta)}{\Gamma(\sigma_1+\delta)}(\varsigma(\nu)-\varsigma(a))^{\sigma_1+\delta-1}$.

We also note that ${}_H\mathsf{D}^{\sigma_1,\sigma_2;\varsigma}(\varsigma(\nu)-\varsigma(a))^{\lambda-1} = 0$.

Lemma 2.2 [19] Let $\vartheta \in \mathrm{L}(a,b)$, $\sigma_1 \in (n-1, n]$ ($n \in \mathbb{N}$), $\sigma_2 \in [0, 1]$, and $\lambda = \sigma_1 + \sigma_2(1-\sigma_1)$ then

$$\left(\mathfrak{I}^{\sigma_1;\varsigma}\ {}_H\mathsf{D}^{\sigma_1,\sigma_2;\varsigma}\vartheta\right)(\nu) = \vartheta(\nu) - \sum_{k=0}^{n}\frac{(\varsigma(\nu)-\varsigma(a))^{\lambda-k}}{\Gamma(\lambda-k+1)}\vartheta_{\varsigma}^{[n-k]}\mathfrak{I}^{(1-\sigma_2)(n-\sigma_1);\varsigma}\vartheta(a),$$

where $\vartheta_{\varsigma}^{[n-k]} = \left(\dfrac{1}{\varsigma'(\nu)}\dfrac{d}{d\nu}\right)^{[n-k]}\vartheta(\nu)$.

Lemma 2.3 Let $\lambda = \sigma_1 + \sigma_2(1 - \sigma_1)$, where $0 < \sigma_1 < 1$, $0 \le \sigma_2 \le 1$, and $\vartheta \in C(\mathbf{J}, \mathbb{R})$. Then, the following ς −Hilfer type of BVP

$$\begin{cases} {}^{H}\mathsf{D}_{a^+}^{\sigma_1,\sigma_2;\varsigma}\vartheta(\nu) = \mathsf{F}(\nu), \nu \in \mathbf{J}, \\ A\vartheta(a) + B\vartheta(b) = C,\ A, B, C \in \mathbb{R}, \end{cases} \quad (2.1)$$

has a solution given by

$$\vartheta(\nu) = \frac{(\varsigma(\nu) - \varsigma(a))^{\lambda-1}}{\Lambda\Gamma(\lambda)}\left[C - B\,\mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\mathsf{F}(b))\right] + \mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\mathsf{F}(\nu), \quad (2.2)$$

where

$$\Lambda = \frac{B}{\Gamma(\lambda)}(\varsigma(b) - \varsigma(a))^{\lambda-1} \neq 0. \quad (2.3)$$

Proof. Set ϑ be a solution of the problem (2.1). Applying $\mathfrak{I}_{a^+}^{\sigma_1;\varsigma}$ on the first equation (2.1) with Lemma 2.2, and setting $\mathfrak{I}_{a^+}^{1-\lambda;\varsigma}\vartheta(a) = c_0$, we obtain

$$\vartheta(\nu) = \frac{c_0}{\Gamma(\lambda)}(\varsigma(\nu) - \varsigma(a))^{\lambda-1} + \mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\mathsf{F}(\nu). \quad (2.4)$$

For determining c_0, we use the boundary value condition $A\vartheta(a) + B\vartheta(b) = C$, and from (2.4) we have

$$C = A(0) + B\left[\frac{c_0}{\Gamma(\lambda)}(\varsigma(b) - \varsigma(a))^{\lambda-1} + \mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\mathsf{F}(b))\right]$$

$$= c_0\frac{B}{\Gamma(\lambda)}(\varsigma(b) - \varsigma(a))^{\lambda-1} + B\mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\mathsf{F}(b).$$

Hence,

$$c_0 = \frac{1}{\Lambda}\left[C - B\,\mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\mathsf{F}(b)\right].$$

Therefore,

$$\vartheta(\nu) = \frac{(\varsigma(\nu) - \varsigma(a))^{\lambda-1}}{\Lambda\Gamma(\lambda)}\left[C - B\,\mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\mathsf{F}(b)\right] + \mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\mathsf{F}(\nu). \quad (2.5)$$

End the proof. □

In this following section, we pay attention to proving the uniqueness and existence of solutions to problem (1.2) via Banach's fixed point theorem (FPT) [26] and Schauder's FPT [27].

