Solving Oscillatory Problems Using Trigonometrically-Fitting Improved Runge-Kutta Nystrom Method

Kasim Abbas Hussain[1,*] and Waleed Jamal Hasan[1]
[1]Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq
[*]Corresponding author Email: kasimmath@gmail.com; waleedjamalhasan1979@gmail.com

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Abstract
In this article, we suggest a new technique for solving oscillatory ordinary differential equations called Trigonometrically Fitted Improved Runge-Kutta Nystrom (TFIRKN4) method, which has three stages and fourth order. The Improved Runge-Kutta Nystrom (IRKN4) method is extended with trigonometric calculations in the proposed approach. The coefficients of the proposed method are based on the frequency and step size. It is discovered that the new method is more precise when compared to the existing Runge-Kutta Nystrom and IRKN4 methods. The number of test problems for the second-order ordinary differential equations (ODEs) is solved to demonstrate the effectiveness of this approach. According to the computational experiments, the TFIRKN4 approach consistently outperforms the IRKN4 and existing Runge-Kutta Nystrom methods.
Keywords: Trigonometrically-fitted, second-order initial value problems, oscillating solutions, improved numerical methods.

1. Introduction

The general form of special initial value problems (IVPs) of second-order is as follows:

$$u''(t) = f(t,u),\ u(t_0) = u_0, \quad u'(t_0) = u'_0. \tag{1}$$

IVPs contains oscillatory character in the solutions. This problem frequently happens in several fields of applied science, for example, quantum chemistry, quantum physical science, astrophysics, and elsewhere [1,2]. The classical approach to solve (1) is converted to the first order ODEs and hence utilizes a suitable technique [3]. On the other hand, (1) can be solved directly by utilizing the hybrid and Runge-Kutta Nystrom strategies [4,5]. There are several RKN methods that have been developed, such as [6,7,8]. In 2012, [9] built Improved Runge-Kutta Nystrom (IRKN) method to solve second IVPs by introducing the novel terms q_{-i} , which picks the $q_i, i \geq 2$ from its prior steps. Exponentially and trigonometrically fitted techniques are used to develop new methods to solve oscillatory problems. In 2019, [10] presented a sixth-order hybrid formula to solve oscillation problems. Furthermore, in 2020, [11] built a two-derivative fifth-order Runge-Kutta formula to resolve oscillatory problems using phase-Lag properties. In addition, [12] derived a 5(3) pair RKN technique with an exponentially-fitted approach to resolve oscillatory second-order IVPs. Moreover, oscillating problems were solved using the developed optimized RKN technique of the sixth order [13].
This paper derives a Trigonometrically-Fitted Improved Runge-Kutta Nystrom (TFIRKN4) fourth-order method to solve oscillatory problems. In Section 2, the TFIRKN4 method is derived. Test problems and numerical comparisons with existing

methods are given in Section 3 to show the capacity of the newly proposed method. Finally, conclusions are presented in Section 4.

2. The Derivation of TFIRKN4 Method

The general IRKN method for solving (1) is expressed as follows: [9]

$$u_{n+1} = u_n + \frac{3h}{2}u'_n + \frac{h}{2}u'_{n-1} + h^2 \sum_{i=2}^{s} d_i(q_i - q_{-i}), \quad (2)$$

$$u'_{n+1} = u'_n + h(b_1 q_1 - b_{-1}q_{-1} + \sum_{i=2}^{s} b_i(q_i - q_{-i}), \quad (3)$$

$$q_1 = f(t_n, u_n), \quad (4)$$

$$q_{-1} = f(t_{n-1}, u_{n-1}), \quad (5)$$

$$q_i = f(t_n + c_i h, u_n + hc_i u'_n + h^2 \sum_{j=1}^{i-1} a_{ij} q_j), \quad (6)$$

$$q_{-i} = f(t_{n-1} + c_i h, u_{n-1} + hc_i u'_{n-1} + h^2 \sum_{j=1}^{i-1} a_{ij} q_{-j}). \quad (7)$$

where c_i, b_i, d_i, b_{-1} and a_{ij} are real numbers and $i, j = 1,2, \dots, s$. The associated Butcher tableau of the IRKN method (2)-(7) is as follows in Table 1:

