Comparison of Some Iterative Schemes for Solution of Nonlinear Equations

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Abstract

The aim of the paper is to determine the best method for solving non-linear equations. We compare three different methods of solving non-linear equations. We used an iterative method to solve nonlinear equations since some nonlinear equations cannot be solved in finite number of steps. Solving the nonlinear equation problems depends on both the cost per iteration and the number of iterations required. We provided illustrative examples to compare the results of all three methods. The results were collected, tabulated, and analyzed in terms of errors, convergence and computational time which imply that the higher the rate of convergence the fastest it will get to approximate root or solution of the equation. The result of the comparison reveals that Muller's method is the best method of solving the nonlinear equation f(x) = 0 containing one variable because of its high rate of convergence and less computations of time.

Keywords: Nonlinear, Iterative, schemes, error, computation time.

Introduction

In real life, we often encounter many problems which are described by numerical Analysis. The behaviors of the solutions of these types are considered as Numerical solutions**.**

The major factors to be considered in coAmparing different Iterative methods are the accuracy of the numerical solutions and its computation time. The method for the solution of nonlinear equations have been presented. The numerical methods of solving nonlinear equations are more important for iterative solution[5]. In engineering analysis finding the solution for the function $f(x)$ for which $f(x) = 0$ is common problem that encountered. The values of $x$in the equation of

$f(x) = 0$ is the zeros of the function. The given equation may have one root or infinite many roots or no root. There are direct methods and indirect/Iterative methods for solving nonlinear (algebraic and transcendental) equations. We apply the direct methods obtained the exact roots in finite number of steps assuming there is no round off errors and can determine the all roots within the same time and for indirect or iterative methods we apply the concept of successive approximations[1]. For solving the nonlinear equations, we start with one or more initial approximation to the root and obtain sequence of iterates which converges to the actual solution of the equations of root [6]. This paper focus on solution of nonlinear equation of one dimension and considering one scalar unknown$x$ as solution of indirect methods. The graph of nonlinear equations cannot be forming the straight line and the variables have never one degree. We approximate the initial approximations; the initial approximation is substituted in the given nonlinear equations to determine the error and the error is used in same manner continuous to improve estimations of the solutions. There are many methods to solve nonlinear equations among them we focus on Modified Newton's methods, Muller's method and Chebyshev method. [2] In the scientific computation the difficult problem is to approximate the solutions of nonlinear equations. The exponential, trigonometric, hyperbolic are considered as examples of transcendental equations that is also referred as nonlinear equations. In solving algebraic and transcendental equations we adopt various numerical approximation methods that the solution is not exact when the coefficient is numerical values [3], professor of mathematics (Rtd) P.S.G College of technology Coimbatore. Numerical methods like Newton's method are often used to obtained the approximation solutions of nonlinear equations in which it is possible to obtain its exact solution by usual algebraic process in Engineering sciences [4].

1.1. Statement of Problem

To compare the iteration schemes of solving nonlinear equations is the one less attention is given and difficult problems in sciences and engineering.

1.2. Objectives of the study.

The purpose of this the study to determine the best iterative scheme for solutions of nonlinear equations.

- To record iteration numbers, computational running time and accurate numerical solution of the three proposed schemes.

- To compare iteration numbers, accurate and computational running time of each schemes to determine the best appropriate schemes for the given equations.

2. The Methods.

The methods we have compared weremodified Newton's Raphson, Muller and Chebyshev method.

2.1. Modified Newton Method.

This scheme is used to find roots of nonlinear equation that is continuous and differentiable, which continuous to find the solution first the initial approximation is given.

Here, the iteration scheme is written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n + a(x_n)f(x_n))} = \phi(x_n) \quad (2.1.1)$$

