Accurate Independent [1, 2] – Domination in Graphs

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Abstract

A [1,2] – dominating set S of G is anindependent [1,2] – dominating set, if the induced subgraph$< S >$has no edges.An independent [1,2] – dominating set S of graph G is an accurate independent [1,2] – dominating set, if $V - S$ has no independent [1, 2] – dominating set of cardinality $|S|$. The cardinality of the minimum accurate independent [1,2] – dominating set is called the accurate independent [1,2] – domination number and is denoted by $i_{a[1,2]}(G)$of G. In this paper, we initiate a study of this new parameter and obtain some results concerning this parameter.

Keywords : independent dominating number, [1,2] –dominating set, accurate [1,2] – dominating set, accurate [1,2] – domination number.

I. INTRODUCTION

Let $G = \{V, E\}$ be a simple finite, undirected and connected graph. Generally, for any graph $N(v)$& $N[v]$ is open and closed neighbourhood of G, for every $v \in V(G)$, the degree of v means the number of edges that incident on v and is denoted by $\deg(v)\, or\, d(v)$.A set of vertices in G is independent, if no two of them are adjacent. The maximum number of vertices in such a set is called the vertex independent number of G and is denoted by $\beta_0(G)$.For more details see [3] and [4].

The diameter $diam(G)$ of a connected graph G is the maximum distance between two vertices of G, that is $diam(G) = max_{u,v\,\epsilon\, V(G)} d_G(u, v)$.

A proper coloring of a graph G is a function from the vertices of a graph to a set of colors such that any two adjacent vertices have different colors. The chromatic number $\chi(G)$ is the minimum number of colors needed in proper coloring of graph.

A set $S \subseteq V$ is dominating set, if every vertex $v \in V(G)$ is adjacent to a vertex or more in it. A dominating set is said to be minimal dominating set, if there is no proper subset in it.

The domination number is the minimum cardinality taken over all such dominating set of G and is denoted by $\gamma(G)$.

An accurate dominating set of G is a dominating set of G such that no $|S|$ - element subset of $V - S$ is a dominating set of G. The accurate domination number of G, denoted by $\gamma_a(G)$, is the cardinality of a smallest accurate dominating set of G.

We call a dominating set of G of cardinality $\gamma(G)$ a $\gamma - set$ of G, and an accurate dominating set of G of cardinality $\gamma_a(G)$ a$\gamma_a - set$ of G.

Since every accurate dominating set of G is a dominating set of G, we note that $\gamma(G) \leq$ *γaG.* The accurate domination in graphs was introduced by Kulli and Kattimani. For more details see [5,6,7& 10].

A dominator coloring of a graph G is a proper coloring in which each vertex of the graph dominates every vertex of some color class. The dominator chromatic number $\chi_d(G)$ is the minimum number of color classes in a dominator coloring of graph G. This concept was introduced by R. Gera at.el [2].

A non – empty subset $S \subseteq V$ is called a [1,2] – dominating set in a graph $G = \{V, E\}$ having the property that for every vertex in v in $V - S$ there is atleast one vertex in S at distance 1 from v and second vertex in S at distance 2 from v. The order of the [1,2] – dominating set of G is called the [1,2] – domination number of G, denoted by $\gamma_{[1,2]}(G)$.

A [1,2] – dominator coloring of a graph G is a proper coloring in which each vertex of the graph dominates every vertex of some color class. The [1,2] – dominator chromatic number $\chi_{[1,2]d}(G)$ is the minimum number of color classes in a [1,2] – dominator coloring of graph G.

A [1,2] – dominating set S of a graph G is an accurate [1,2] – dominating set, if $V - S$ has no [1,2] – dominating set of cardinality |S|. The accurate [1,2] – domination number $\gamma_{a[1,2]}(G)$ is the smallest [1,2] – dominating set of G. Since every accurate [1,2] – dominating set is a [1,2] – dominating set of G. For more details see [13].

II. ACCURATE INDEPENDENT [1,2] – DOMINATION IN GRAPHS

Anindependent dominating set S is a set of vertices of a graph G, if no two vertices in S are adjacent. The independent domination number $i(G)$ is the minimum cardinalities of an independent dominating setof G.

An independent dominating set S of a graph G is an accurate independent dominating set,if $V - S$ has no independent dominating set of cardinality |S|. The order of the smallest accurate independent dominating set of G is called the accurate independent domination number of G, denoted by $i_a(G)$. This concept was introduced by Kulli [7].

