# Some Statistical Properties of the Laplace Distribution

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**Abstract**

The main aim of this paper is to use the Lauricella series to extend the beta function, and then through this function we investigate some statistical properties. Moreover, we use the Lauricella series in order to diagnose the Riemann - Liouville fractional factor extensionand study some of its properties.

**Keywords:** Statistical distribution;Lauricella series; Laplace distribution.

## 1 Introduction

The beta function is given by [1,12,17]

$$B(P_1,P_2) = \int_0^1 \zeta^{P_1-1}(1-\zeta)^{P_2-1} d\zeta = \frac{\Gamma(P_1)\Gamma(P_2)}{\Gamma(P_1)+\Gamma(P_2)} \quad (1)$$

where $\Psi(P_2), \Psi(P_2) > 0, \Psi$is the real part of the function. The Gauss hypergeometric andconfluent hypergeometric functions are defined as

$$2H_1(P_1;P_2;P_3;v) = \sum_{i=0}^{\infty} \frac{(P_1)_i (P_2)_i v^i}{(P_3)_i \, i!} \quad (2)$$

where $|v| < 1, P_1, P_2, P_3 \in C;\ P_3 \neq 0,-1,-2,\ldots$ and

$$\delta_1(P_2;P_3;v) = \sum_{i=0}^{\infty} \frac{(P_2)_i v^i}{(P_3)_i \, i!} \quad (3)$$

The integral representations of hypergeometric and confluent hypergeometric function are [9,10,16]

$$2H_1(P_1;P_2;P_3;v) = \frac{\Gamma(P_3)}{\Gamma(P_2)\Gamma(P_3-P_2)} \int_0^1 \zeta^{P_2-1}(1-\zeta)^{P_3-P_2-1}(1-i\zeta)^{-P_1} d\zeta \quad (4)$$

where $\Psi(P_3) > \Psi(P_2) > 0, |\arg(1-i)| < \pi,$

$$\delta_1(P_2;P_3;v) = \frac{\Gamma(P_3)}{\Gamma(P_2)\Gamma(P_3-P_2)}\int_0^1 \zeta^{P_2-1}(1-\zeta)^{P_3-P_2-1}e^{i\zeta}\,d\zeta \qquad (5)$$

where $\Psi(P_3) > \Psi(P_2) > 0$

Goswami et al. [2,11,16,18] defined an extension of beta function as follows:

$$B_{\lambda_1\lambda_2}^{\mu_1\mu_2}(P_1,P_2) = \int_0^1 \zeta^{P_1-1}(1-\zeta)^{P_2-1}2H_1\left(\mu_1,\mu_2;\frac{-\lambda_1}{\zeta}-\frac{\lambda_2}{1-\zeta}\right)d\zeta \qquad (6)$$

where $\min\{\Psi(\lambda_1),\Psi(\lambda_2)\}, \min\{\Psi(P_1),\Psi(P_2)\}, \mu_1 \in C$ and $\mu_2 \neq 0-1,-2,\dots$

The Appell's double hypergeometric functions are defined as follows [19,20]:

$$H_1(\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4,\kappa,\iota) = \sum_{i,j=0}^{\infty}\frac{(\varsigma_1)_{i+j}(\varsigma_2)_i(\varsigma_3)_j}{(\varsigma_4)_{i+j}}\frac{\kappa^i\iota^j}{i!\,j!} \qquad (7)$$

Convergence conditions for the Appell series are as follows:

1) $H_2$ converges for $|k| + |\iota| < 1$;

2) $H_1$ and $H_3$ converge when $|k| < 1$ and $|\iota| < 1$;

3) $H_4$ converges when $|\sqrt{k}| + |\sqrt{\iota}| < 1$.

The classical Riemann–Liouville fractional integral of order $O \in C$ with $\Psi(O) > 0$ of a function g is given by [13,19]

$$[Z_w^O h](w) = \frac{1}{\Gamma(O)}\int_0^w h(\zeta)(w-\zeta)^{O-1}d\zeta \qquad (8)$$

The classical Riemann–Liouville fractional integral of order $O \in C$ with $\Psi(O) < 0$ of a function g is given by [3,8,9]

$$[E_w^O h](w) = \frac{1}{\Gamma(-O)}\int_0^w h(\zeta)(w-\zeta)^{-O-1}d\zeta \qquad (9)$$

**2extended beta functionality**

Several authors have studied generalizations of beta function and Hypergeometric functions (see, for example, [4,5,12,15]). In this part, we will define an extension of the beta function using the Lauricella function.

**Definition 2.2** The extensions of beta function by using Lauricella series [6,7,14] respectively, are defined as follows:

$$B_{\lambda_1,\lambda_2,\dots,\lambda_n}^{H_K^{(n)}}(P_1,P_2) = \int_0^1 \zeta^{P_1-1}(1-\zeta)^{P_2-1}H_K^{(n)}\left(\varsigma_1^1,\varsigma_2^1,\dots,\varsigma_n^1;\varsigma_1^2,\varsigma_2^2,\dots,\varsigma_n^2;\frac{\lambda_1}{\zeta^v},\frac{\lambda_2}{\zeta^v},\dots,\frac{\lambda_n}{\zeta^v}\right)d\zeta \qquad (8)$$

where $\Psi(\varsigma_1), \Psi(\varsigma_1^1), \Psi(\varsigma_2^1), \dots, \Psi(\varsigma_n^1); \Psi(\varsigma_1^2), \Psi(\varsigma_2^2), \dots, \Psi(\varsigma_n^2) > 0, \Psi(\lambda_1), \Psi(\lambda_2), \dots, \Psi(\lambda_n) \geq 0, \Psi(P_1), \Psi(P_2) > 0$.

$$B_{\lambda_1,\lambda_2,\dots,\lambda_n}^{H_T^{(n)}}(P_1, P_2) = \int_0^1 \zeta^{P_1-1}(1-\zeta)^{P_2-1} H_T^{(n)}\left(\varsigma_1, \varsigma_2, \dots, \varsigma_n; \varsigma_1^1, \varsigma_2^1, \dots, \varsigma_n^1; \varsigma_1^2; \frac{\lambda_1}{\zeta^v}, \frac{\lambda_2}{\zeta^v}, \dots, \frac{\lambda_n}{\zeta^v}\right) d\zeta \qquad (9)$$

where $\Psi(\varsigma_1), \Psi(\varsigma_2), \dots, \Psi(\varsigma_n); \Psi(\varsigma_1^1), \Psi(\varsigma_2^1), \dots, \Psi(\varsigma_n^1), \Psi(\varsigma_1^2) \geq 0, v \geq 0, \Psi(P_1), \Psi(P_2) > 0$.