Consider$\phi'(x) = 1 - \frac{f'(x)}{f'\left(x+a(x)_{f(x)}\right)} + \frac{f(x)f''(x+a(x)f(x))(1+a'(x)f(x)+a(x)f'(x)}{[f'(x+a(x)f(x))]2}$ (2.1.2)

and$\phi''(x) = \frac{f''(x)}{f'(x+a(x)f(x))} + \frac{2f'(x)f''(x+a(x)f(x))[1+a'(x)f(x)+a(x)f'(x)]}{[f'(x+a(x)f(x))^2}$

$$-2\frac{f(x)[f''(x+a(x)f(x))]^2[1+a'(x)f(x)+a(x)f'(x))]^2}{[f'(x+a(x)f(x))]^2} + \frac{f(x)f'''(x+a(x)f(x))[1+a'(x)f(x)+a(x)f'(x)]^2}{[f'(x+a(x)f(x))]^2} +$$

$$\frac{[f(x)]^2 f''(x+a9x)f(x)a''(x)}{[f'(x+a(x)f(x))]^2} + \frac{f(x)f''(x+a(x) \langle (x))[2a'(x)f'(x)+a(x)f''(x)]}{[f'x+a(x)f9x))]^2} \quad (2.1.3)$$

suppose $\xi$is root of f(x)=0, f(ξ)=0 and therefore $\phi(\xi) = \xi and \phi'(\xi)$.

Eq(2.13)

$$\phi''(\xi) = \frac{f''(\xi)}{f'(\xi)} + \frac{2f'(\xi)f''(\xi)[1+a(\xi)f'(\xi)]}{[f'(\xi)]^2} = \frac{f''(\xi)[1+2a(\xi)f'(\xi)]}{f'(\xi)} \quad (2.1.4)$$

If a$(\xi) = \frac{-1}{2f'(\xi)}$,then $\phi''(\xi) = 0$

The Modified Newton Raphson scheme is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'[x_n + a(x_n) + f(x_n)]} \quad (2.1.5)$$

where a$(x_n) = \frac{-1}{2f'(x_n)}$

the equation (2.1.4) can also be written as

$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{[f'(x_n)]^2 - f(x_n)f''(x_n)}$ (2.1.6)

Suppose that ,$\phi(\xi) = \xi, then \phi'(\xi) = 0 \ and \phi''(\xi) = 0$ (2.1.7)

Rate of convergence

Suppose that$\xi$is the root of $f(x) = 0$ and

let$\epsilon_n = x_n - \xi$

Hence $x_{n+1} = \phi(x_n) = \phi(\epsilon_n + \xi)$

Or $\epsilon_{n+1} = \frac{\epsilon_n^3}{3!}\phi'''(\xi) + O(\epsilon_n^4)$ (2.1.8)

neglect the ϵ_n^4Eq(2.1.8) reduces to

$$\epsilon_{n+1} = A\epsilon_n^3$$

In which A=$\frac{\phi'''(\xi)}{3!}$ (2.1.9)

Equation (2.1.9) is the modified Newton's Raphson scheme's rate of convergence that cubic.

2.2. Muller's Method

Let the quadratics polynomial be f(x)=$ax^2 + bx + c$ (2.2.1)

If Eq(2.2.1) passes through the point $(x_{n-2}, f_{n-2}), (x_{n-1}, f_{n-1}) and (x_2, f_2)$, then

$$ax_{n-2}^2 + bx_{n-2} + c = f_{n-2}$$

$$ax_{n-1}^2 + bx_{n-1} + c = f_{n-1}$$

$ax_n^2 + bx_n + c = f_n$ (2.2.2)

Eq(2.2.2), be coming written in determinant form

$$\begin{vmatrix} f(x) & x^2 & x & 1 \\ f_{n-2} & x_{n-2}^2 & x_{n-2} & 1 \\ f_{n-1} & x_{n-1}^2 & x_{n-1} & 1 \\ f_n & x_n^2 & x_n & 1 \end{vmatrix} = 0 \quad (2.2.3)$$

Eq(2.2.3) become written in function form

$$f(x)=\frac{(x-x_{n-1)(x-x_n)}}{(x_{n-2}-x_{n-1})(x_{n-2}-x_n)}f_{n-2}+\frac{(x-x_{n-2)(x-x_n)}}{(x_{n-1}-x_{n-2})(x_{n-1}-x_n)}f_{n-1}+\frac{(x-x_{n-1)(x-x_{n-1})}}{(x_n-x_{n-2})(x_n-x_{n-1})}f_n \quad (2.2.4).$$

Suppose h=x-x_n,$h_n = x_n - x_{n-1}$ and $h_{n-1} = x_{n-1} - x_{n-2}$

Now, Eq(2.2.4) becomes

$$\frac{h(h+h_n)}{h_{n-1}(h_{n-1}+h_n)}f_{n-2}-\frac{h(h+h_n+h_{n-1}}{h_n h_{n-1}}f_{n-1}+\frac{(h+h_n)(h+h_n+h_{n-1})}{h_n(h_n+h_{n-1})}f_n = 0 \quad (2.2.5)$$

Noting f(x)=0.