An independent [1,2] – dominating set S of a graph G is an accurate independent [1,2] – dominating set, if $V - S$ has no independent [1,2] – dominating set of cardinality |S|. The minimum cardinality taken over all such accurate independent [1,2] – dominating set is called the accurate independent [1,2] – domination number of G and is denoted by $i_{a[1,2]}(G)$.

We denote the path and cycle on n vertices by P_n and C_n respectively. We denote by K_n the complete graph on n vertices, and by $K_{m,n}$ the complete bipartite graph with partite sets of size m and n. Let $\lfloor x \rfloor$ denote the least integer greater than as equal to $x$and $\lceil x \rceil$ least integer greater than or equal to x.

III. MAIN RESULTS

We obtain exact values $i_{a[1,2]}(G)$ for some standard graphs.

Observation3.1.

- **i)** Evert accurate independent [1,2] – dominating set is independent [1,2] – dominating set. Hence it is minimal [1,2] – dominating set.
- **ii)** Every minimal accurate independent [1,2] – dominating set is a maximal independent [1,2] – dominating set.

Proposition 3.1. For any nontrivial connected graph G, $\gamma_{[1,2]}(G) \leq i_{a[1,2]}(G)$.

Proof: Clearly every accurate independent [1,2] – dominating set of G is a [1,2] – dominating set of G.

Proposition 3.2. If G contains an isolated vertex, then every accurate [1,2] – dominating set is an accurate independent [1,2] – dominating set.

Proposition 3.3. For graphs P_p, W_p and $K_{m,n}$ there are

i) $i_{a[1,2]}(P_p) = \lceil p/3 \rceil$ if $p \geq 3$

ii) $i_{a[1,2]}(W_p) = 1$ if $p \geq 5$

iii) $i_{a[1,2]}(K_{m,n}) = m$ for $1 \leq m < n$.

Theorem 3.1. For any graph G, $i_{a[1,2]}(G) \leq p - \gamma_{[1,2]}(G)$

Proof: Let S be the minimal [1,2] – dominating set of G. Then there exist at least one accurate independent [1,2] – dominating set in $(V - S)$ and by proposition 3.1,

$$i_{a[1,2]}(G) \leq |V| - |S| \leq p - \gamma_{[1,2]}(G).$$

Notice that the path P_4 achieves this bounds.

Theorem 3.2. For any graph G,

$$\lceil p/(\Delta + 1) \rceil \leq i_{a[1,2]}(G) \leq \lfloor p\Delta / (\Delta + 1) \rfloor$$

and these bounds are sharp.

Proof: It is known that $p/(\Delta + 1) \leq \gamma_{[1,2]}(G)$ and by proposition 3.1, we see that the lower bound holds. By theorem 3.1,

$$\begin{aligned} i_{a[1,2]}(G) &\leq p - \gamma_{[1,2]}(G) \\ &\leq p - p/(\Delta + 1) \\ &\leq p\Delta / (\Delta + 1) \end{aligned}$$

Notice that the path $P_p, p \geq 3$ achieves this lower bound.

Proposition 3.4. If $G = K_{m_1, m_2, m_3, \dots, m_r}$, $r \geq 3$, then

$$i_{a[1,2]}(G) = m_1 \text{ if } m_1 < m_2 < m_3 \dots < m_r .$$

Theorem 3.3. For any graph G without isolated vertices $\gamma_{a[1,2]}(G) \leq i_{a[1,2]}(G)$if $G \neq K_{m_1,m_2,m_3,\dots,m_r}$, $r \geq 3$. Furthermore, the equality holds if $G = P_p(p \neq 4, p \geq 3), W_p(p \geq 5)$ or $K_{m,n}$for $1 \leq m < n$.

Proof: Since we have $\gamma_{[1,2]}(G) \leq \gamma_{a[1,2]}(G)$ and by the proposition 3.1,

$$\gamma_{a[1,2]}(G) \leq i_{a[1,2]}(G).$$

Let $\gamma_{a[1,2]}(G) \leq i_{a[1,2]}(G)$. If $G = K_{m_1,m_2,m_3,\dots,m_r}$, $r \geq 3$ then by proposition 3.4, $i_{a[1,2]}(G) = m_1$ if $m_1 < m_2 < m_3 \dots < m_r$ and also accurate independent [1,2] – domination number is $\lfloor p/2 \rfloor + 1$ i.e, $\gamma_{a[1,2]}(G) = \lfloor p/2 \rfloor + 1 > m_1 = i_{a[1,2]}(G)$, a contradiction.

Corollary 3.1. For any graph G, $i_{a[1,2]}(G) = \gamma_{a[1,2]}(G)$ if $diam(G) = 2$.