$$B_{\lambda_1,\lambda_2,\dots,\lambda_n}^{H_Q^{(n)}}(P_1, P_2) = \int_0^1 \zeta^{P_1-1}(1-\zeta)^{P_2-1} H_Q^{(n)}\left(\varsigma_1, \varsigma_1^1; \varsigma_1^2, \dots \varsigma_n^2; \frac{\lambda_1}{\zeta^v}, \frac{\lambda_2}{\zeta^v}, \dots, \frac{\lambda_n}{\zeta^v}\right) d\zeta \qquad (10)$$

where $\Psi(\varsigma_1),; \Psi(\varsigma_1^1), \Psi(\varsigma_1^2), \dots, \Psi(\varsigma_n^2) \geq 0, \Psi(\lambda_1), \Psi(\lambda_2), \dots, \Psi(\lambda_n) \geq 0,, \Psi(P_1), \Psi(P_2) \geq 0, v \geq 0, \Psi(P_1), \Psi(P_2) > 0$.

$$B_{\lambda_1,\lambda_2,\dots,\lambda_n}^{H_S^{(n)}}(P_1, P_2) = \int_0^1 \zeta^{P_1-1}(1-\zeta)^{P_2-1} H_S^{(n)}\left(\varsigma_1, \varsigma_1^1; \varsigma_2^1, \dots, \varsigma_n^1, \varsigma_1^2; \frac{\lambda_1}{\zeta^v}, \frac{\lambda_2}{\zeta^v}, \dots, \frac{\lambda_n}{\zeta^v}\right) d\zeta \qquad (11)$$

where $\Psi(\varsigma_1),; \Psi(\varsigma_1^1), \Psi(\varsigma_2^1), \dots, \Psi(\varsigma_n^1); \Psi(\varsigma_1^2) \geq 0, \Psi(\lambda_1), \Psi(\lambda_2), \dots, \Psi(\lambda_n) \geq 0,, \Psi(P_1), \Psi(P_2) \geq 0, v \geq 0, \Psi(P_1), \Psi(P_2) > 0$.

## 3 Some properties of the extended beta function

In this section, we will derive iteration relationships and integral representations of the new extended beta function. We start this section by this proposition .

**Proposition 3.1** The extension of beta function involving Lauricella function series $H_K^{(n)}(.)$check as follows:

$$B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1, 2-P_2) = \sum_{i=0}^{\infty} \frac{(P_2)_{i+1}}{(i+1)!} B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1+i, 2) \qquad (12)$$

Proof : LHS =

$$B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1, 2-P_2) = \int_0^1 \zeta^{P_1-1}(1-\zeta)^{P_2-1} H_K^{(n)}$$

$$\left(\varsigma_1^1, \varsigma_2^1, \dots, \varsigma_n^1; \varsigma_1^2, \varsigma_2^2, \dots, \varsigma_n^2; \frac{\lambda_1}{\zeta^v}, \frac{\lambda_2}{\zeta^v}, \dots, \frac{\lambda_n}{\zeta^v}\right) d\zeta$$

Let $(1-\zeta)^{-P_2} = \sum_{i=0}^{\infty} \frac{\zeta^{i+1}}{(i+1)!}(P_2)_{i+1}$

So we implies
$B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1, 2-P_2) = \int_0^1 \zeta^{P_1-1} \sum_{i=0}^{\infty} \frac{\zeta^{i+1}}{(i+1)!}(P_2)_{i+1} H_K^{(n)}\left(\varsigma_1^1, \varsigma_2^1, \dots, \varsigma_n^1; \varsigma_1^2, \varsigma_2^2, \dots, \varsigma_n^2; \frac{\lambda_1}{\zeta^v}, \frac{\lambda_2}{\zeta^v}, \dots, \frac{\lambda_n}{\zeta^v}\right) d\zeta$

$$B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1, 2-P_2) = \sum_{i=0}^{\infty} \frac{(P_2)_{i+1}}{(i+1)!} \int_0^1 \zeta^{P_1+i-1} H_K^{(n)}\left(\varsigma_1^1, \varsigma_2^1, \dots, \varsigma_n^1; \varsigma_1^2, \varsigma_2^2, \dots, \varsigma_n^2; \frac{\lambda_1}{\zeta^v}, \frac{\lambda_2}{\zeta^v}, \dots, \frac{\lambda_n}{\zeta^v}\right) d\zeta$$

$= \sum_{i=0}^{\infty} \frac{(P_2)_{i+1}}{(i+1)!} B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1+i, 2) =$ RHS.

Therefore $B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1, 2-P_2) = \sum_{i=0}^{\infty} \frac{(P_2)_{i+1}}{(i+1)!} B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1+i, 2)$.

**Proposition 3.2** The extension of beta function involving Lauricella function series$H_K^{(n)}(.)$check as follows:

$$B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1, P_2) = \sum_{i=0}^{\infty} B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1+i+1, P_2+2) \qquad (13)$$

Proof

LHS $=B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1, P_2) = \int_0^1 \zeta^{P_1-1}(1-\zeta)^{P_2-1} H_K^{(n)}\left(\varsigma_1^1, \varsigma_2^1, \dots, \varsigma_n^1; \varsigma_1^2, \varsigma_2^2, \dots, \varsigma_n^2; \frac{\lambda_1}{\zeta^v}, \frac{\lambda_2}{\zeta^v}, \dots, \frac{\lambda_n}{\zeta^v}\right) d\zeta$

Let $(1-\zeta)^{P_2-1} = (1-\zeta)^{P_2+1} \sum_{i=0}^{\infty} \zeta^{i+2}$

So we implies
$B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1, P_2) = \int_0^1 \zeta^{P_1-1}(1-\zeta)^{P_2+1} \sum_{i=0}^{\infty} \zeta^{i+2} \left(\varsigma_1^1, \varsigma_2^1, \dots, \varsigma_n^1; \varsigma_1^2, \varsigma_2^2, \dots, \varsigma_n^2; \frac{\lambda_1}{\zeta^v}, \frac{\lambda_2}{\zeta^v}, \dots, \frac{\lambda_n}{\zeta^v}\right) d\zeta$

$$B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1, P_2) = \sum_{i=0}^{\infty} \int_0^1 \zeta^{P_1+i+1}(1-\zeta)^{P_2+2} H_K^{(n)}\left(\varsigma_1^1, \varsigma_2^1, \dots, \varsigma_n^1; \varsigma_1^2, \varsigma_2^2, \dots, \varsigma_n^2; \frac{\lambda_1}{\zeta^v}, \frac{\lambda_2}{\zeta^v}, \dots, \frac{\lambda_n}{\zeta^v}\right) d\zeta$$

$= \sum_{i=0}^{\infty} B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1+i+1, P_2+2) \quad =$ RHS.

Therefore $B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1, P_2) = \sum_{i=0}^{\infty} B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1+i+1, P_2+2)$.