Table 1. IRKN method

0					
c_2	a_{21}				
c_3	a_{31}	a_{32}			
.	.	.	.		
.	.	.	.		
.	.	.	.		
c_s	a_{s1}	a_{s2}	...	a_{ss-1}	
b_{-1}	b_1	b_2	...	b_{ss-1}	b_s
		d_2	...	d_{ss-1}	d_s

The general form of the three stages of the IRKN4 fourth-order method is illustrated by:

$$u_{n+1} = u_n + \frac{3h}{2}u'_n + \frac{h}{2}u'_{n-1} + h^2(d_2(q_2 - q_{-2}) + d_3(q_3 - q_{-3})), \quad (8)$$

$$u'_{n+1} = u'_n + h(b_1 q_1 - b_{-1}q_{-1} + b_2(q_2 - q_{-2}) + b_3(q_3 - q_{-3})), \quad (9)$$

$$q_1 = f(t_n, u_n), \quad (10)$$

$$q_{-1} = f(t_{n-1}, u_{n-1}), \quad (11)$$

$$q_2 = f(t_n + c_2 h, u_n + hc_2 u'_n + h^2 a_{21} q_1), \quad (12)$$

$$q_{-2} = f(t_n + c_2 h, u_{n-1} + hc_2 u'_{n-1} + h^2 a_{21} q_{-1}), \quad (13)$$

$$q_3 = f(t_n + c_3 h, u_n + hc_3 u'_n + h^2(a_{31} q_1 + a_{32} q_2), \quad (14)$$

$$q_{-3} = f(t_n + c_3 h, u_{n-1} + hc_3 u'_{n-1} + h^2(a_{31} q_{-1} + a_{32} q_{-2}), \quad (15)$$

If $u(t_{\mathrm{n}})$ is integrate exactly by IRKN4 method (8)-(15), then we have;

$$u_{\mathrm{n}} = u(t_{\mathrm{n}}) = \mathrm{e}^{\mathrm{i}\omega t_{\mathrm{n}}}, \quad (16)$$

$$u'_n = \mathrm{i}\omega \mathrm{e}^{\mathrm{i}\omega t_{\mathrm{n}}}, \quad (17)$$

$$u''_n = -\omega^2 \mathrm{e}^{\mathrm{i}\omega t_{\mathrm{n}}} = f(t_n, u_n), \quad (18)$$

$$u_{\mathrm{n}-1} = u(t_{\mathrm{n}-1}) = \mathrm{e}^{\mathrm{i}\omega(t_{\mathrm{n}}-h)}, \quad (19)$$

$$u'_{n-1} = \mathrm{i}\omega \mathrm{e}^{\mathrm{i}\omega(t_\mathrm{n}-h)}, \quad (20)$$

$$u''_{n-1} = -\omega^2 \mathrm{e}^{\mathrm{i}\omega(t_\mathrm{n}-h)} = f(t_{n-1}, u_{n-1}), \quad (21)$$

$$u_{\mathrm{n}+1} = \mathrm{e}^{\mathrm{i}\omega(t_\mathrm{n}+h)}, \quad (22)$$

Using Euler formula $e^{iv} = \cos(v) + isin(v)$, and substituting the equations (16)-(22) into equation (8)-(15). We detach the real and the imaginary part, therefore, we gain the trigonometrically-fitting order conditions:

$$\cos(v) = 1 + \frac{1}{2}\, v\, sin(v) - v^2 \sum_{i=2}^{3} d_i(\cos(c_i v) - \cos(c_i v - v)), \quad (23)$$

$$\sin(v) = \frac{3}{2} v - \frac{1}{2}\, v\, cos(v) - v^2 \sum_{i=2}^{3} d_i(\sin(c_i v) - \sin(c_i v - v)), \quad (24)$$

$$\cos(v) = 1 - b_{-1}\, v\, sin(v) - v \sum_{i=2}^{3} b_i(\sin(c_i v) - \sin(c_i v - v)), \quad (25)$$

$$\sin(v) = b_1 v - b_{-1}\, v\, cos(v) - v \sum_{i=2}^{3} b_i(\cos(c_i v) - \cos(c_i v - v)). \quad (26)$$

where $v = \omega h$. In this work, the three-stages of order four IRKN4 method is expressed as follows in Table 2 [9]:

Table 2. IRKN4 method

0			
$\frac{1}{4}$	$\frac{1}{32}$		
$\frac{3}{4}$	0	$\frac{9}{32}$	
$\frac{1}{18}$	$\frac{19}{18}$	$-\frac{1}{6}$	$\frac{11}{18}$
		$\frac{7}{24}$	$\frac{1}{8}$

To find the parameters of the TFIRKN4 method, we utilize the additional equations of order conditions for the IRKN4 method;

$$b_1 - b_{-1} = 1, \quad (27)$$

$$b_2 + b_3 + b_{-1} = \frac{1}{2}, \quad (28)$$

Therefore, we have six equations with six unknowns. Let $c_2 = \frac{1}{4}$, and $c_3 = \frac{3}{4}$ from Table 2, and choose $b_{-1}, b_1, b_2, b_3, d_2$ and d_3 as free parameters. Solving equations (23)-(28) simultaneously, we obtain

$$b_{-1} = \frac{2 - 2\cos(v) - v\sin\left(\frac{1}{4}v\right) - v\sin\left(\frac{3}{4}v\right)}{2v\left(\sin(v) - \sin\left(\frac{1}{4}v\right) - \sin\left(\frac{3}{4}v\right)\right)}, \quad (29)$$

$$b_1 = \frac{2 + 2v\sin(v) - 2\cos(v) - 3v\sin\left(\frac{1}{4}v\right) - 3v\sin\left(\frac{3}{4}v\right)}{2v\left(\sin(v) - \sin\left(\frac{1}{4}v\right) - \sin\left(\frac{3}{4}v\right)\right)}, \quad (30)$$

$$b_2 = \frac{-1}{4v\left(-\cos\left(\frac{3}{4}v\right) + \cos\left(\frac{1}{4}v\right)\right)\left(\sin(v) - sin\left(\frac{1}{4}v\right) - sin\left(\frac{3}{4}v\right)\right)} \Big(2\cos(v)^2 - 4\cos(v) +$$

$$2v\sin(v) + 2 - \cos\left(\frac{1}{4}v\right) v\sin(v) - 2\cos\left(\frac{1}{4}v\right) cos(v) + 2\cos\left(\frac{1}{4}v\right) +$$

$$\cos\left(\frac{3}{4}v\right) v\sin(v) + 2\cos\left(\frac{3}{4}v\right) cos\,(v) - 2\cos\left(\frac{3}{4}v\right) - 2\ \sin(v)^2 + 2\sin(v)\ sin\left(\frac{1}{4}v\right) + 2\sin(v)\ sin\left(\frac{3}{4}v\right) + v\cos(v)\ sin\left(\frac{1}{4}v\right) + v\cos(v)\ sin\left(\frac{3}{4}v\right) - 3\,v\sin\left(\frac{1}{4}v\right) - 3\,v\sin\left(\frac{3}{4}v\right)\ \Big), \tag{31}$$

$$b_3 = \frac{1}{4v\left(-\cos\left(\frac{3}{4}v\right)+\cos\left(\frac{1}{4}v\right)\right)\left(\sin(v)-sin\left(\frac{1}{4}v\right)-sin\left(\frac{3}{4}v\right)\right)}\Big(2\cos(v)^2 - 4\cos(v) + 2v\sin(v) + 2 + \cos\left(\frac{1}{4}v\right) v\sin(v) + 2\cos\left(\frac{1}{4}v\right) cos\,(v) - 2\cos\left(\frac{1}{4}v\right) - \cos\left(\frac{3}{4}v\right) v\sin(v) - 2\cos\left(\frac{3}{4}v\right) cos\,(v) + 2\cos\left(\frac{3}{4}v\right) - 2\ \sin(v)^2 + 2\sin(v)\ sin\left(\frac{1}{4}v\right) + 2\sin(v)\ sin\left(\frac{3}{4}v\right) + v\cos(v)\ sin\left(\frac{1}{4}v\right) + v\cos(v)\ sin\left(\frac{3}{4}v\right) - 3\,v\sin\left(\frac{1}{4}v\right) - 3\,v\sin\left(\frac{3}{4}v\right)\ \Big), \tag{32}$$