According to Lemma 2.3, Now, we introduce $\mathrm{T}:C(\mathbf{J},\mathbb{R})\to C(\mathbf{J},\mathbb{R})$ as operator define by

$$(\mathrm{T}\vartheta)(\nu)=\frac{(\varsigma(\nu)-\varsigma(a))^{\lambda-1}}{\Lambda\Gamma(\lambda)}\Big[C-B\,\mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\varpi(b,\vartheta(b),\mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\vartheta(b))\Big] \qquad (2.6)$$
$$+\mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\varpi(\nu,\vartheta(\nu),\mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\vartheta(\nu)).$$

It should be observed that the Integrodifferential-type problem (1.2) has a solution ϑ if and only if T has fixed points. Hence, for suitability purpose, we are setting the constant:

$$\mathrm{Y}:=\frac{(\varsigma(b)-\varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1+1)}+B\left(\frac{(\varsigma(b)-\varsigma(a))^{\lambda-1}}{|\Lambda|\Gamma(\lambda)}\right)\left(\frac{(\varsigma(b)-\varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1+1)}\right). \qquad (2.7)$$

3 Integrodifferential-type problem (1.2)

Some essential assumptions are presented as follows:

($\mathbf{H}_1$): There exists $\zeta\in(0, 1)$ such that

$$\left|\varpi(\nu,\vartheta_1,\vartheta_1^*)-\varpi(\nu,\vartheta_2,\vartheta_2^*)\right|\le\zeta\left(\left|\vartheta_1-\vartheta_2\right|+\left|\vartheta_1^*-\vartheta_2^*\right|\right),$$

for any $\vartheta_1,\vartheta_1^*,\vartheta_2,\vartheta_2^*\in\mathbb{R}$ and $\zeta\in\mathbf{J}$.

($\mathbf{H}_2$): Let $\varpi\in C(\mathbf{J}\times\mathbb{R}^2,\mathbb{R})$ be a function such that $\varpi(.,\vartheta(.),\mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\vartheta(.))\in C(\mathbf{J}\times\mathbb{R}^2)$ and $\Theta\in C(\mathbf{J},\mathbb{R}^+)$ such that

$$\left|\varpi(\nu,\vartheta,\vartheta^*)\right|\le\Theta(\nu),\ \ \forall(\nu,\vartheta,\vartheta^*)\in\mathbf{J}\times\mathbb{R}^2.$$

Theorem 3.1 Suppose that (H_1) holds. If

$$\zeta\zeta_1\mathrm{Y}<1, \qquad (3.1)$$

where Y is defined by (2.7) and $\zeta_1=1+\frac{(\varsigma(b)-\varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1+1)}$, then the integrodifferential-type problem (1.2) has a unique solution on $\mathbf{J}$.

Proof. We convert (1.2) into a FPT, i.e., $\vartheta=\mathrm{T}\vartheta$ such that $\mathrm{T}:C(\mathbf{J},\mathbb{R})\to C(\mathbf{J},\mathbb{R})$ defined by (2.6).

observe that the fixed points of T are solutions of (1.2). By applying the Banach theorem [26], we will proof that T has a unique fixed point. Indeed, we put $\sup_{\tau\in\mathbf{J}}\left|\varpi(\tau,0,0)\right|=\mathrm{M}<\infty$ and choose

$$\kappa\ge\frac{\mathrm{MY}+\frac{(\varsigma(b)-\varsigma(a))^{\lambda-1}}{|\Lambda|\Gamma(\lambda)}|C|}{1-\mathrm{Y}\zeta\zeta_1}.$$

First, we prove that $TS_\kappa \subset S_\kappa$, where

$$S_\kappa = \{\vartheta \in C(\mathbf{J}, \mathbb{R}) : \|\vartheta\| \le \kappa\}. \qquad (3.2)$$

By using (H_1), we obtain

$$\left|\varpi(\nu, \vartheta(\nu), \Im_{a^+}^{\sigma_1;\varsigma}\vartheta(\nu))\right| \le \left|\varpi(\nu, \vartheta(\nu), \Im_{a^+}^{\sigma_1;\varsigma}\vartheta(\nu)) - \varpi(\nu, 0, 0)\right| + \left|\varpi(\nu, 0, 0)\right|$$

$$\le \zeta\left|\vartheta(\nu)\right| + \zeta\left|\Im_{a^+}^{\sigma_1;\varsigma}\vartheta(\nu)\right| + M$$

$$\le \zeta\|\vartheta\|\left(1 + \frac{(\varsigma(b) - \varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1 + 1)}\right) + M.$$

For any $\vartheta \in S_\kappa$, we get

$$\left|(T\vartheta)(\nu)\right| \le \sup_{\nu \in \mathbf{J}}\left\{\frac{(\varsigma(\nu) - \varsigma(a))^{\lambda-1}}{|\Lambda|\Gamma(\lambda)}\left[C + B\, \Im_{a^+}^{\sigma_1;\varsigma}\left|\varpi(\tau, \vartheta(\tau), \Im_{a^+}^{\sigma_1;\varsigma}\vartheta(\tau))\right|(b))\right]\right.$$

$$\left. + \Im_{a^+}^{\sigma_1;\varsigma}\left|\varpi(\tau, \vartheta(\tau), \Im_{a^+}^{\sigma_1;\varsigma}\vartheta(\tau))\right|(\nu)\right\}$$