**Proposition 3.3** The extension of beta function involving Lauricella function series$H_K^{(n)}(.)$ check as follows:

$$B_{\lambda_1\lambda_2}^{H_K^{(n)}}(\rho+1, \rho-i+1) = \sum_{i=0}^{\infty} C_i^l B_{\lambda_1\lambda_2}^{H_K^{(n)}}(\rho+l+1, \rho-l+1) \qquad (14)$$

where $C_i^l = \frac{l!}{i!(l-i)!}$

Proof:

Since $B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1, P_2) = B_{\lambda_1\lambda_2}^{H_K^{(n)}}(\rho+1, \rho-i) + B_{\lambda_1\lambda_2}^{H_K^{(n)}}(\rho, \rho-i+1)$

Let$P_1 = \rho+1, P_2 = \rho-i+1$ then

$$B_{\lambda_1\lambda_2}^{H_K^{(n)}}(\rho+1,\rho-i+1) = B_{\lambda_1\lambda_2}^{H_K^{(n)}}(\rho+2,\rho-i+1) + B_{\lambda_1\lambda_2}^{H_K^{(n)}}(\rho,\rho-i+2)$$

So by substituting n = 1, 2, . . . in above equation we get

$$B_{\lambda_1\lambda_2}^{H_K^{(n)}}(\rho+1,\rho-i+1) = B_{\lambda_1\lambda_2}^{H_K^{(n)}}(\rho+2,\rho+2) + B_{\lambda_1\lambda_2}^{H_K^{(n)}}(\rho+1,\rho+2)$$

$$B_{\lambda_1\lambda_2}^{H_K^{(n)}}(\rho+1,\rho-i+2) = B_{\lambda_1\lambda_2}^{H_1}(\rho+3,\rho+3) + B_{\lambda_1\lambda_2}^{H_K^{(n)}}(\rho+2,\rho+2)$$

$$B_{\lambda_1\lambda_2}^{H_K^{(n)}}(\rho+1,\rho-i+3) = B_{\lambda_1\lambda_2}^{H_K^{(n)}}(\rho+4,\rho+3) + B_{\lambda_1\lambda_2}^{H_K^{(n)}}(\rho+3,\rho+4)$$

and we continue the process until we get the result.

## 4 Statistical distribution involving extended beta function

In this section, In this section, we get some statistics properties in extension of beta function in .We derive the results formean, variance, moment generating function.

**Definition 4.1** We construct a distribution of a new extended beta function involving Lauricella function series by:

$$h(\zeta) = \begin{cases} \frac{1}{B_{\lambda_1\lambda_2}^{H_1}(P_1+1,P_2+1)}\zeta^{P_1-1}(1-\zeta)^{P_2-1}H_K^{(n)}\left(\varsigma_1^1,\varsigma_2^1,\dots,\varsigma_n^1;\varsigma_1^2,\varsigma_2^2,\dots,\varsigma_n^2;\frac{(\lambda_1+1)}{\zeta^{v+1}},\frac{(\lambda_2+1)}{\zeta^{v+1}},\dots,\frac{(\lambda_n+1)}{\zeta^{v+1}}\right)d\zeta\ , o < \xi < 1 \\ 0 \quad , \text{otherwise} \end{cases} \quad (15)$$

$\Psi(\varsigma_1),\Psi(\varsigma_1^1),\Psi(\varsigma_2^1),\dots,\Psi(\varsigma_n^1);\Psi(\varsigma_1^2),\Psi(\varsigma_2^2),\dots,\Psi(\varsigma_n^2) > 0,\Psi(\lambda_1),\Psi(\lambda_2),\dots,\Psi(\lambda_n) \geq 0,\Psi(P_1),\Psi(P_2) > 0.$

the mean of the extended beta distribution defined as:

$$E(Z^i) = \int_0^1 Z^i f(Z^i)d\zeta = \frac{B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1+\Theta+1,P_2+1)}{B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1+1,P_2+1)} \quad (16)$$

where $\Theta \in R$ and Z is any random variable.
Also the Variance of the distribution defined as:

$$\sigma^2 = E(Z^2) - \left(E(Z)\right)^2 = \frac{B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1+1,P_2+1)\, B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1+3,P_2+1) - \left(B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1+2,P_2+1)\right)^2}{\left(B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1+1,P_2+1)\right)^2}$$

(17)

the distribution of Moment generating function defined as:

$$M(\zeta)=\sum_{0}^{\infty}E(Z^{i})\frac{\zeta^{i+1}}{(i+1)!}=\frac{1}{B_{\lambda_1\lambda_2}^{H_1}(P_1+1,P_2+1)}\sum_{i=0}^{\infty}\ B_{\lambda_1\lambda_2}^{H_K^{(n)}}(P_1+i+1,P_2+1)\frac{\zeta^{i+1}}{(i+1)!}$$

(18)

**5-Some properties of Riemann–Liouville fractional operators**

In this section, we derive the Riemann–Liouville fractional operators using the Lauricella function series $H_K^{(n)}(.)$ and investigate its some properties.

**Definition 5.1** The integral of extendedRiemann–Liouville fractional defined as follows:

$$I_\delta^\beta[g(z):\lambda_1\lambda_2] = \frac{1}{\Gamma(\beta)}\int_0^z g(\zeta)\,(z-\zeta)^\beta H_K^{(n)}\left(\varsigma_1^1,\varsigma_2^1,\dots,\varsigma_n^1;\varsigma_1^2,\varsigma_2^2,\dots,\varsigma_n^2;\frac{(\lambda_1+1)z^{\nu+1}}{\zeta^{\nu+1}},\frac{(\lambda_2+1)z^{\nu+1}}{\zeta^{\nu+1}},\dots,\frac{(\lambda_n+1)z^{\nu+1}}{\zeta^{\nu+1}}\right)d\zeta \quad (19)$$

where
$\Psi(\beta)>0;\Psi(\varsigma_1),\Psi(\varsigma_1^1),\Psi(\varsigma_2^1),\dots,\Psi(\varsigma_n^1);\Psi(\varsigma_1^2),\Psi(\varsigma_2^2),\dots,\Psi(\varsigma_n^2)>0,\Psi(\lambda_1),\Psi(\lambda_2),\dots,\Psi(\lambda_n)\geq 0,\nu\geq 0.$
and

$$I_\delta^\beta[g(z):\lambda_1\lambda_2] = \frac{1}{\Gamma(\beta)}\int_0^z g(\zeta)\,(z-\zeta)^\beta H_T^{(n)}\left(\varsigma_1,\varsigma_2,\dots,\varsigma_n;\varsigma_1^1,\varsigma_2^1,\dots,\varsigma_n^1;\varsigma_1^2;\frac{(\lambda_1+1)z^{\nu+1}}{\zeta^{\nu+1}},\frac{(\lambda_2+1)z^{\nu+1}}{\zeta^{\nu+1}},\dots,\frac{(\lambda_n+1)z^{\nu+1}}{\zeta^{\nu+1}}\right)d\zeta \ (20)$$

where
$\Psi(\beta)>0;\Psi(\varsigma_1),\Psi(\varsigma_2),\dots,\Psi(\varsigma_n);\Psi(\varsigma_1^1),\Psi(\varsigma_2^1),\dots,\Psi(\varsigma_n^1),\Psi(\varsigma_1^2)\geq 0,v\geq 0,\Psi(P_1),\Psi(P_2)>0.$
and