$$d_2 = \frac{-1}{4\left(v^2\left(\sin\left(\frac{1}{4}v\right)+\sin\left(\frac{3}{4}v\right)\right)\left(\cos\left(\frac{1}{4}v\right)-\cos\left(\frac{3}{4}v\right)\right)\right)}\Big(2\sin(v)\cos\left(\frac{1}{4}v\right) - 2\sin(v)\cos\left(\frac{3}{4}v\right) - 3v\cos\left(\frac{1}{4}v\right) + 3v\cos\left(\frac{3}{4}v\right) + v\cos(v)\cos\left(\frac{1}{4}v\right) - v\cos(v)\cos\left(\frac{3}{4}v\right) + 2\sin\left(\frac{1}{4}v\right)\ \cos(v) - 2\sin\left(\frac{1}{4}v\right) + \sin\left(\frac{1}{4}v\right)\ v\sin(v) + 2\sin\left(\frac{3}{4}v\right)\ \cos(v) - 2\sin\left(\frac{3}{4}v\right) + \sin\left(\frac{3}{4}v\right)\ v\sin(v)\Big), \tag{33}$$

$$d_3 = \frac{-1}{4\left(v^2\left(\sin\left(\frac{1}{4}v\right)+\sin\left(\frac{3}{4}v\right)\right)\left(\cos\left(\frac{1}{4}v\right)-\cos\left(\frac{3}{4}v\right)\right)\right)}\Big(2\sin(v)\cos\left(\frac{1}{4}v\right) - 2\sin(v)\cos\left(\frac{3}{4}v\right) - 3v\cos\left(\frac{1}{4}v\right) + 3v\cos\left(\frac{3}{4}v\right) + v\cos(v)\cos\left(\frac{1}{4}v\right) - v\cos(v)\cos\left(\frac{3}{4}v\right) - 2\sin\left(\frac{1}{4}v\right)\ \cos(v) + 2\sin\left(\frac{1}{4}v\right) - \sin\left(\frac{1}{4}v\right)\ v\sin(v) - 2\sin\left(\frac{3}{4}v\right)\ \cos(v) + 2\sin\left(\frac{3}{4}v\right) - \sin\left(\frac{3}{4}v\right)\ v\sin(v)\Big). \tag{34}$$

Thus, for the small value of v, the corresponding Taylor series expansions are presented as follows:

$$b_{-1} = \frac{1}{18} - \frac{1}{2160}\,v^2 - \frac{29}{2903040}\,v^4 - \frac{71}{464486400}\,v^6 - \frac{2603}{1177194332160}\,v^8 - \frac{8127943}{257099242143744000}\,v^{10} + \cdots, \tag{35}$$

$$b_1 = \frac{19}{18} - \frac{1}{2160}\,v^2 - \frac{29}{2903040}\,v^4 - \frac{71}{464486400}\,v^6 - \frac{2603}{1177194332160}\,v^8 - \frac{8127943}{257099242143744000}\,v^{10} + \cdots, \tag{36}$$

$$b_2 = -\frac{1}{6} + \frac{1}{576}\,v^2 - \frac{29}{258048}\,v^4 - \frac{311}{123863040}\,v^6 - \frac{102233}{1569592442880}\,v^8 + \cdots, \tag{37}$$

$$b_3 = \frac{11}{18} - \frac{11}{8640}\,v^2 + \frac{203}{1658880}\,v^4 + \frac{707}{265420800}\,v^6 + \frac{317111}{4708777328640}\,v^8 + \cdots, \tag{38}$$

$$d_2 = \frac{7}{24} - \frac{7}{11520}\,v^2 + \frac{1243}{15482880}\,v^4 + \frac{5809}{2477260800}\,v^6 + \frac{2292401}{31391848857600}\,v^8 + \cdots, \tag{39}$$

$$d_3 = \frac{1}{8} + \frac{7}{3840}\,v^2 + \frac{127}{1720320}\,v^4 + \frac{1951}{825753600}\,v^6 + \frac{152891}{2092789923840}\,v^8 + \cdots, \tag{40}$$

For $v \to 0$, TFIRKN4 method is reduced to the original IRKN4 method.

3. Numerical Results and Discussion

To assess the accuracy of the TFIRKN4 method, we have presented four problems that have vacillating solutions. The maximum absolute error is.

$$MaxError = max(|u(t_n) - u_n|).$$

where $u(t_n)$ is the exact solution and u_n is the numerical solution. Figs. 1 to 4 reveal the competence curves of $Log_{10}(MaxError)$ against the step length h. In the comparison, we used the following methods:

- **TFIRKN4:** The new IRKN formula derived herein.
- **IRKN4:** The three-stage fourth-order IRKN4 given in [9].
- **RKN4G:** RKN4 method is presented in [14].
- **RKN4X:** RKN 4 code is given in [15].
- **RKN4D:** RKN4 code is given in [16].