Let $\lambda = \frac{h}{h_n}$, $\lambda_n = \frac{h_n}{h_{n-1}}$,$and \delta_n = 1 + \lambda_n$

The equation (2.2.5) now reduces to the following form

$$\lambda^2(f_{n-1}\lambda_n^2 f_{n-1}\lambda_n\delta_n + f_n\lambda_n)\delta^{-1} + \lambda\{f_{n-2}\lambda_n^2 - f_{n-2}\delta_n^2 + f_n(\lambda_n + \delta_n)\}\delta_n^{-1} + f_n = 0(2.2.6)$$

Or $\lambda^2 c_n + \lambda g_n + \delta_n f_n = 0(2.2.5)$

Where $g_n = \lambda_n^2 f_{n-2} - \delta_n^2 f_{n-1} + (\lambda_n + \delta_n)f_n$

$$c_n = \lambda_n(\lambda_n f_{n-2} - \delta_n f_{n-1} + f_n)$$

Equation (2.2.7) can be written as

$$\delta_n f_n\left(\frac{1}{\lambda^2}\right) + \frac{g_n}{\lambda} + \lambda_n = 0 \quad (2.2.7)$$

Solving Eq(2.2.8) for $1/\lambda$, we obtain

$$\lambda = -\frac{2\delta_n f_n}{g_n \pm \sqrt{g_n^2 - \delta_n f_n c_n}} \quad (2.2.8)$$

The sign in the denominator of ((2.2.9)is$\pm$according as $g_n > 0 or g_n < 0$

Hence $\lambda = \frac{x-x_n}{x_n-x_{n-1}}$

$x_{n+1} = x_n + (x_n - x_{n-1})\lambda(2.2.9)$

Now the equation $x_{n+1} = x_n + (x_n - x_{n-1})\lambda_{n+1}$ (2.2.9)

Where $\lambda_{n+1} = \frac{x-x_n}{x_n-x_{n-1}}$

Alternative ,$x_{n+1} = x_n - \frac{2a_0}{a_1 \pm \sqrt{a_{1-4a_0a_1}^2}}$, n=2 3,… (2.2.10)

Where $a_0 = \frac{1}{D}[(x_n - x_{n-2})(f_n - f_{n-1}) - (x_n - x_{n-1})(f_n - f_{n-2})$ (2.2.11)

$a_2 = f_n$ (2.2.12)

D =[$(x_n - x_{n-1})(x_n - x_{n-2})(x_{n-1} - x_{n-2})$] (2.2.13)

Rate of Convergence

On substituting $x_m = \xi + \epsilon_m, m = n-2, n-1, andexpandingf(\xi + \epsilon_m)\xi$in (2.2.11)to (2.2.12) , we obtain

$$D = [(\epsilon_n - \epsilon_{n-1})(\epsilon_n - \epsilon_{n-2})(\epsilon_{n-1} - \epsilon_{n-2})]$$

$$a_2 = \epsilon_{n0} f^{'}(\xi) + 1/2\epsilon_n^2 f^{''}(\xi) + 1/6\epsilon_k^3 f'''(\xi) + ..$$

$$a_1 = f^{'}(\xi) + f^{''}(\xi)\epsilon_n + 1/6\{2\epsilon_n^2 + \epsilon_n\epsilon_{n-1} + \epsilon_n\epsilon_{n-2} + \epsilon n - 1\epsilon_{n-2}\}f'''(\xi) + ...$$

$$a_0 = 1/2f''(\xi) + 1/6\{\epsilon_n + \epsilon_{n-1} + \epsilon_{n-2}\}f'''(\xi)$$

So,$a_{1-4a_0a_2}^2 = [f^{'}(\xi)]^2 - 1/2(\epsilon_n\epsilon_{n-1} + \epsilon_n\epsilon_{n-2} + \epsilon n - 1\epsilon_{n-2})f^{'}(\xi)f^{'''}(\xi)$ ...

$$a_1 + \sqrt{a_{1-4a_0a_2}^2} = 2f^{'}(\xi)[1 + 1/2\epsilon_n C_2 + 1/6(\epsilon_n^2 - \epsilon n - 1\epsilon_{n-2})C_3..]$$