$$I_\delta^\beta[g(z):\lambda_1\lambda_2] = \frac{1}{\Gamma(\beta)}\int_0^z g(\zeta)\,(z-\zeta)^\beta H_Q^{(n)}\left(\varsigma_1,\varsigma_1^1;\varsigma_1^2,\dots\varsigma_n^2;\frac{(\lambda_1+1)z^{\nu+1}}{\zeta^{\nu+1}},\frac{(\lambda_2+1)z^{\nu+1}}{\zeta^{\nu+1}},\dots,\frac{(\lambda_n+1)z^{\nu+1}}{\zeta^{\nu+1}}\right)d\zeta \ (21)$$

where
$\Psi(\beta)>0;\Psi(\varsigma_1),;\Psi(\varsigma_1^1),\Psi(\varsigma_1^2),\dots,\Psi(\varsigma_n^2)\geq 0,\Psi(\lambda_1),\Psi(\lambda_2),\dots,\Psi(\lambda_n)\geq 0,,\Psi(P_1),\Psi(P_2)\geq 0,v\geq 0,\Psi(P_1),\Psi(P_2)>0.$
Finally:

$$I_\delta^\beta[g(z):\lambda_1\lambda_2] = \frac{1}{\Gamma(\beta)}\int_0^z g(\zeta)\,(z-\zeta)^\beta H_S^{(n)}\left(\varsigma_1,\varsigma_1^1;\varsigma_2^1,\dots,\varsigma_n^1,\varsigma_1^2;\frac{(\lambda_1+1)z^{\nu+1}}{\zeta^{\nu+1}},\frac{(\lambda_2+1)z^{\nu+1}}{\zeta^{\nu+1}},\dots,\frac{(\lambda_n+1)z^{\nu+1}}{\zeta^{\nu+1}}\right)d\zeta \quad (22)$$

where $\Psi(\beta)>0;\Psi(\varsigma_1),;\Psi(\varsigma_1^1),\Psi(\varsigma_2^1),\dots,\Psi(\varsigma_n^1);\Psi(\varsigma_1^2)\geq 0,\Psi(\lambda_1),\Psi(\lambda_2),\dots,\Psi(\lambda_n)\geq 0,,\Psi(P_1),\Psi(P_2)\geq 0,v\geq 0,\Psi(P_1),\Psi(P_2)>0.$

**Definition 5.2** The derivative of extended Riemann–Liouville fractional defined as follows:

$$D_\delta^\beta[g(z):\lambda_1\lambda_2] = \frac{1}{\Gamma(-\beta)}\int_0^z g(\zeta)\,(z-\zeta)^\beta H_K^{(n)}\left(\varsigma_1^1,\varsigma_2^1,\dots,\varsigma_n^1;\varsigma_1^2,\varsigma_2^2,\dots,\varsigma_n^2;\frac{(\lambda_1+1)z^{\nu+1}}{\zeta^{\nu+1}},\frac{(\lambda_2+1)z^{\nu+1}}{\zeta^{\nu+1}},\dots,\frac{(\lambda_n+1)z^{\nu+1}}{\zeta^{\nu+1}}\right)d\zeta \quad (23)$$

where
$\Psi(\beta)<0;\Psi(\varsigma_1),\Psi(\varsigma_1^1),\Psi(\varsigma_2^1),\dots,\Psi(\varsigma_n^1);\Psi(\varsigma_1^2),\Psi(\varsigma_2^2),\dots,\Psi(\varsigma_n^2)>0,\Psi(\lambda_1),\Psi(\lambda_2),\dots,\Psi(\lambda_n)\geq 0,\nu\geq 0.$
and

$$I_\delta^\beta[g(z):\lambda_1\lambda_2] = \frac{1}{\Gamma(-\beta)}\int_0^z g(\zeta)\,(z-\zeta)^\beta H_T^{(n)}\left(\varsigma_1,\varsigma_2,\dots,\varsigma_n;\varsigma_1^1,\varsigma_2^1,\dots,\varsigma_n^1;\varsigma_1^2;\frac{(\lambda_1+1)z^{\nu+1}}{\zeta^{\nu+1}},\frac{(\lambda_2+1)z^{\nu+1}}{\zeta^{\nu+1}},\dots,\frac{(\lambda_n+1)z^{\nu+1}}{\zeta^{\nu+1}}\right)d\zeta \quad (24)$$

where
$\Psi(\beta)<0;\Psi(\varsigma_1),\Psi(\varsigma_2),\dots,\Psi(\varsigma_n);\Psi(\varsigma_1^1),\Psi(\varsigma_2^1),\dots,\Psi(\varsigma_n^1),\Psi(\varsigma_1^2)\geq 0,v\geq 0,\Psi(P_1),\Psi(P_2)>0.$
and

$$I_\delta^\beta[g(z):\lambda_1\lambda_2] = \frac{1}{\Gamma(-\beta)}\int_0^z g(\zeta)\,(z-\zeta)^\beta H_Q^{(n)}\left(\varsigma_1,\varsigma_1^1;\varsigma_1^2,\dots\varsigma_n^2;\frac{(\lambda_1+1)z^{\nu+1}}{\zeta^{\nu+1}},\frac{(\lambda_2+1)z^{\nu+1}}{\zeta^{\nu+1}},\dots,\frac{(\lambda_n+1)z^{\nu+1}}{\zeta^{\nu+1}}\right)d\zeta \quad (21)$$

where
$\Psi(\beta)<0;\Psi(\varsigma_1),;\Psi(\varsigma_1^1),\Psi(\varsigma_1^2),\dots,\Psi(\varsigma_n^2)\geq 0,\Psi(\lambda_1),\Psi(\lambda_2),\dots,\Psi(\lambda_n)\geq 0,,\Psi(P_1),\Psi(P_2)\geq 0,v\geq 0,\Psi(P_1),\Psi(P_2)>0.$
Finally:

$$I_\delta^\beta[g(z):\lambda_1\lambda_2] = \frac{1}{\Gamma(-\beta)}\int_0^z g(\zeta)\,(z-\zeta)^\beta H_S^{(n)}\left(\varsigma_1,\varsigma_1^1;\varsigma_2^1,\dots,\varsigma_n^1,\varsigma_1^2;\frac{(\lambda_1+1)z^{\nu+1}}{\zeta^{\nu+1}},\frac{(\lambda_2+1)z^{\nu+1}}{\zeta^{\nu+1}},\dots,\frac{(\lambda_n+1)z^{\nu+1}}{\zeta^{\nu+1}}\right)d\zeta \quad (22)$$

where $\Psi(\beta) < 0; \Psi(\varsigma_1), ; \Psi(\varsigma_1^1), \Psi(\varsigma_2^1), \dots, \Psi(\varsigma_n^1); \Psi(\varsigma_1^2) \geq 0, \Psi(\lambda_1), \Psi(\lambda_2), \dots, \Psi(\lambda_n) \geq 0, , \Psi(P_1), \Psi(P_2) \geq 0, v \geq 0, \Psi(P_1), \Psi(P_2) > 0$.