$$\le \frac{(\varsigma(\nu) - \varsigma(a))^{\lambda-1}}{|\Lambda|\Gamma(\lambda)}\left[C + B\, \Im_{a^+}^{\sigma_1;\varsigma}\left(\zeta\|\vartheta\|\left(1 + \frac{(\varsigma(b) - \varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1 + 1)}\right) + M\right)(b)\right]$$

$$+ \Im_{a^+}^{\sigma_1;\varsigma}\left(\zeta\|\vartheta\|\left(1 + \frac{(\varsigma(b) - \varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1 + 1)}\right) + M\right)(\nu)$$

$$\le \frac{(\varsigma(\nu) - \varsigma(a))^{\lambda-1}}{|\Lambda|\Gamma(\lambda)}\left[C + B\left(\frac{(\varsigma(b) - \varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1 + 1)}\right)\left(\zeta\|\vartheta\|\left(1 + \frac{(\varsigma(b) - \varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1 + 1)}\right) + M\right)\right]$$

$$+ \left(\frac{(\varsigma(\nu) - \varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1 + 1)}\right)\left(\zeta\|\vartheta\|\left(1 + \frac{(\varsigma(b) - \varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1 + 1)}\right) + M\right)$$

$$\le \frac{(\varsigma(\nu) - \varsigma(a))^{\lambda-1}}{|\Lambda|\Gamma(\lambda)}C + \left\{B\left(\frac{(\varsigma(\nu) - \varsigma(a))^{\lambda-1}}{|\Lambda|\Gamma(\lambda)}\right)\left(\frac{(\varsigma(b) - \varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1 + 1)}\right)\right.$$

$$\left. + \left(\frac{(\varsigma(b) - \varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1 + 1)}\right)\right\} \times \left(\zeta\|\vartheta\|\left(1 + \frac{(\varsigma(b) - \varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1 + 1)}\right) + M\right)$$

$$\le Y(\zeta\zeta_1\kappa + M) + \frac{(\varsigma(b) - \varsigma(a))^{\lambda-1}}{|\Lambda|\Gamma(\lambda)}|C|$$

$$\le \kappa.$$

This means that $TS_\kappa \in S_\kappa$. i. e $TS_\kappa \subset S_\kappa$.

Next, For each $\vartheta, \vartheta^{å} \in C(\mathbf{J}, \mathbb{R})$ and $v \in \mathbf{J}$, we have

$$\left|(T\vartheta)(\nu)-(T\vartheta^{å})(\nu)\right|$$

$$\leq B\frac{(\varsigma(\nu)-\varsigma(a))^{\lambda-1}}{|\Lambda|\Gamma(\lambda)}\mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\left(\left|\varpi(\tau,\vartheta(\tau),\mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\vartheta(\tau))\right|-\left|\varpi(\tau,\vartheta^{å}(\tau),\mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\vartheta^{å}(\tau))\right|\right)(b)$$

$$+\mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\left(\left|\varpi(\tau,\vartheta(\tau),\mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\vartheta(\tau))\right|-\left|\varpi(\tau,\vartheta^{å}(\tau),\mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\vartheta^{å}(\tau))\right|\right)(\nu)$$

$$\leq B\frac{(\varsigma(\nu)-\varsigma(a))^{\lambda-1}}{|\Lambda|\Gamma(\lambda)}\mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\,\zeta\left\|\vartheta-\vartheta^{å}\right\|\left(1+\frac{(\varsigma(b)-\varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1+1)}\right)$$

$$+\mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\,\zeta\left\|\vartheta-\vartheta^{å}\right\|\left(1+\frac{(\varsigma(b)-\varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1+1)}\right)$$

$$\leq\left[B\zeta\zeta_1\left(\frac{(\varsigma(\nu)-\varsigma(a))^{\lambda-1}}{|\Lambda|\Gamma(\lambda)}\right)\left(\frac{(\varsigma(b)-\varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1+1)}\right)\right]\left\|\vartheta-\vartheta^{å}\right\|$$

$$+\zeta\zeta_1\frac{(\varsigma(\nu)-\varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1+1)}\left\|\vartheta-\vartheta^{å}\right\|$$

$$\leq\zeta\zeta_1\left[B\left(\frac{(\varsigma(\nu)-\varsigma(a))^{\lambda-1}}{|\Lambda|\Gamma(\lambda)}\right)\left(\frac{(\varsigma(b)-\varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1+1)}\right)+\frac{(\varsigma(b)-\varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1+1)}\right]\left\|\vartheta-\vartheta^{å}\right\|$$

$$\leq\zeta\zeta_1\Upsilon\left\|\vartheta-\vartheta^{å}\right\|.$$

which leads us to $\left\|T\vartheta-T\vartheta^{å}\right\|\leq\zeta\zeta_1\Upsilon\left\|\vartheta-\vartheta^{å}\right\|$. By (3.1), T is a contraction. Then, a unique solution exists on **J** for (1.2) due of the Banach's FPT [26]. Hence, end the proof. □

The following theorem relies to Schauder's fixed point technique [27].

