Comparison between Bayesian and Classical Estimators of Birnbaum-Saunders Distribution Parameters with Application

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Abstract
Estimating the parameters of any distribution or model is a main process that helps to make good understanding of the variables behavior under study. **Birnbaum and Saunders** introduced a two-parameter lifetime distribution to model fatigue life. Since then, extensive work submitted on this model providing different interpretations, constructions, generalizations, inferential method. In our study, we estimate the parameters of it by using classical methods moreover, we use Bayesian estimation, comparison wasmade by using Mean Square Errors **MSE** and Mean od Absolute Percentage Errors **MAPE.**

Keywords: Birnbaum-Saunders, Bayesian Estimation, Fatigue Time , Jeffrey's prior

1- Introduction

From all statistical distributions, we can see that the normal distribution is the most one that appears in practice. Some distributions have been developed from the normal distribution.one of them is the Two-parameter Birnbaum-Saunders (BS) is such distribution established by making transformation from random variable has a standard normal distribution. The Birnbaum-Saunders (BS) distributionrepresent a natural physical model of a fatigue failure produced under loading[1].

The Birnbaum-Saunders (BS) distribution probability density function could be written as follows:

$$\boldsymbol{f(t;\alpha,\beta)} = \frac{\mathbf{1}}{\mathbf{2}\sqrt{\mathbf{2}\pi\alpha\beta}}\left[\left(\frac{\beta}{t}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right]\mathbf{Exp}\left[-\frac{\mathbf{1}}{\alpha^2}\left(\frac{t}{\beta}+\frac{\beta}{t}-\mathbf{2}\right)\right]\ldots\mathbf{(1)}$$

Whereα, β are the shape and scale parameters and $t, \alpha, \beta > 0$the formula in (1) was given by Birnbaum and Saunders [2] that fetched the worth of thisdistribution.

The Birnbaum and Saunders distribution has appeared in several scientific works with many features. like Fletcher [6] who derived the properties of the cumulative distribution function and Konstantinowsky [7] who use this distribution as a life time distribution with industrial data and Desmond[4] who offered a general formula of this distribution and use it in biological studies and also,he supported the physical model of this distribution and its application in industry and Biology.

Our study consists of 7 sections, the first section is the introduction and the second section is Maximum likelihood estimation, the 3ed section is the Bayesian estimation and the 4th section is the data under study while 5th section is the comparison methodsand the 6th section is the results and discussion and the 7th section is the conclusions recommendations.

2- Maximum Likelihood Estimation

The maximum likelihood (ML) estimationfor the both shape and scale parameters were present byby Birnbaum and Saunders [1]. and the asymptotic properties of the distributions were found by Engelhardt [5]. Now for the PDF in (1) we obtain the log-likelihood function to be as follows.

$$\boldsymbol{L}\ \mathbf{ln}\,\boldsymbol{f(t{:}\,\alpha,\beta)} = -\boldsymbol{n}\,\mathbf{ln}(\boldsymbol{\alpha}) - \boldsymbol{n}\,\mathbf{ln}(\boldsymbol{\beta}) + \sum_{i=1}^{n}\mathbf{ln}\left[\left(\frac{\beta}{ti}\right)^{\frac{1}{2}} + \left(\frac{\beta}{ti}\right)^{\frac{3}{2}}\right] - \frac{\mathbf{1}}{\alpha^2}\sum_{i=1}^{n}\left(\frac{ti}{\beta}+\frac{\beta}{ti}-\mathbf{2}\right)\ ..\ \mathbf{(2)}$$

For simplicity in obtaining the parameters from (2) we consider the following sample arithmetic mean and the sample harmonic mean [2].

$$\bar{\boldsymbol{t}} = \frac{\mathbf{1}}{\boldsymbol{n}}\sum_{i=1}^{n}\boldsymbol{t_i} \qquad \ldots\ \mathbf{(3)}$$

$$\boldsymbol{ht} = \left[\frac{\mathbf{1}}{\boldsymbol{n}}\sum_{i=1}^{n}\boldsymbol{t_i^{-1}}\right]^{-\mathbf{1}} \qquad \ldots\ \mathbf{(4)}$$

And by differentiating (2) with respect to α and equalizing by zero we can obtain the following [13].

$$\boldsymbol{\alpha^2 = \frac{t}{\beta} + \frac{\beta}{ht} - 2} \quad \text{… (5)}$$