**Theorem 5.3.** if $\Psi(\beta) > 0; \Psi(\varsigma_1), \Psi(\varsigma_1^1), \Psi(\varsigma_2^1), \dots, \Psi(\varsigma_n^1); \Psi(\varsigma_1^2), \Psi(\varsigma_2^2), \dots, \Psi(\varsigma_n^2) > 0, \Psi(\lambda_1), \Psi(\lambda_2), \dots, \Psi(\lambda_n) \geq 0, \nu \geq 0$.Then

$$I_\delta^\beta[g(z) = z^\Theta : \lambda_1\lambda_2] = \frac{1}{\Gamma(\beta)} \beta_{\lambda_1,\lambda_2}^{H_K^{(n)}}(\Theta + 1, \beta + 1) z^{\Theta+\beta+1} (23)$$

Proof : by definition 5.1

$$I_\delta^\beta[g(z) : \lambda_1\lambda_2] = \frac{1}{\Gamma(\beta)} \int_0^z \zeta^\Theta (z - \zeta)^\beta H_K^{(n)}\left(\varsigma_1^1, \varsigma_2^1, \dots, \varsigma_n^1; \varsigma_1^2, \varsigma_2^2, \dots, \varsigma_n^2; \frac{(\lambda_1 + 1)z^{\nu+1}}{\zeta^{\nu+1}}, \frac{(\lambda_2 + 1)z^{\nu+1}}{\zeta^{\nu+1}}, \dots, \frac{(\lambda_n + 1)z^{\nu+1}}{\zeta^{\nu+1}}\right) d\zeta \quad (24)$$

let $\zeta = z\tau$ then equation (24) becomes

$$I_\delta^\beta[g(z) = z^\Theta : \lambda_1\lambda_2] = = \frac{1}{\Gamma(\beta)} \zeta^{\Theta+\beta} \int_0^1 \tau^\Theta (z - z)^\beta H_K^{(n)}\left(\varsigma_1^1, \varsigma_2^1, \dots, \varsigma_n^1; \varsigma_1^2, \varsigma_2^2, \dots, \varsigma_n^2; \frac{(\lambda_1 + 1)z^{\nu+1}}{z\tau^{\nu+1}}, \frac{(\lambda_2 + 1)z^{\nu+1}}{z\tau^{\nu+1}}, \dots, \frac{(\lambda_n + 1)z^{\nu+1}}{z\tau^{\nu+1}}\right)$$

Therefore $I_\delta^\beta[g(z) = z^\Theta : \lambda_1\lambda_2] = \frac{1}{\Gamma(\beta)} \beta_{\lambda_1,\lambda_2}^{H_K^{(n)}}(\Theta + 1, \beta + 1) z^{\Theta+\beta+1}$ .

**Theorem 5. 4.**If $\Psi(\beta) > 0; \Psi(\varsigma_1), \Psi(\varsigma_2), \dots, \Psi(\varsigma_n); \Psi(\varsigma_1^1), \Psi(\varsigma_2^1), \dots, \Psi(\varsigma_n^1), \Psi(\varsigma_1^2) \geq 0, v \geq 0, \Psi(P_1), \Psi(P_2) > 0$.Then

$$I_\delta^\beta[g(z) = z^\Theta : \lambda_1\lambda_2] = \frac{1}{\Gamma(\beta)} \beta_{\lambda_1,\lambda_2}^{H_T^{(n)}}(\Theta + 1, \beta + 1) z^{\Theta+\beta+1} (25)$$

Proof : as same as proof ofTheorem 5. 3.

**Theorem 5. 5.** If $\Psi(\beta) > 0; \Psi(\varsigma_1), ; \Psi(\varsigma_1^1), \Psi(\varsigma_1^2), \dots, \Psi(\varsigma_n^2) \geq 0, \Psi(\lambda_1), \Psi(\lambda_2),$

$\dots, \Psi(\lambda_n) \geq 0, , \Psi(P_1), \Psi(P_2) \geq 0, v \geq 0, \Psi(P_1), \Psi(P_2) > 0$.Then

$$I_\delta^\beta[g(z) = z^\Theta : \lambda_1\lambda_2] = \frac{1}{\Gamma(\beta)} \beta_{\lambda_1,\lambda_2}^{H_Q^{(n)}}(\Theta + 1, \beta + 1) z^{\Theta+\beta+1} (26)$$

Proof : as same as proof ofTheorem 5. 3.

**Theorem 5. 6.** If$\Psi(\beta) > 0; \Psi(\varsigma_1), ; \Psi(\varsigma_1^1), \Psi(\varsigma_2^1), \dots, \Psi(\varsigma_n^1); \Psi(\varsigma_1^2) \geq 0, \Psi(\lambda_1), \Psi(\lambda_2)$ $, \dots, \Psi(\lambda_n) \geq 0, , \Psi(P_1), \Psi(P_2) \geq 0, v \geq 0, \Psi(P_1), \Psi(P_2) > 0$. Then

$I_{\delta}^{\beta}[g(z)=z^{\Theta}:\lambda_1\lambda_2]=\frac{1}{\Gamma(\beta)}\beta_{\lambda_1,\lambda_2}^{H_S^{(n)}}(\Theta+1,\beta+1)z^{\Theta+\beta+1}(27)$

Proof : as same as proof ofTheorem 5. 3.

**Theorem 5. 7.** if $\Psi(\beta)>0;\Psi(\varsigma_1),\Psi(\varsigma_1^1),\Psi(\varsigma_2^1),\dots,\Psi(\varsigma_n^1);\Psi(\varsigma_1^2),\Psi(\varsigma_2^2),\dots,\Psi(\varsigma_n^2)>0,\Psi(\lambda_1),\Psi(\lambda_2),\dots,\Psi(\lambda_n)\geq 0,\nu\geq 0$and $g(\zeta)=\sum_{k=0}^{\infty}\chi_{k+1}\zeta^{k+1}$,$|\chi_{k+1}|<1$.Then

1) $I_{\delta}^{\beta}[g(z):\lambda_1\lambda_2]=\frac{1}{\Gamma(\beta)}\sum_{k=0}^{\infty}\chi_{k+1}\beta_{\lambda_1,\lambda_2}^{H_K^{(n)}}(k+2,\beta+1)z^{k+\beta+1}$ (28)

2) $I_{\delta}^{\beta}[g(z)\zeta^{\omega}:\lambda_1\lambda_2]=\frac{\zeta^{\omega+\beta+2}}{\Gamma(\beta)}\sum_{k=0}^{\infty}\chi_{k+1}\beta_{\lambda_1,\lambda_2}^{H_K^{(n)}}(k+2,\beta+1)z^{k+1}$ (29)