Theorem 3.2 Suppose that (H_2) holds. Then the ς-Hilfer problem (1.2) has at least one solution in $C(\mathbf{J},\mathbb{R}^+)$.

Proof. We will complete the proof in three stages.

Stage1: We prove that $TS_\kappa\subset S_\kappa$, where S_κ is defined in (3.2), which proved in Theorem 3.1.

Stage 2: We have to prove the continuity of T. Assume that $\{\vartheta_n\}$ is a sequence such that ϑ_n

$\rightarrow\vartheta$ in S_κ as $n\rightarrow\infty$. Then, for each $\nu\in\mathbf{J}$, we have

$$\left|(\mathrm{T}\vartheta_n)(\nu)-(\mathrm{T}\vartheta)(\nu)\right|$$

$$\leq \frac{(\varsigma(\nu)-\varsigma(a))^{\lambda-1}}{\Lambda\Gamma(\lambda)} B\, \Im_{a^+}^{\sigma_1;\varsigma}\left|\varpi(\tau,\vartheta_n(\tau),\Im_{a^+}^{\sigma_1;\varsigma}\vartheta_n(\tau))(b)-\varpi(\tau,\vartheta(\tau),\Im_{a^+}^{\sigma_1;\varsigma}\vartheta(\tau))(b)\right|$$

$$+\Im_{a^+}^{\sigma_1;\varsigma}\left|\varpi(\tau,\vartheta_n(\tau),\Im_{a^+}^{\sigma_1;\varsigma}\vartheta_n(\tau))(\nu)-\varpi(\tau,\vartheta(\tau),\Im_{a^+}^{\sigma_1;\varsigma}\vartheta(\tau))(\nu)\right|$$

$$\leq \frac{(\varsigma(\nu)-\varsigma(a))^{\lambda-1}}{\Lambda\Gamma(\lambda)} B\, \frac{(\varsigma(b)-\varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1+1)}\left\|\varpi(.,\vartheta_n(.),\Im_{a^+}^{\sigma_1;\varsigma}\vartheta_n(.))-\varpi(.,\vartheta(.),\Im_{a^+}^{\sigma_1;\varsigma}\vartheta(.))\right\|$$

$$+\frac{(\varsigma(\nu)-\varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1+1)}\left\|\varpi(.,\vartheta_n(.),\Im_{a^+}^{\sigma_1;\varsigma}\vartheta_n(.))-\varpi(.,\vartheta(.),\Im_{a^+}^{\sigma_1;\varsigma}\vartheta(.))\right\|$$

$$\leq \frac{(\varsigma(b)-\varsigma(a))^{\lambda-1}}{\Lambda\Gamma(\lambda)} B\, \frac{(\varsigma(b)-\varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1+1)}\left\|\varpi(.,\vartheta_n(.),\Im_{a^+}^{\sigma_1;\varsigma}\vartheta_n(.))-\varpi(.,\vartheta(.),\Im_{a^+}^{\sigma_1;\varsigma}\vartheta(.))\right\|$$

$$+\frac{(\varsigma(b)-\varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1+1)}\left\|\varpi(.,\vartheta_n(.),\Im_{a^+}^{\sigma_1;\varsigma}\vartheta_n(.))-\varpi(.,\vartheta(.),\Im_{a^+}^{\sigma_1;\varsigma}\vartheta(.))\right\|.$$

$$\leq\left[\left(\frac{(\varsigma(b)-\varsigma(a))^{\lambda-1}}{\Lambda\Gamma(\lambda)}\right)\left(\frac{(\varsigma(b)-\varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1+1)}\right)B+\frac{(\varsigma(b)-\varsigma(a))^{\sigma_1}}{\Gamma(\sigma_1+1)}\right]$$

$$\times\left\|\varpi(.,\vartheta_n(.),\Im_{a^+}^{\sigma_1;\varsigma}\vartheta_n(.))-\varpi(.,\vartheta(.),\Im_{a^+}^{\sigma_1;\varsigma}\vartheta(.))\right\|.$$

By continuity of function ϖ , we achieve that

$$\left\|(\mathrm{T}\vartheta_n)-(\mathrm{T}\vartheta)\right\|\to 0 \text{ at } n\to 0,$$

which means that, T is continuous on S_κ.

Stage 3: We have to show the relatively compact of T. From stage 1, we have $\mathrm{T}\mathrm{S}_\kappa\subset\mathrm{S}_\kappa$, which gives that $\mathrm{T}\mathrm{S}_\kappa$ is uniformly bounded.