Proposition 3.5. For any graph G without isolated vertices $i_{[1,2]}(G) \leq i_{a[1,2]}(G)$. Furthermore, the equality holds if $G = P_p(p \geq 3), W_p(p \geq 5)$ or $K_{m,n}$for $1 \leq m < n$.

Proof: Every accurate independent [1,2] – dominating set is an independent [1,2] – dominating set. Thus results holds.

Definition 3.1. The double star $S_{m,n}$ is the graph obtained by joining the centres of two stars $K_{1,n}$ and $K_{1,m}$ with an edge.

Proposition 3.6. For any graph G, $i_{a[1,2]}(G) \leq \beta_0(G)$. Furthermore, the equality holds if $G = S_{m,n}$.

Proof: Since every minimal accurate independent [1,2] – dominating set is an maximal independent dominating set. Thus results holds.

Theorem 3.4. For any graph G, $i_{a[1,2]}(G) \leq p - \alpha_0(G)$.

Proof: Let S be a vertex cover of G. Then $V - S$ is an accurate independent [1,2] – dominating set. Then $i_{a[1,2]}(G) \leq |V| - |S| \leq p - \alpha_0(G)$.

Corollary 3.1. For any graph G, $i_{a[1,2]}(G) \leq p - \beta_0(G) + 2$.

Theorem 3.5. If G is any nontrivial connected graph containing exactly one vertex of degree $\Delta(G) = p - 1$, then $\gamma_{[1,2]}(G) = i_{a[1,2]}(G) = 1$.

Proof: Let G be any nontrivial connected graph containing exactly one vertex v of degree $\deg(v) = p - 1$. Let S be a minimal [1,2] – dominating set of G containing vertex of degree $\deg(v) = 1$. Then S is a minimum [1,2] – dominating set of G i.e.,

$$|S| = \gamma_{[1,2]}(G) = 1. \tag{1}$$

Also $V - S$ has no [2,2] – dominating set of some cardinality $|S|$. Therefore,

$$|S| = i_{a[1,2]}(G). \tag{2}$$

Hence, by (1) and (2) $\gamma_{[1,2]}(G) = i_{a[1,2]}(G) = 1$.

Theorem 3.6. If G is a connected graph with p vertices then $i_{a[1,2]}(G) = p/2$if and only if $G = H \circ K_1$, where H is any nontrivial connected graph.

Proof: Let S be any minimal accurate independent [1,2] – dominating set with $|S| = p/2$. If $G \neq H \circ K_1$ then there exist at least one vertex $v_i \in V(G)$ which is neither a pendent vertex nor a support vertex. Then there exist a minimal accurate independent [1,2] – dominating set S' containing v_i such that

$$|S'| \leq |S| - \{v_i\} \leq p/2 - \{v_i\} \leq p/2 - 1,$$

which is a contradiction to minimality of S.

Conversely, ley l be the set of all pendent vertices in $G = H \circ K_1$ such that $|l| = p/2$. If $G = H \circ K_1$, then there exist a minimal accurate independent [1,2] – dominating set, $S \subseteq V(G)$ containing all pendent vertices of G. Hence $|S| = |l| = p/2$.

Now we characterize the trees for which $i_{a[1,2]}(T) = p - \Delta(T)$.

Theorem 3.7. For any tree T, $i_{a[1,2]}(T) = p - \Delta(T)$ if and only if T is a wounded spider and $T \neq K_1, K_{1,1}$.

Proof: Suppose T is wounded spider. Then it is easy to verify that $i_{a[1,2]}(T) = p - \Delta(T)$.

Conversely, suppose T is a tree with $i_{a[1,2]}(T) = p - \Delta(T)$. Let v be the vertex of maximum degree $\Delta(T)$ and u be a vertex in $N(v)$ which has degree 1. If $T - N[v] = \phi$ then T is the star $K_{1,n}$, $n \geq 2$. Thus T is a double wounded spider. Assume now there is at least one vertex in $T - N[v]$. Let S be a maximal independent set of $\langle T - N[v] \rangle$. Then either $S \cup \{v\}$ or $S \cup \{u\}$ is an accurate independent [1,2] – dominating set. Furthermore, $N(v)$ is an accurate independent [1,2] – dominating set.