Again, by differentiating (2) with respect to β and equalizing by zero we can obtain the following nonlinear function.

$$\boldsymbol{\beta^2 - \beta(2ht + R) + ht(\bar{t} + R)} \quad \text{… (6)}$$

Where R can be written as the following formula

$$\boldsymbol{R = \left[\frac{1}{n}\sum_{i=1}^{n}(\beta + t_i)^{-1}\right]^{-1}} \quad \text{… (7)}$$

Where $\boldsymbol{u} > \boldsymbol{0}$and here, we can see that the value of $\hat{\beta}$is the positive root of the nonlinear equation in (6).
And finally, we can put αas follows [2].

$$\boldsymbol{\hat{\alpha} = \left(\frac{t}{\hat{\beta}} + \frac{\hat{\beta}}{ht} - 2\right)^{1/2}} \quad \text{… (8)}$$

3- Bayes Estimation

The Bayesian estimation is significant and has fascinated a lot of attention lately. The Bayesian approach begins with determining a prior distribution function for the parameters under study. we have no knowledge about the parameters so we used the non-Informative prior function. The most common prior distribution function is Jeffrey's prior. Jeffrey [8] proposes using the square root of the determinate of the Fisher information matrix as a prior distribution function for the parameters such that [9].

$$\boldsymbol{u(\alpha,\beta) = \sqrt{\det(I_{(\alpha,\beta)})}} \quad \text{… (9)}$$

where:

$$\boldsymbol{I_{(\alpha,\beta)} = \begin{bmatrix} E\left(\frac{\partial^2 \log(f_{(x)})}{\partial^2\alpha^2}\right) & E\left(\frac{\partial^2 \log(f_{(x)})}{\partial\alpha\partial\beta}\right) \\ E\left(\frac{\partial^2 \log(f_{(x)})}{\partial\alpha\partial\beta}\right) & E\left(\frac{\partial^2 \log(f_{(x)})}{\partial^2\beta^2}\right) \end{bmatrix}} \text{… (10)}$$

Here, we may write posterior distribution function according to Bayes theory, in which the joint density function of α, β is [12]:

$$\boldsymbol{f(\alpha,\beta \backslash t_1, t_2, \ldots . t_n) = \frac{f(t_1,t_2,\ldots t_n \backslash \alpha,\beta)u(\alpha,\beta)}{\int_0^\infty \int_0^\infty f(t,t,\ldots t_n \backslash \alpha,\beta)u(\alpha,\beta)d\beta d\alpha} \ldots (11)}$$

By using the likelihood function:

$$\boldsymbol{f(\alpha,\beta \backslash t_1, t_2, \ldots . t_n) = \frac{L(t_1,t_2,\ldots t_n \backslash \alpha,\beta)u(\alpha,\beta)}{\int_0^\infty \int_0^\infty L(t_1,t_2,\ldots t_n \backslash \alpha,\beta)u(\alpha,\beta)d\beta d\alpha} \ldots (12)}$$

Achcar[1] use Laplace transformation to obtain $\boldsymbol{u(\alpha,\beta)}$for the non-informative

Jeffrey's prior function will as the following.

$$\boldsymbol{u(\alpha,\beta) = \frac{1}{\alpha\beta}\left(\frac{1}{\alpha^2}+\frac{1}{4}\right)^{\frac{1}{2}}} \qquad \ldots \mathbf{(13)}$$

While we can write the likelihood function of the Birnbaum-Saunders (BS) distribution as follows.

$$\boldsymbol{Lf(t:\alpha,\beta) = \left(\frac{1}{2\sqrt{2\pi\alpha\beta}}\right)^n \prod_{i=1}^n \left[\left(\frac{\beta}{ti}\right)^{\frac{1}{2}} + \left(\frac{\beta}{ti}\right)^{\frac{3}{2}}\right] \exp -\frac{1}{\alpha^2}\sum_{i=1}^n \left(\frac{ti}{\beta}+\frac{\beta}{ti}-2\right) \ldots (14)}$$

Then we can write posterior distribution function [11].