To prove the equicontinuous of T in S_κ , let $\vartheta\in\mathrm{S}_\kappa$ and v_1, v_2

$\in$**J** with $v_1<v_2$. Then,

$$\left|(\mathrm{T}\vartheta)(\nu_2)-(\mathrm{T}\vartheta)(\nu_1)\right|$$

$$\leq\frac{1}{\Gamma(\sigma_1)}\left|\int_a^{\nu_1}\varsigma'(\tau)\left[(\varsigma(\nu_2)-\varsigma(\tau))^{\sigma_1-1}-\left((\varsigma(\nu_1)-\varsigma(\tau))^{\sigma_1-1}\right)\right]\varpi(\tau,\vartheta(\tau),\Im_{a^+}^{\sigma_1;\varsigma}\vartheta(\tau))d\tau\right.$$

$$\left.+\int_{\nu_1}^{\nu_2}\varsigma'(\tau)\left[(\varsigma(\nu_2)-\varsigma(\tau))^{\sigma_1-1}\right]\varpi(\tau,\vartheta(\tau),\Im_{a^+}^{\sigma_1;\varsigma}\vartheta(\tau))d\tau\right|$$

$$\leq\frac{\|\varpi\|}{\Gamma(\sigma_1+1)}\left[2((\varsigma(\nu_2)-\varsigma(\nu_1))^{\sigma_1}+\left|\left(\varsigma(\nu_2)-\varsigma(a)\right)^{\sigma_1}-\left(\varsigma(\nu_1)-\varsigma(a)\right)^{\sigma_1}\right|\right]$$

$$\leq\frac{\Theta}{\Gamma(\sigma_1+1)}\left[2((\varsigma(\nu_2)-\varsigma(\nu_1))^{\sigma_1}+\left|\left(\varsigma(\nu_2)-\varsigma(a)\right)^{\sigma_1}-\left(\varsigma(\nu_1)-\varsigma(a)\right)^{\sigma_1}\right|\right].$$

As $v_2 - v_1 \to 0$, we obtain

$$\left|(T\vartheta)(\nu_2) - (T\vartheta)(\nu_1)\right| \to 0, \text{ for any } \vartheta \in S_\kappa. \quad (3.3)$$

Note that the right-hand side of the inequality (3.3) lead to zero as $\nu_2 \to \nu_1$ and independent of ϑ. Consequently, in view of the previous stages, so, by Arzela-Ascoli theorem, T is relatively compact, and hence it is completely continuous. As a result of Schauder's FPT [26], we deduce that our problem (1.2) has at least one solution $\vartheta \in C(\mathbf{J}, \mathbb{R}^+)$. The proof is completed. □

4 UH Stability Analysis:

In the following part, we will discuss two kinds of stability results of the problem (1.2), namely Ulam-Hyers (UH) and generalized Ulam-Hyers (GUH) stability.

Definition 4.1 The problem (1.2) is UH stable if there exists a constant $K_\varpi > 0$ such that for each $\varepsilon > 0$ and every solution $\bar{\vartheta} \in C(\mathbf{J}, \mathbb{R})$ of the inequalities

$$\left| {}^{H}D_{a^+}^{\sigma_1,\sigma_2;\varsigma} \bar{\vartheta}(\nu) - \varpi(\nu, \bar{\vartheta}(\nu), \mathfrak{I}_{a^+}^{\sigma_1;\varsigma} \bar{\vartheta}(\nu)) \right| \le \varepsilon, \text{ for all } \nu \in \mathbf{J}, \quad (4.1)$$

there exists a unique solution $\vartheta \in C(\mathbf{J}, \mathbb{R})$ of the problem (1.2), which satisfies

$$\left|\vartheta(\nu) - \bar{\vartheta}(\nu)\right| \le K_\varpi \varepsilon. \quad (4.2)$$

Definition 4.2 The problem (1.2) is GUH stable if there exists $\Psi \in C([0,\infty),[0,\infty))$, $\Psi(0) = 0$, such that for every solution $\bar{\vartheta} \in C(\mathbf{J}, \mathbb{R})$ of the inequality

$$\left| {}^{H}D_{a^+}^{\sigma_1,\sigma_2;\varsigma} \bar{\vartheta}(\nu) - \varpi(\nu, \bar{\vartheta}(\nu), \mathfrak{I}_{a^+}^{\sigma_1;\varsigma} \bar{\vartheta}(\nu)) \right| \le \varepsilon, \ \nu \in \mathbf{J}, \quad (4.3)$$

there exists a unique solution $\vartheta \in C(\mathbf{J}, \mathbb{R})$ for the problem (1.2), such that

$$\left|\bar{\vartheta}(\nu) - \vartheta(\nu)\right| \le \Psi(\varepsilon), \ \nu \in \mathbf{J}.$$

Remark 4.1 $\bar{\vartheta} \in C(\mathbf{J}, \mathbb{R})$ satisfies the inequality (4.1) if and only if there exists a function $h \in C(\mathbf{J}, \mathbb{R})$ with