The connectivity of T implies that each vertex in $V-N[v]$ must be adjacent to at least one vertex in $N(v)$. Moreover if any vertex in $V-N[v]$is adjacent to two or more vertices in $N(v)$, then a cycle is formed. Hence each vertex in $V-N[v]$ is adjacent to exactly one vertex in $N(v)$. To show that $\Delta(T)+1$ vertices are necessary to dominate T, there must be at least one vertex in $N(v)$ which are not adjacent to any vertex in $V-N[v]$ and each vertex in $N(v)$ has either 0 or 1 neighbours in $V-N[v]$. Thus T is wounded spider.

Proposition 3.7. If G is a path $P_p,\ p\geq 3$then$\gamma_{[1,2]}(P_p)=i_{a[1,2]}(P_p)$.

IV. ACCURATE INDEPENDENT [1,2] – DOMINATION OF SOME GRAPH FAMILIES

In this section accurate independent [1,2] – domination of fan graph, double fan graph, helm graph, gear graph are considered. We also obtain the corresponding relation between other dominating parameters and dominator coloring of the above graph families.

Definition 4.1.A fan graph, denoted by F_n can be constructed by joining n copies of the cycle c_3 with a common vertex.

Observation 4.1. Let F_n be a fan. Then,

i) $F_n$is a planar undirected graph with $2n+1$vertices and $3n$ edges.

ii) $F_n$has exactly one vertex with $\Delta(F_n)=p-1$.

iii) $diam(F_n)=2$.

Theorem 4.1. For a fan graph $F_n, n\geq 2,\ \chi_{[1,2]d}(F_n)=3$.

Proposition 4.1. For a fan graph $F_n, n\geq 2,\ i_{a[1,2]}(F_n)=1$.

Proof: By Observation 4.1 (ii) and Theorem 3.3, results holds.

Proposition 4.2. For a fan graph $F_n, n\geq 2,\ i_{a[1,2]}(F_n)<\chi_{[1,2]d}(F_n)$.

Proof: By Proposition 4.1 and Theorem 4.1, we know that $\chi_{[1,2]d}(F_n) = 3$. This implies that $i_{a[1,2]}(F_n) < \chi_{[1,2]d}(F_n)$.

Definition 4.2.A double fan graph, denoted by $F_{2,n}$ isomorphic to $P_n + 2K_1$.

Observation 4.2. Let $F_{2,n}$ be a double fan. Then,

i) $F_{2,n}$is a planar undirected graph with $n + 2$ vertices and $3n - 2$ edges.

ii) $diam(F_{2,n}) = 2$.

Theorem 4.2. For a double fan graph $F_{2,n}, n \geq 2$, $\chi_{[1,2]d}(F_{2,n}) = 3$.

Theorem 4.2. For a double fan graph $F_{2,n}, n \geq 2$, $i_{a[1,2]}(F_{2,2}) = 2$, $i_{a[1,2]}(F_{2,3}) = 1$, $i_{a[1,2]}(F_{2,5}) = 3$ and $i_{a[1,2]}(F_{2,n}) = 2$ if $n \geq 7$.

Proof: Our proof is divided into four cases following

Case 1. If $n = 2$and $n \geq 7$, then $F_{2,n}, n \geq 2$ has only one accurate independent [1,2] – dominating set S of $|S| = 2$. Hence $i_{a[1,2]}(F_{2,2}) = 2$.

Case 2. If $n = 3$, then $F_{2,n}$ has exactly one vertex of $\Delta(G) = p - 1$. Then by Theorem 3.5, $i_{a[1,2]}(F_{2,3}) = 1$.

Case 3. If $n = 5$ and S be an independent [1,2] – dominating set of $G$with $|S| = 2$, then $(V - S)$ also has an independent [1,2] – dominating set of cardinality 2. Hence S is not accurate. Let S_1 be an independent [1,2] – dominating set with $|S_1| = 3$, then $V - S_1$ has no independent [1,2] – dominating set of cardinality 3. Then S_1 is accurate. Hence, $i_{a[1,2]}(F_{2,5}) = 3$.

Case 4. If $n = 5 \; and \; 6$, there does not exist accurate independent [1,2] – dominating set.

Proposition 4.3. For a double fan graph $F_{2,n}, n \geq 7$,

$$\gamma_{[1,2]}(F_{2,n}) = i_{[1,2]}(F_{2,n}) = \gamma_{a[1,2]}(F_{2,n}) = i_{a[1,2]}(F_{2,n}) = 2.$$

Proof: Let $F_{2,n}, n \geq 7$ be a double fan graph. Then $2K_1$ forms a minimal [1,2] – dominating set of $F_{2,n}$ such that $\gamma_{[1,2]}(F_{2,n}) = 2$, since this dominating set is independent and in $(V - S)$there is no independent [1,2] – dominating set of cardinality 2 it is both independent and accurate independent [1,2] – dominating set. Also accurate [1,2] – dominating set. Hence,

$$\gamma_{[1,2]}(F_{2,n}) = i_{[1,2]}(F_{2,n}) = \gamma_{a[1,2]}(F_{2,n}) = i_{a[1,2]}(F_{2,n}) = 2.$$

Proposition 4.4. For a double fan graph $F_{2,n}, n \geq 7$,

$$i_{a[1,2]}(F_{2,n}) < \chi_{[1,2]d}(F_{2,n}).$$

Proof: The proof follows by Theorem 4.2 and 4.3.