Where $C_i = \frac{f^{(i)}(\xi)}{f'(\xi)}$,i=1 2 3……

Hence using (2.2.10) we get

$$\epsilon_{n+1} = 1/6(\epsilon_n\epsilon_{n-1}\epsilon_{n-2}C_3) + .$$

Muller's error is given by

$\epsilon_{n+1} = C\epsilon_n\epsilon_{n-1}\epsilon_{n-2}$ (2.2.14)

Where C= 1/6$C_3 = 1/6\frac{f'''(\xi)}{f'(\xi)}$ (2.2.15)

Now we want a relation of the form

ϵ_{n+1}=$A\epsilon_n^p$ (2.2.16)

Where A and p are to be determined

From (2.2.16) $\epsilon_n = A\epsilon_{n-1}^p or \epsilon_{n-1} = A^{-1/p}\epsilon_n^{1/p}$

$$\epsilon_{n-1} = A\epsilon_{n-2}^p or \epsilon_{n-2} = A^{-1/p}\epsilon_{n-1}^{1/p} = A^{-(\frac{1}{p}+\frac{1}{p^2})}\epsilon_n^{1/p^2}$$

Substituting the values of $\epsilon_{n+1}, \epsilon_{n-1}$ and ϵ_{n-2}in (2.2.13)

$\epsilon_n^p = CA^{-(1+\frac{2}{p}+\frac{1}{p^2})}\epsilon_n^{1+\frac{1}{p}+1/p^2}$ (2.2.17)

$F(p) = p^3 - p^2 - p - 1 = 0$(2.2.18)

Suppose that $F(p) = 0$,p is the smallest root (1,2) .

With initial approximation $p_0 = 2$.

$$p_1 = 1.8571, p_2 = 1.8395, p_3 = 1.8393 \; ....$$

Therefore the root of the equation (2.2.18) is $p = 1.84$ (approximation) .

2.3 The Chebyshev'sMethod.

For $f(x) = 0$,the function f(x) in the neighborhood of x_n as

$f(x)$=f$(x_n) - x_n$ f'(x_n)+… (2.3.1)

Equation (2.2.3) gives

$x = x_n - \frac{f(x_n)}{f'(x_n)}$ (2.3.2)

Equation(2.2.3) of the $(n+1)^{th}$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ (2.3.3)

Then we obtain

0=f(x)=f$(x_n) + (x - x_n)f'(x_n) + \frac{(x-x_n)^2 f''(x_n)}{2}$ (2.3.4)

Hence f(x_{n+1}) =f$(x_n) + (x_{n+1} - x_n)f'(x_n) + \frac{(x_{n+1}-x_n)^2 f''(x_n)}{2}$=0 (2.3.5)

Substituting the value of$x_{n+1} - x_n$from (2.2.34) to the last term and we obtain

$\mathrm{f}(x_n) + (x_{n+1} - x_n)f'(x_n) + \frac{[f(x_n)]^2 f''(x_n)}{2\,[f'(x_n)]^2} = 0$ (2.3.6)

Hence $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1[f(x_n)]^2 f''(x_n)}{2\,[f'(x_n)]^3}$ (2.3.7).

Rate of convengence

From the Equ(2.3.7)

Substuting $x_n = \xi + \epsilon_n$ and expanding $(x_n), f'(x_n), f''(x_n)$ near ξ in the equation

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}[\frac{f((x_n)}{f'(x_n)}]^2 \frac{f''(x_n)}{f'(x_n)}$

(2.2.8)

Now $\frac{f(x_n)}{f'(x_n)} = \frac{f(\xi+\epsilon_n)}{f'(\xi+\epsilon_n)} = \frac{\epsilon_n f'(\xi)+1/2\epsilon_n{}^2 f''(\xi)+1/6\epsilon_n{}^3 f'''(\xi)+..}{f(\xi)+\epsilon_n f''(\xi)+1/2\epsilon_n{}^2 f'''(\xi)}$

$[\epsilon_n + 1/2C_2\epsilon_n{}^2 + 1/6C_3\epsilon_n{}^3]\mathrm{x}[1 + C_2\epsilon_n + 1/2C_3\epsilon_n{}^2 +]^{-1}$

$=[\epsilon_n + 1/2C_2\epsilon_n{}^2 + 1/6C_3\epsilon_n{}^3]\mathrm{x}[1 - C_2\epsilon_n + (1/2C_2{}^2 - 1/2C_3)\epsilon_n{}^2 +]$