(1) $|h(\nu)| \le \varepsilon$, $v \in \mathbf{J}$,
(2) For all $v \in \mathbf{J}$,

$$ {}^{H}\mathrm{D}_{a^+}^{\sigma_1,\sigma_2;\varsigma}\bar{\vartheta}(\nu)=\varpi(\nu,\bar{\vartheta}(\nu),\Im_{a^+}^{\sigma_1;\varsigma}\bar{\vartheta}(\nu))+|h(\nu)|. $$

Lemma 4.1 If $\vartheta\in C(\mathbf{J},\mathbb{R})$ is a solution to inequality (4.1), then ϑ satisfies the following inequality

$$ |\vartheta(\nu)-\Theta_{\vartheta}|\le\varepsilon\Delta, $$

where

$$ \Theta_{\bar{\vartheta}}=\Im_{a^+}^{\sigma_1;\varsigma}\varpi(\tau,\vartheta(\tau),\Im_{a^+}^{\sigma_1;\varsigma}\vartheta(\tau))(\nu) $$
$$ +\frac{(\varsigma(\nu)-\varsigma(a))^{\lambda-1}}{\Lambda\Gamma(\lambda)}\left[C-B\,\Im_{a^+}^{\sigma_1;\varsigma}\varpi(\tau,\vartheta(\tau),\Im_{a^+}^{\sigma_1;\varsigma}\vartheta(\tau))(b))\right], $$

and

$$ \Delta=\frac{(\varsigma(b)-\varsigma(a))^{\sigma_1-1}}{\Gamma(\sigma_1)}+\frac{(\varsigma(\nu)-\varsigma(a))^{\lambda-1}}{\Lambda\Gamma(\lambda)}\left[\frac{|C|}{\varepsilon}+B\frac{(\varsigma(b)-\varsigma(a))^{\sigma_1-1}}{\Gamma(\sigma_1)}\right]. $$

Proof. According to Remark 4.1, we get

$$ \begin{cases} {}^{H}\mathrm{D}_{a^+}^{\sigma_1,\sigma_2;\varsigma}\vartheta(\nu)=\varpi(\nu,\vartheta(\nu),\Im_{a^+}^{\sigma_1;\varsigma}\vartheta(\nu))+|h(\nu)| \\ A\vartheta(a)+B\vartheta(b)=C,\ A,B,C\in\mathbb{R},\ 0\le\sigma_2\le1 \end{cases}. $$

Then, by Lemma 2.3, we get

$$ \Theta_{\bar{\vartheta}}=\Im_{a^+}^{\sigma_1;\varsigma}\varpi(\tau,\vartheta(\tau),\Im_{a^+}^{\sigma_1;\varsigma}\vartheta(\tau))(\nu) $$
$$ +\frac{(\varsigma(\nu)-\varsigma(a))^{\lambda-1}}{\Lambda\Gamma(\lambda)}\left[C-B\,\Im_{a^+}^{\sigma_1;\varsigma}\varpi(\tau,\vartheta(\tau),\Im_{a^+}^{\sigma_1;\varsigma}\vartheta(\tau))(b))\right], $$

$$ \vartheta(\nu)=\Theta_{\vartheta} $$
$$ +\Im_{a^+}^{\sigma_1;\varsigma}h(\nu)+\frac{(\varsigma(\nu)-\varsigma(a))^{\lambda-1}}{\Lambda\Gamma(\lambda)}\left[C-B\,\Im_{a^+}^{\sigma_1;\varsigma}h(b)\right] $$

which implies

$$\left|\vartheta(\nu)-\Theta_{\vartheta}\right| \leq \varepsilon \frac{(\varsigma(\nu)-\varsigma(a))^{\sigma_1-1}}{\Gamma(\sigma_1)}+\varepsilon \frac{(\varsigma(\nu)-\varsigma(a))^{\lambda-1}}{\Lambda\Gamma(\lambda)}$$

$$\times\left[\frac{|C|}{\varepsilon}+B\frac{(\varsigma(b)-\varsigma(a))^{\sigma_1-1}}{\Gamma(\sigma_1)}\right]$$

$$\leq \varepsilon\left(\frac{(\varsigma(b)-\varsigma(a))^{\sigma_1-1}}{\Gamma(\sigma_1)}+\frac{(\varsigma(\nu)-\varsigma(a))^{\lambda-1}}{\Lambda\Gamma(\lambda)}\left[\frac{|C|}{\varepsilon}+B\frac{(\varsigma(b)-\varsigma(a))^{\sigma_1-1}}{\Gamma(\sigma_1)}\right]\right)$$

$$\leq \varepsilon\Delta.$$

□

Theorem 4.1 Suppose that (H_1)-(H_2) hold. Under the Lemma 4.1 , the following equation

$${}^{H}\mathsf{D}_{a^+}^{\sigma_1,\sigma_2;\varsigma}\vartheta(\nu)=\varpi(\nu,\vartheta(\nu),\mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\vartheta(\nu)),\ \nu\in\mathbf{J}, \quad (4.4)$$

is UH as well as GUH stable provided that $\zeta\zeta_1\Upsilon < 1$.