Definition 4.2. For $n \geq 4$, the wheel W_n is defined to be the graph $W_n = C_{n-1} + K_1$. Also it is defined as $W_{1,n} = C_n + K_1$.

Definition 4.2.A helm H_n is the graph obtain from $W_{1,n}$ by attaching a pendent edge at each vertex of the $n - cycles$.

Observation 4.3.AH_n is a planar undirected graph with $2n + 1$ vertices and $3n$ edges.

Theorem 4.4. For a helm graph $H_n, n \geq 3,\ \chi_{[1,2]d}(H_n) = n + 1$.

Proposition 4.5. For a helm graph $H_n, n \geq 3,\ i_{a[1,2]}(H_n) = n$.

Proof: Let $H_n, n \geq 3$ be a helm graph. Then there exist a minimal independent [1,2] – dominating set$S$ with $|S| = n$ and $V - S$ has no independent [1,2] – dominating set of cardinality n. Hence S is accurate. Therefore,$i_{a[1,2]}(H_n) = n$.

Proposition 4.6. For a helm graph $H_n, n \geq 3$

$$\gamma_{[1,2]}(H_n) = i_{[1,2]}(H_n) = \gamma_{a[1,2]}(H_n) = i_{a[1,2]}(H_n) = n.$$

Proposition 4.7. For a helm graph $H_n, n \geq 3$

$$i_{a[1,2]}(H_n) = \chi_{[1,2]d}(H_n) - 1.$$

Proof: Applying Proposition 4.5, $i_{a[1,2]}(H_n) = n = n + 1 - 1 = \chi_{[1,2]d}(H_n) - 1$ by Theorem 4.4, $\chi_{[1,2]d}(H_n) = n + 1$. Hence the proof.

Definition 4.3.A gear graph G_n also known as a bipartite wheel graph, is a wheel graph $W_{1,n}$ with a vertex added between each pair of adjacent vertices of the outer cycle.

Observation 4.3. Agear graph G_n is a planar undirected graph with $2n + 1$ vertices and $3n$ edges.

Theorem 4.5. For a gear graph $G_n, n \geq 3,\ \chi_{[1,2]d}(G_n) = \lceil 2n/3 \rceil + 2$.

Theorem 4.5. For a gear graph $G_n, n \geq 3,\ i_{a[1,2]}(G_n) = n$.

Proof: It is clear from the definition of gear graph G_n is obtained from wheel graph $W_{1,n}$ with a vertex added between each pair of adjacent vertices of the outer cycle of wheel graph $W_{1,n}$. These n vertices forms an independent [1,2] – dominating set in G_n such that $V - S$ has no independent [1,2] – dominating set of cardinality n. Therefore, the set S with cardinality n is an accurate independent [1,2]- dominating set of G_n. Therefore,$i_{a[1,2]}(G_n) = n$.

Corollary 4.1. For a gear graph $G_n, n \geq 3,\ \gamma_{[1,2]}(G_n) = i_{[1,2]}(G_n) = n - 1$.

Proposition 4.8. For a gear graph $G_n, n \geq 3,\ \gamma_{a[1,2]}(G_n) = i_{a[1,2]}(G_n)$.

Proposition 4.9. For a gear graph $G_n, n \geq 3,\ i_{a[1,2]}(G_n) = \ \gamma_{[1,2]}(G_n) = i_{[1,2]}(G_n) + 1$.

Proof: Applying Theorem 4.6 and Corollary 4.1, we know that

$$i_{a[1,2]}(G_n) = n$$

$$= n + 1 - 1$$

$$= \gamma_{[1,2]}(G_n) + 1$$

$$= i_{[1,2]}(G_n) + 1.$$

V. CONCLUSION

In this paper, we introduce a new parameter $i_{a[1,2]}(G)$ of a graph G known as accurate independent [1,2] – domination number. We computed the exact values of the accurate independent [1,2] – domination number for some graphs. This works reduces the difficulties of traditional methods and makes the manipulations simple and easier.

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