$$\boldsymbol{f(\alpha,\beta \backslash t_1, t_2, \ldots . t_n) = \frac{\left(\frac{1}{2\sqrt{2\pi\alpha\beta}}\right)^n \prod_{i=1}^n \left[\left(\frac{\beta}{ti}\right)^{\frac{1}{2}} + \left(\frac{\beta}{ti}\right)^{\frac{3}{2}}\right] \exp -\frac{1}{\alpha^2}\sum_{i=1}^n \left(\frac{ti}{\beta}+\frac{\beta}{ti}-2\right)\frac{1}{\alpha\beta}\left(\frac{1}{\alpha^2}+\frac{1}{4}\right)^{\frac{1}{2}}}{\iint_0^\infty \left(\frac{1}{2\sqrt{2\pi\alpha\beta}}\right)^n \prod_{i=1}^n \left[\left(\frac{\beta}{ti}\right)^{\frac{1}{2}} + \left(\frac{\beta}{ti}\right)^{\frac{3}{2}}\right] \exp -\frac{1}{\alpha^2}\sum_{i=1}^n \left(\frac{ti}{\beta}+\frac{\beta}{ti}-2\right)\frac{1}{\alpha\beta}\left(\frac{1}{\alpha^2}+\frac{1}{4}\right)^{\frac{1}{2}} d\alpha d\beta} \ldots (15)}$$

So, the Bayes estimators (BE) for the parameters will be:

$$BE_\alpha = E(\alpha \backslash t_1, t_2, \dots . t_n) = \int_0^\infty \alpha f(\alpha, \beta \backslash t_1, t_2, \dots . x_n) d\alpha \qquad \dots (16)$$

$$BE_\beta = E(\beta \backslash t_1, t_2, \dots . t_n) = \int_0^\infty \beta f(\alpha, \beta \backslash x_1, x_2, \dots . x_n) d\beta \qquad \dots (17)$$

4- Comparison Methods

There are many statistical methods that we can use to make comparison and here we selected the MSE Mean Squared Errors and the Mean AbsolutePercentage Error MAPE, which is given by [11].

$$MSE = \frac{1}{n} \sum_{i=1}^{n} [t_i - \hat{t}_i]^2 \dots (18)$$

$$MAPE = \frac{1}{n} \sum_{i=1}^{n} \left| \frac{t_i - \hat{t}_i}{t} \right| \qquad \dots (19)$$

5- Data and Results

Here we choose to use two different data from pervious experiments we calculate the Maximum Likelihood Estimators in equations (6),(8) and we use numerical Laplace's approximation method to solve the integrals in (16),(17) then we calculate MSE and MAPE as in Equations (18),(19), and we use MATLAB program to obtain results we choose two data sets from the application of Birnbaum-Saunders (BS) distribution

5-1 Fatigue Data

This data wasgiven by Birnbaum and Saunders [3] which represent the life time of cut aluminum cylinder under certain pressure as follows.

Data

70 90 96 97 99 100 103 104 104 105 107 108 108 108 109 109 112 112 113 114 114 114116 119 120 120 120 121 121 123 124 124 124 124 124 128 128 129 129 130 130 130 131 131131 131 131 132 132 132 133 134 134 134 134 134 136 136 137 138 138 138 139 139 141 141142 142 142 142 142 142 144 144 145 146 148 148 149 151 151 152 155 156 157 157 157 157158 159 162 163 163 164 166 166 168 170 174 196 212

Table (1)
ML and BE estimators for Fatigue Data
With ME and MAPE

Methods	$\hat{\alpha}$	$\hat{\beta}$	**MSE**	**MAPE**
ML	0.5432	140.13	2.345	0.654
BE	0.118	126.18	0.723	0.411

5-2 Ball Bearings

This data representsthe fatigues life in hours of ten ball bearings of a certain type which was originally given by McCool [10]

152.7 172.0172.5 173.3 193.0 204.7 216.5 234.9 262.6 422.6

Table (2)
ML and BE estimators for Ball Bearings
With ME and MAPE

Methods	$\hat{\alpha}$	$\hat{\beta}$	**MSE**	**MAPE**
ML	0.1942	207.11	3.284	0.954
BE	0.0118	166.11	0.911	0.503

6- Conclusions and Recommendations

From results in both of table (1), (2) we can notice that the Bayesian Estimation BE of the scale and shape parameters of Birnbaum-Sounders distribution is better than the Maximum Likelihood Estimation MLE due the values of MSE and MAPE.

We highly recommend to use Bayesian estimation for the Birnbaum-Sounders distribution parameters and use different informative and non-informative loss functions for better results.

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