Proof : 1) by definition 5.1 ,we get

$$I_{\delta}^{\beta}[g(z):\lambda_1\lambda_2]=\frac{1}{\Gamma(\beta)}\int_0^z\sum_{k=0}^{\infty}\chi_{k+1}\zeta^{k+1}(z-\zeta)^{\beta}H_K^{(n)}$$
$$\left(\varsigma_1^1,\varsigma_2^1,\dots,\varsigma_n^1;\varsigma_1^2,\varsigma_2^2,\dots,\varsigma_n^2;\frac{(\lambda_1+1)z^{\nu+1}}{\zeta^{\nu+1}},\frac{(\lambda_2+1)z^{\nu+1}}{\zeta^{\nu+1}},\dots,\frac{(\lambda_n+1)z^{\nu+1}}{\zeta^{\nu+1}}\right)d\zeta$$

So we get

$$I_{\delta}^{\beta}[g(z):\lambda_1\lambda_2]=\frac{1}{\Gamma(\beta)}\sum_{k=0}^{\infty}\chi_{k+1}\int_{k=0}^{\infty}\zeta^{k+1}(z-\zeta)^{\beta}H_K^{(n)}$$
$$\left(\varsigma_1^1,\varsigma_2^1,\dots,\varsigma_n^1;\varsigma_1^2,\varsigma_2^2,\dots,\varsigma_n^2;\frac{(\lambda_1+1)z^{\nu+1}}{\zeta^{\nu+1}},\frac{(\lambda_2+1)z^{\nu+1}}{\zeta^{\nu+1}},\dots,\frac{(\lambda_n+1)z^{\nu+1}}{\zeta^{\nu+1}}\right)d\zeta \quad (30)$$

Hence by using eq (21) we get the result.

2) by definition 5.1,we get

$$I_{\delta}^{\beta}[g(z)\zeta^{\omega}:\lambda_1\lambda_2]=\frac{1}{\Gamma(\beta)}\sum_{k=0}^{\infty}\chi_{k+1}\int_0^z\zeta^{k+1}(z-\zeta)^{\beta}H_K^{(n)}$$
$$\left(\varsigma_1^1,\varsigma_2^1,\dots,\varsigma_n^1;\varsigma_1^2,\varsigma_2^2,\dots,\varsigma_n^2;\frac{(\lambda_1+1)z^{\nu+1}}{\zeta^{\nu+1}},\frac{(\lambda_2+1)z^{\nu+1}}{\zeta^{\nu+1}},\dots,\frac{(\lambda_n+1)z^{\nu+1}}{\zeta^{\nu+1}}\right)d\zeta$$

So we get

$$I_{\delta}^{\beta}[g(z):\lambda_1\lambda_2]=\frac{1}{\Gamma(\beta)}\int_0^z\sum_{k=0}^{\infty}\zeta^{k+\omega+1}\chi_{k+1}(z-\zeta)^{\beta}H_K^{(n)}$$
$$\left(\varsigma_1^1,\varsigma_2^1,\dots,\varsigma_n^1;\varsigma_1^2,\varsigma_2^2,\dots,\varsigma_n^2;\frac{(\lambda_1+1)z^{\nu+1}}{\zeta^{\nu+1}},\frac{(\lambda_2+1)z^{\nu+1}}{\zeta^{\nu+1}},\dots,\frac{(\lambda_n+1)z^{\nu+1}}{\zeta^{\nu+1}}\right)d\zeta \quad (31)$$

Hence by using eq (21) we get the result.

**Theorem 5. 8.**If $\Psi(\beta)>0;\Psi(\varsigma_1),\Psi(\varsigma_2),\dots,\Psi(\varsigma_n);\Psi(\varsigma_1^1),\Psi(\varsigma_2^1),\dots,\Psi(\varsigma_n^1),\Psi(\varsigma_1^2)\geq 0,\nu\geq 0,\Psi(P_1),\Psi(P_2)>0$and $g(\zeta)=\sum_{k=0}^{\infty}\chi_{k+1}\zeta^{k+1}$,$|\chi_{k+1}|<1$.Then

1) $I_{\delta}^{\beta}[g(z):\lambda_1\lambda_2]=\frac{1}{\Gamma(\beta)}\sum_{k=0}^{\infty}\chi_{k+1}\beta_{\lambda_1,\lambda_2}^{H_T^{(n)}}(k+2,\beta+1)z^{k+\beta+1}$ (32)

2) $I_{\delta}^{\beta}[g(z)\zeta^{\omega}:\lambda_1\lambda_2]=\frac{\zeta^{\omega+\beta+2}}{\Gamma(\beta)}\sum_{k=0}^{\infty}\chi_{k+1}\beta_{\lambda_1,\lambda_2}^{H_T^{(n)}}(k+2,\beta+1)z^{k+1}$ (33)

Proof : as same as proof ofTheorem 5. 7.

**Theorem 5. 9.** If $\Psi(\beta) > 0; \Psi(\varsigma_1), ; \Psi(\varsigma_1^1), \Psi(\varsigma_1^2), \ldots, \Psi(\varsigma_n^2) \geq 0, \Psi(\lambda_1), \Psi(\lambda_2),$ $\ldots, \Psi(\lambda_n) \geq 0, , \Psi(P_1), \Psi(P_2) \geq 0, v \geq 0, \Psi(P_1), \Psi(P_2) > 0$and $g(\zeta) = \sum_{k=0}^{\infty} \chi_{k+1} \zeta^{k+1}, |\chi_{k+1}| < 1$.Then

1) $I_\delta^\beta[g(z): \lambda_1 \lambda_2] = \frac{1}{\Gamma(\beta)} \sum_{k=0}^{\infty} \chi_{k+1} \beta_{\lambda_1,\lambda_2}^{H_Q^{(n)}}(k+2, \beta+1) z^{k+\beta+1}$ (34)

2) $I_\delta^\beta[g(z)\zeta^\omega: \lambda_1 \lambda_2] = \frac{\zeta^{\omega+\beta+2}}{\Gamma(\beta)} \sum_{k=0}^{\infty} \chi_{k+1} \beta_{\lambda_1,\lambda_2}^{H_Q^{(n)}}(k+2, \beta+1) z^{k+1}$ (35)

Proof : as same as proof ofTheorem 5. 7.

**Theorem 5.10.** If $\Psi(\beta) > 0; \Psi(\varsigma_1), ; \Psi(\varsigma_1^1), \Psi(\varsigma_2^1), \ldots, \Psi(\varsigma_n^1); \Psi(\varsigma_1^2) \geq 0, \Psi(\lambda_1), \Psi(\lambda_2)$ $, \ldots, \Psi(\lambda_n) \geq 0, , \Psi(P_1), \Psi(P_2) \geq 0, v \geq 0, \Psi(P_1), \Psi(P_2) > 0.$ and $g(\zeta) = \sum_{k=0}^{\infty} \chi_{k+1} \zeta^{k+1}, |\chi_{k+1}| < 1$.Then

1) $I_\delta^\beta[g(z): \lambda_1 \lambda_2] = \frac{1}{\Gamma(\beta)} \sum_{k=0}^{\infty} \chi_{k+1} \beta_{\lambda_1,\lambda_2}^{H_S^{(n)}}(k+2, \beta+1) z^{k+\beta+1}$ (36)

2) $I_\delta^\beta[g(z)\zeta^\omega: \lambda_1 \lambda_2] = \frac{\zeta^{\omega+\beta+2}}{\Gamma(\beta)} \sum_{k=0}^{\infty} \chi_{k+1} \beta_{\lambda_1,\lambda_2}^{H_S^{(n)}}(k+2, \beta+1) z^{k+1}$ (37)

Proof : as same as proof ofTheorem 5. 7.