$=\epsilon_n - 1/2C_2\epsilon_n{}^2 \quad + (1/2C_2{}^2 - 1/2C_3)\epsilon_n{}^3$

Where $C_i = \frac{f^{(i)}(\xi)}{f'(\xi)}$,i=1 2 3……

Also $[\frac{f((x_n)}{f'(x_n)} = \epsilon_n{}^2 - C_2\epsilon_n{}^3$

Now $\frac{f''(x_n)}{f'(x_n)} = \frac{f''(\xi+\epsilon_n)}{f'(\xi+\epsilon_n)} = \frac{f''(\xi)+\epsilon_n f'(\xi)+1/2\epsilon_n{}^2 f''(\xi)+1/6\epsilon_n{}^3 f'''(\xi)+..}{f(\xi)+\epsilon_n f''(\xi)+1/2\epsilon_n{}^2 f'''(\xi)..}$

$=\frac{f''(\xi)}{f'(\xi)}[1+\frac{C_3\epsilon_n}{C_2}+\ldots][1+(C_2\epsilon_n+\ldots]$

$=C_2[1+\frac{C_3\epsilon_n}{C_2}+\ldots][1-C_2\epsilon_n + \cdots]$

$=C_2 + (C_3 - C_2^2)\epsilon_n + \cdots$

Substituting (2.2.39)

$$\epsilon_{n+1} = \epsilon_n - [\epsilon_n - 1/2C_2\epsilon_n{}^2 + (\frac{1C_2^2 -}{2}\ 1/3C_3\epsilon_n{}^3 + \cdots.]$$

$$-1/2[\epsilon_n{}^2 - C_2\epsilon_n{}^3 + \ldots][C_2 + \left(C_3 - C_2^2\right)\epsilon_n + \cdots]$$

$=[1/2C_2^2 - 1/6C_3]\epsilon_n{}^3 + o(\epsilon_n{}^4)$
(2.3.8)

Chebyshev method has the rate convergence 3

3. Results.

For the three method solve the following examples in table form.

Example1.: Solving the equation $f(x) = x^3 - 7x^2 + 11x - 5$;correct to six decimal places.

Table 1. Numerical result of the three methods.

Modified Newton's Method			Muller's Method		Chebyshev method	
Iteration No	x_{n+1}	e_{n+1}	x_{n+1}	e_{n+1}	x_{n+1}	e_{n+1}
0	0	-	1.250000		0	
1	0.909091	0.9090909	1.033333	-0.216667	0.586026	0.5860255
2	0.999001	0.0899101	1.037044	0.003711	0.837301	0.2512752
3	1.000012	0.0009989	1.008927	-0.028117	0.937781	0.1004807
4	1.000023	0.0000001	1.000777	-0.00815	0.976488	0.0387071
Elapsed time	4.190822 seconds.		0.071692 seconds		5.093625 seconds.	

4. Discussion

As presented in the table above, the results obtained have been compared with the result obtained by the three methods. For Iterative method we start with the initial approximation convergences to exact solution, or to better approximation repeating computation cycle for achieving required accuracy. When we compare the rate of convergence of numerical method that have higher rate of convergence may consider as it reaches the solution with less number of iteration to another method which has higher number of iteration with slower convergence. Using the intermediate value theorem, we can check that the weather the nonlinear equation of $f(x) = 0$ has solution or not with one variable. For the methods presented in this paper, we discovered that Muller's method is the best of iterative method among the method we compare to solve non-linear equations for single variable that converges accurately to the solution of the given nonlinear equation with initial approximation that close to the roots of the equation and Muller's method is best method considering the computational time. From the presented method we have compared Chebyshev method converges slowly when we compare with modified Newton's method andMuller's method respectively**.**

5. Conclusion

We tested and compared three different methods of solving the nonlinear equations; namely Muller's method, Modified Newton method and Chebyshev method. The Chebyshev method fail before it converges to the root of nonlinear equations. Furthermore, it converges more slowly when we compare to the modified Newton method and Muller's method respectively. The third method of solving nonlinear equations is the Muller's method. This method yields low error, high convergence rate and less computational time. Comparatively, the Muller's method is the most efficient method in finding the roots of nonlinear equation f(x) = 0 appeared in this paper.

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