Proof. $\overline{\vartheta}\in C(\mathbf{J},\mathbb{R})$ be a function satisfies (4.1), let $\vartheta\in C(\mathbf{J},\mathbb{R})$ be a unique solution to the next problem

$$\begin{cases} {}^{H}\mathsf{D}_{a^+}^{\sigma_1,\sigma_2;\varsigma}\vartheta(\nu)=\varpi(\nu,\vartheta(\nu),\mathfrak{I}_{a^+}^{\sigma_1;\varsigma}\vartheta(\nu)) \ \ \nu\in\mathbf{J}, \\ A\vartheta(a)+B\vartheta(b)=A\overline{\vartheta}(a)+B\overline{\vartheta}(b), \\ \text{where } A,B,C\in\mathbb{R},\ 0\leq\sigma_2\leq 1 \end{cases}.$$

Then, by Lemma 2.3, we get

$$\vartheta(\nu)=\Theta_{\vartheta}.$$

Now, by Theorem 3.1, we have

$$\left\|\vartheta-\overline{\vartheta}\right\|=\sup_{\nu\in\mathbf{J}}\left|\vartheta(\nu)-\Theta_{\overline{\vartheta}}\right|\leq\sup_{\nu'\in\mathbb{R}}\left|\vartheta(\nu)-\Theta_{\vartheta}\right|+\sup_{\nu'\in\mathbb{R}}\left|\Theta_{\vartheta}-\Theta_{\overline{\vartheta}}\right|$$

$$\leq\varepsilon\Delta+\zeta\zeta_1\Upsilon\left\|\vartheta-\overline{\vartheta}\right\|.$$

Thus,

$$\left\|\vartheta-\overline{\vartheta}\right\|\leq K_{\varpi}\varepsilon,$$

where

$$K_{\varpi}=\frac{\Delta}{1-\zeta\zeta_1\Upsilon}>0,$$

which main that the problem (4.4) is UH stability. Now, by choosing $\Psi(\varepsilon) = K_{\varpi}\varepsilon$ such that $\Psi(0) = 0$, then the problem (4.4) is GUH stability. □

5 Examples

Example 5.1 For $\nu \in [0, 1]$, we consider the following problem of BVP for a generalized Hilfer-type integrodifferentialequation:

$$
{}^{H}\mathrm{D}_{0^+}^{\frac{1}{2},\frac{1}{2};\frac{1}{2}}\vartheta(\nu) = \begin{cases} \left[\dfrac{1}{3}e^{\sqrt{\nu}+1} + \dfrac{2+|\vartheta(\nu)| + \left|\Im_{0^+}^{\frac{1}{2};\frac{1}{2}}\vartheta(\nu)\right|}{8e^{2-\nu}\left(1+|\vartheta(\nu)|+\left|\Im_{0^+}^{\frac{1}{2};\frac{1}{2}}\vartheta(\nu)\right|\right)}\right], \nu \in [0,1], \\ \vartheta(0)+\vartheta(1) = 0, \end{cases} \quad (5.1)
$$

Set:

$$
\varpi(\nu, \vartheta_1, \vartheta_2) = \left[\frac{1}{3}e^{\sqrt{\nu}+1} + \frac{2+\vartheta_1+\vartheta_2}{8e^{2-\nu}(1+\vartheta_1+\vartheta_2)}\right], \nu \in [0,1], \vartheta_1, \vartheta_2 \in \mathbb{R}^+,
$$

with $\sigma_1 = \frac{1}{2}, \sigma_2 = \frac{1}{2}$,$A = B = 1, C = 0$, and $\varsigma = \frac{1}{2}$. Clearly, the function $\varpi \in C([0,1])$. For each $\vartheta_1, \vartheta_2, \vartheta_1^*, \vartheta_2^* \in \mathbb{R}^+$ and $\nu \in [0, 1]$

$$
|\varpi(\nu, \vartheta_1, \vartheta_2) - \varpi(\nu, \vartheta_1^*, \vartheta_2^*)| = \left|\frac{2+\vartheta_1+\vartheta_2}{8e^{2-\nu}(1+\vartheta_1+\vartheta_2)} - \frac{2+\vartheta_1^*+\vartheta_2^*}{8e^{2-\nu}(1+\vartheta_1^*+\vartheta_2^*)}\right|
$$

$$
\leq \frac{1}{8e^{2-\nu}}\left(|\vartheta_1-\vartheta_1^*| + |\vartheta_2-\vartheta_2^*|\right)
$$

$$
\leq \frac{1}{8e}\left(|\vartheta_1-\vartheta_1^*| + |\vartheta_2-\vartheta_2^*|\right).
$$

Hence, the condition (H_1) is satisfied with $\zeta = \frac{1}{8e}$. It is easy to verify that $\zeta\zeta_1\Upsilon = 0.13193 < 1$, where $\zeta_1 = 1.7979$, $\Lambda = 0.970\ 45$, $\lambda = \frac{3}{4}$, and $\Upsilon = 1.5958$. Since all the hypotheses of Theorem 3.1 are satisfied, therefore problem (5.1) has a unique solution.