**Theorem 5. 11.** if $\Psi(\beta) < 0; \Psi(\varsigma_1), \Psi(\varsigma_1^1), \Psi(\varsigma_2^1), \ldots, \Psi(\varsigma_n^1); \Psi(\varsigma_1^2), \Psi(\varsigma_2^2), \ldots, \Psi(\varsigma_n^2) > 0, \Psi(\lambda_1), \Psi(\lambda_2), \ldots, \Psi(\lambda_n) \geq 0, \nu \geq 0$. Then

$$I_\delta^\beta[g(z) = z^\Theta: \lambda_1 \lambda_2] = \frac{1}{\Gamma(-\beta)} \beta_{\lambda_1,\lambda_2}^{H_K^{(n)}}(\Theta+1, -\beta+1) z^{\Theta-\beta+1} (38)$$

Proof : Proof Using Definition 5.2 and following the same way as in Theorem 5.3, we get the result.

**Theorem 5. 12.** If $\Psi(\beta) > 0; \Psi(\varsigma_1), \Psi(\varsigma_2), \ldots, \Psi(\varsigma_n); \Psi(\varsigma_1^1), \Psi(\varsigma_2^1), \ldots, \Psi(\varsigma_n^1), \Psi(\varsigma_1^2) \geq 0, v \geq 0, \Psi(P_1), \Psi(P_2) > 0$.Then

$$I_\delta^\beta[g(z) = z^\Theta: \lambda_1 \lambda_2] = \frac{1}{\Gamma(-\beta)} \beta_{\lambda_1,\lambda_2}^{H_T^{(n)}}(\Theta+1, -\beta+1) z^{\Theta-\beta+1} (39)$$

Proof : as same as proof ofTheorem 5. 12.

**Theorem 5. 13.** If $\Psi(\beta) < 0; \Psi(\varsigma_1), ; \Psi(\varsigma_1^1), \Psi(\varsigma_1^2), \ldots, \Psi(\varsigma_n^2) \geq 0, \Psi(\lambda_1), \Psi(\lambda_2),$

$\ldots, \Psi(\lambda_n) \geq 0, , \Psi(P_1), \Psi(P_2) \geq 0, v \geq 0, \Psi(P_1), \Psi(P_2) > 0$.Then

$$I_\delta^\beta[g(z) = z^\Theta: \lambda_1 \lambda_2] = \frac{1}{\Gamma(-\beta)} \beta_{\lambda_1,\lambda_2}^{H_Q^{(n)}}(\Theta+1, -\beta+1) z^{\Theta-\beta+1} (40)$$

Proof : it's clear.

**Theorem 5. 14.** If $\Psi(\beta) < 0; \Psi(\varsigma_1), ; \Psi(\varsigma_1^1), \Psi(\varsigma_2^1), \ldots, \Psi(\varsigma_n^1); \Psi(\varsigma_1^2) \geq 0, \Psi(\lambda_1), \Psi(\lambda_2)$ $, \ldots, \Psi(\lambda_n) \geq 0, , \Psi(P_1), \Psi(P_2) \geq 0, v \geq 0, \Psi(P_1), \Psi(P_2) > 0.$ Then

$$I_{\delta}^{\beta}[g(z) = z^{\Theta}: \lambda_1\lambda_2] = \frac{1}{\Gamma(-\beta)} \beta_{\lambda_1,\lambda_2}^{H_S^{(n)}}(\Theta + 1, -\beta + 1) z^{\Theta-\beta+1} (41)$$

Proof : Obvious.

**Theorem 5. 15.** if $\Psi(\beta) < 0; \Psi(\varsigma_1), \Psi(\varsigma_1^1), \Psi(\varsigma_2^1), \dots, \Psi(\varsigma_n^1); \Psi(\varsigma_1^2), \Psi(\varsigma_2^2), \dots, \Psi(\varsigma_n^2) > 0, \Psi(\lambda_1), \Psi(\lambda_2), \dots, \Psi(\lambda_n) \geq 0, \nu \geq 0$and $g(\zeta) = \sum_{k=0}^{\infty} \chi_{k+1} \zeta^{k+1}, |\chi_{k+1}| < 1$.Then

1) $I_{\delta}^{\beta}[g(z): \lambda_1\lambda_2] = \frac{1}{\Gamma(-\beta)} \sum_{k=0}^{\infty} \chi_{k+1} \beta_{\lambda_1,\lambda_2}^{H_K^{(n)}}(k + 2, -\beta + 1) z^{k-\beta+1}$ (42)

2) $I_{\delta}^{\beta}[g(z)\zeta^{\omega}: \lambda_1\lambda_2] = \frac{\zeta^{\omega-\beta+2}}{\Gamma(-\beta)} \sum_{k=0}^{\infty} \chi_{k+1} \beta_{\lambda_1,\lambda_2}^{H_K^{(n)}}(k + 2, -\beta + 1) z^{k+1}$ (43)

Proof : Proof Using Definition 5.2 and following the same way as in Theorem 5.7, we get the result.

**Theorem 5. 16.** If $\Psi(\beta) < 0; \Psi(\varsigma_1), \Psi(\varsigma_2), \dots, \Psi(\varsigma_n); \Psi(\varsigma_1^1), \Psi(\varsigma_2^1), \dots, \Psi(\varsigma_n^1), \Psi(\varsigma_1^2) \geq 0, \nu \geq 0, \Psi(P_1), \Psi(P_2) > 0$and $g(\zeta) = \sum_{k=0}^{\infty} \chi_{k+1} \zeta^{k+1}, |\chi_{k+1}| < 1$.Then

1) $I_{\delta}^{\beta}[g(z): \lambda_1\lambda_2] = \frac{1}{\Gamma(-\beta)} \sum_{k=0}^{\infty} \chi_{k+1} \beta_{\lambda_1,\lambda_2}^{H_T^{(n)}}(k + 2, -\beta + 1) z^{k-\beta+1}$ (44)

2) $I_{\delta}^{\beta}[g(z)\zeta^{\omega}: \lambda_1\lambda_2] = \frac{\zeta^{\omega-\beta+2}}{\Gamma(-\beta)} \sum_{k=0}^{\infty} \chi_{k+1} \beta_{\lambda_1,\lambda_2}^{H_T^{(n)}}(k + 2, -\beta + 1) z^{k+1}$ (45)

as same as proof ofTheorem 5. 15.