Example 5.2 For $\nu \in [0, 1]$, we consider the following problem of BVP for a generalized Hilfer-type integrodifferential equation:

$$\begin{cases} {}^{H}\mathrm{D}_{0^+}^{\frac{1}{3},\frac{1}{2};\frac{3}{2}}\vartheta(\nu) = \dfrac{|\vartheta(\nu)| + \cos\left|\mathfrak{I}_{0^+}^{\frac{3}{2};1}\vartheta(\nu)\right|}{30(\nu+2)(1+|\vartheta(\nu)|)}, \\ \vartheta(0)+\vartheta(1) = 0, \end{cases} \qquad (5.2)$$

where $A = B = 1$, $C = 0$.

Set:

$$\varpi(\nu,\vartheta_1,\vartheta_2) = \frac{\vartheta_1 + \cos\vartheta_2}{30(\nu+2)(1+\vartheta_1)},\ \nu\in[0,1],\ \vartheta_1,\vartheta_2 \in \mathbb{R}^+,$$

with $\sigma_1 = \frac{1}{3}, \sigma_2 = \frac{1}{2}$ and $\varsigma = \frac{3}{2}$. Now, for each $\vartheta_1,\vartheta_2,\vartheta_1^*,\vartheta_2^* \in \mathbb{R}^+$ and $\nu \in [0, 1]$

$$\left|\varpi(\nu,\vartheta_1,\vartheta_2) - \varpi(\nu,\vartheta_1^*,\vartheta_2^*)\right| = \left|\frac{\vartheta_1+\cos\vartheta_2}{30(\tau+2)(1+\vartheta_1)} - \frac{\vartheta_1^*+\cos\vartheta_2^*}{30(\tau+2)(1+\vartheta_1^*)}\right|$$

$$\leq \frac{1}{30}\left(\left|\vartheta_1-\vartheta_1^*\right| + \left|\vartheta_2-\vartheta_2^*\right|\right).$$

Hence, the condition (H_1) is satisfied with $\zeta = \frac{1}{30}$. It is easy to check that $\zeta\zeta_1\Upsilon \approx 0.07835 < 1$, where $\zeta_1 = 1.8611$, $\Lambda = 1.3820$, $\lambda = \frac{2}{3}$, $\Upsilon = 1.2631$. It follows from Theorem 3.1 that problem (5.2) has a unique solution.

We can observe that all the required conditions of Theorem 4.1 are satisfied. Hence, the proposed problem (5.1) is UH and GUH stable.

In view of Theorem 4.1 , for $\varepsilon > 0$, any solution $\bar{\vartheta} \in C([0,1],\mathbb{R})$ satisfies the inequality

$$\left| {}^{H}\mathrm{D}_{0^+}^{\frac{1}{2},\frac{1}{2};\frac{1}{2}}\bar{\vartheta}(\nu) = \left[\frac{1}{3}e^{\sqrt{\nu}+1} + \frac{2+|\bar{\vartheta}(\nu)| + \left|\mathfrak{I}_{0^+}^{\frac{1}{2};\frac{1}{2}}\bar{\vartheta}(\nu)\right|}{8e^{2-\nu}\left(1+|\bar{\vartheta}(\nu)| + \left|\mathfrak{I}_{0^+}^{\frac{1}{2};\frac{1}{2}}\bar{\vartheta}(\nu)\right|\right)}\right]\right| \leq \varepsilon,\ \nu\in[0,1],$$

there exists a solution $\vartheta \in C([0,1],\mathbb{R})$ for the problem (5.1) such that

$\left|\bar{\vartheta}(\nu)-\vartheta(\nu)\right| \le K_{\varpi}\varepsilon,\ \nu \in [0,1],$

where $K_{\varpi}=\frac{\Delta}{1-\zeta\zeta_1\Upsilon}=\frac{1.5958}{1-0.13193}=1.8383>1.$ Moreover, if we set $K_{\varpi}\varepsilon = \Psi(\varepsilon)$, and $\Psi(0) = 0$, then

$\left|\bar{\vartheta}(\nu)-\vartheta(\nu)\right| \le \Psi(\varepsilon),\ \nu \in [0,1].$

6 Conclusions:

ς-HFD a general fractional operator, is of large use because of its broad freedom to cover a lot of classic fractional operators. In this study, we considered the frame of ς-Hilfer for the problem (1.2). First, the uniqueness and existence of solutions for the proposed problem were examined. Next, the stability of the ς-Hilfer type BVP (1.2) has been obtained by applying some mathematical methods. Moreover, Schauder's and Banach's FPTs have been applied. Finally, we have presented some examples. Applying these examinations, other qualitative analyses of the solution like stability results can be discussed, and this is what we desire to think about in future studies.

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