**Theorem 5. 17.** If $\Psi(\beta) < 0; \Psi(\varsigma_1), ; \Psi(\varsigma_1^1), \Psi(\varsigma_1^2), \dots, \Psi(\varsigma_n^2) \geq 0, \Psi(\lambda_1), \Psi(\lambda_2), \dots, \Psi(\lambda_n) \geq 0, , \Psi(P_1), \Psi(P_2) \geq 0, \nu \geq 0, \Psi(P_1), \Psi(P_2) > 0$and $g(\zeta) = \sum_{k=0}^{\infty} \chi_{k+1} \zeta^{k+1}, |\chi_{k+1}| < 1$.Then

1) $I_{\delta}^{\beta}[g(z): \lambda_1\lambda_2] = \frac{1}{\Gamma(-\beta)} \sum_{k=0}^{\infty} \chi_{k+1} \beta_{\lambda_1,\lambda_2}^{H_Q^{(n)}}(k + 2, -\beta + 1) z^{k-\beta+1}$ (46)

2) $I_{\delta}^{\beta}[g(z)\zeta^{\omega}: \lambda_1\lambda_2] = \frac{\zeta^{\omega-\beta+2}}{\Gamma(-\beta)} \sum_{k=0}^{\infty} \chi_{k+1} \beta_{\lambda_1,\lambda_2}^{H_Q^{(n)}}(k + 2, -\beta + 1) z^{k+1}$ (47)

Proof : as same as proof ofTheorem 5. 15.

**Theorem 5.18.** If $\Psi(\beta) < 0; \Psi(\varsigma_1), ; \Psi(\varsigma_1^1), \Psi(\varsigma_2^1), \dots, \Psi(\varsigma_n^1); \Psi(\varsigma_1^2) \geq 0, \Psi(\lambda_1), \Psi(\lambda_2), \dots, \Psi(\lambda_n) \geq 0, , \Psi(P_1), \Psi(P_2) \geq 0, \nu \geq 0, \Psi(P_1), \Psi(P_2) > 0$. and $g(\zeta) = \sum_{k=0}^{\infty} \chi_{k+1} \zeta^{k+1}, |\chi_{k+1}| < 1$.Then

1) $I_{\delta}^{\beta}[g(z): \lambda_1\lambda_2] = \frac{1}{\Gamma(-\beta)} \sum_{k=0}^{\infty} \chi_{k+1} \beta_{\lambda_1,\lambda_2}^{H_S^{(n)}}(k + 2, -\beta + 1) z^{k-\beta+1}$ (48)

2) $I_{\delta}^{\beta}[g(z)\zeta^{\omega}: \lambda_1\lambda_2] = \frac{\zeta^{\omega-\beta+2}}{\Gamma(-\beta)} \sum_{k=0}^{\infty} \chi_{k+1} \beta_{\lambda_1,\lambda_2}^{H_S^{(n)}}(k + 2, -\beta + 1) z^{k+1}$ (49)

Proof: it's clear.

**Conclusion:**

In this article, we have discussed some extensions of the beta functionas the new extension to the beta function was Using the Lauricella series, and the properties of the beta function are studied. Moreover, the statistics the distribution was determined using the newly defined and median beta function, variance, moment generating function and were discussed here. Moreover, the extension for Riemann - Liouville fractional coefficients using the Lauricella series it was defined and its various characteristics are discussed using the new extension from the beta function.

Finally, we derived and investigate the Riemann–Liouville fractional operators using the Lauricella function series $\mathrm{H}_{\mathrm{K}}^{(\mathrm{n})}(.)$ with some of the features we got.

**References**

1. Mubeen, S., Rahman, G., Nisar, K.S., Choi, J., Arshad, M.: An extended beta function and its properties. Far East J. Math. Sci. 102(2017), 1545–1557.
2. Goswami, A., Jain, S., Agarwal, P., Araci, S.: A note on the new extended beta and Gauss hypergeometric functions. Appl. Math. Inf. Sci. 12(1) (2018), 139–144.
3. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, vol. 204(2006).
4. Choi, J., Rathie, A.K., Parmar, R.K.: Extension of extended beta, hypergeometric and confluent hypergeometric functions. Honam Math. J. 36(2) (2014), 357–385.
5. Shadab, M., Jabee, S., Choi, J.: An extension of beta function and its application. Far East J. Math. Sci. 103(1) (2018) , 235–251.
6. Agarwal R, Jain S, Agarwal RP. Solution of fractional Volterra integral equation and non-homogeneous time fractional heat equation using integral transform of pathway type. Progr Fract Differ Appl. (2015) 1:145–55.doi: 10.12785/pfda/010301
7. Agarwal, R., Jain, S., Agarwal, R.P., Baleanu, D.: A remark on the fractional integral operators and the image formulas of generalized Lommel–Wright function. Front. Phys. 6, 79 (2018).
8. Srivasatava, H., Agarwal, R., Jain, S.: Integral transform and fractional derivative formulas involving the extended generalized hypergeometric functions and probability distributions. Math. Methods Appl. Sci. 40(1) (2016), 255–273.
9. Kumar D. Solution of fractional kinetic equation by a class of integral transform of pathway type. J Math Phys. (2013) 54:043509. doi: 10.1063/1.4800768
10- Agarwal .R, Jain.S and Agarwal .R. P, Solution of fractional Volterra integral equation and non-homogeneous time fractional heat equation using integral transform of pathway type, Progr. Fract. Different. Appl. 1 (2015), 145–155.
11. Choi .J and Agarwal .P, Certain integral transforms for the incomplete functions, Appl. Math. Inform. Sci. 9 (2015), 2161–2167.
12. Debnath.L and Bhatta.D, Integral Transforms and Their Applications, Third edition, Chapman and Hall (CRC Press), Taylor and Francis Group, London and New York, 2014.
13. Kang .H.-C. and P .C,An, Differentiation formulas of some hypergeometric functions with respect to all parameters, Appl. Math. Comput. 258 (2015), 454–464; see also Erratum, Appl. Math. Comput. 273 (2016), 1122–1122.
14. Maširević.D. J., Parmar. R. K. and Pogány.T. K., ( *p*, *q*) -extended Bessel andmodified Bessel functions of the first kind, Results Math. 72 (2017), 617-632.DOI: 10.1007/ s00025-016-0649-1.
15. Özergin E., Özarslan .M. A. and Altin .A., Extension of gamma, beta and hypergeometric functions, J. Comput. Appl. Math. 235 (2011), 4601-4610.

16. Haubold. H.J. and Kumar.D., Extension of thermonuclear functions through the pathway model including Maxwell Boltzman and Tsallis distributions, Astroparticle Physics, 29 (2008), 70-76.
17. Manoj. S., Mohd. F.A. and Renu. J, Advanced generalized fractional kinetic equation in astrophysics, Progress in Fractional Differentiation and Applications .1(1)(2015), 65-75.
18.Thomas.R. and Fehmi.C., An immersed finite element method with integral equation correction, Int. J. Numer. Meth. Engng.86(2010), 93-114.
19. S¸ahin.R. ,An extension of some Lauricella hypergeometric functions, AIP Conf. Proc. 1558 (2013) 1140–1143.
20. Srivastava. H.M., . Parmar. R.K, and Chopra .P., A class of extended fractional derivative operators and associated generating relations involving hypergeometric functions, Axioms I (2012) 238–258.