Problem 1: [17]

$$u''(t) = -u(t) + t,\ \ u(0) = 1,\ \ u'(t) = 2.$$

Exact solution: $u(t) = \sin(t) + \cos(t) + t.$

Problem 2: [18]

$$u_1''(t) = -\frac{101}{2}u_1(t) + \frac{99}{2}u_2(t) + \frac{93}{2}\ cos(2t) - \frac{99}{2}\ sin(2t), u_1(0) = 0, u_1'(0) = -10,$$

$$u_2''(t) = \frac{99}{2}\ u_1(t) - \frac{101}{2}\ u_2(t) + \frac{93}{2}\ sin(2t) - \frac{99}{2}\ cos(2t),\ u_2(0) = 1, u_2'(0) = 12.$$

Exact solution:

$$u_1(t) = -\cos(10t) - \sin(10t) + \cos(2t),$$
$$u_2(t) = \cos(10t) + \sin(10t) + \sin(2t).$$

Problem 3: [19]

$$u_1''(t) = -400\, u_1(t) + (400\ + 0.0025)e^{-0.05t},\ u_1(0) = 1.1, u_1'(0) = -0.05,$$
$$u_2''(t) = -400\, u_2(t) + (400\ + 0.0025)e^{-0.05t},\ u_2(0) = 1.0, u_2'(0) = 1.95$$

Exact solution:

$$u_1(t) = 0.1\cos(20\ t) + e^{-0.05\ t},$$
$$u_1(t) = 0.1\sin(20\ t) + e^{-0.05\ t}.$$

Problem 4: [20]

$$u_1''(t) + u_1(t) = 0.001\ \cos(t),\ \ u_1(0) = 1,\ \ u_1'(0) = 0,$$
$$u_2''(t) + u_2(t) = 0.001\ \sin(t),\ \ u_2(0) = 0,\ \ u_2'(0) = 0.9995.$$

Exact solution:

$$u_1(t) = \cos(t) + 0.0005\ t\sin(t),$$
$$u_2(t) = \sin(t) - 0.0005\ t\cos(t).$$

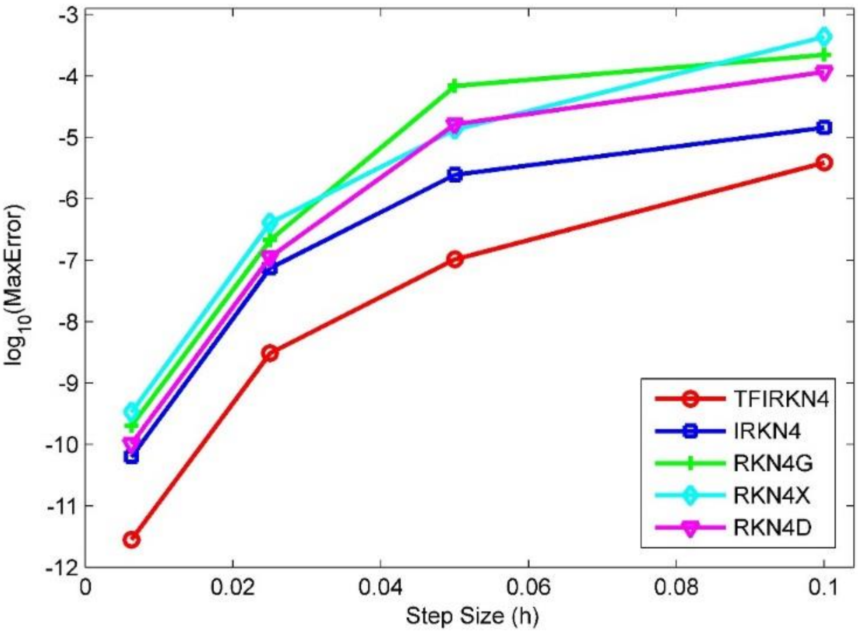

Fig. 1: Competence curves with step size $h = \frac{0.1}{2^r}, r = 0,1,2,4.$ for problem1

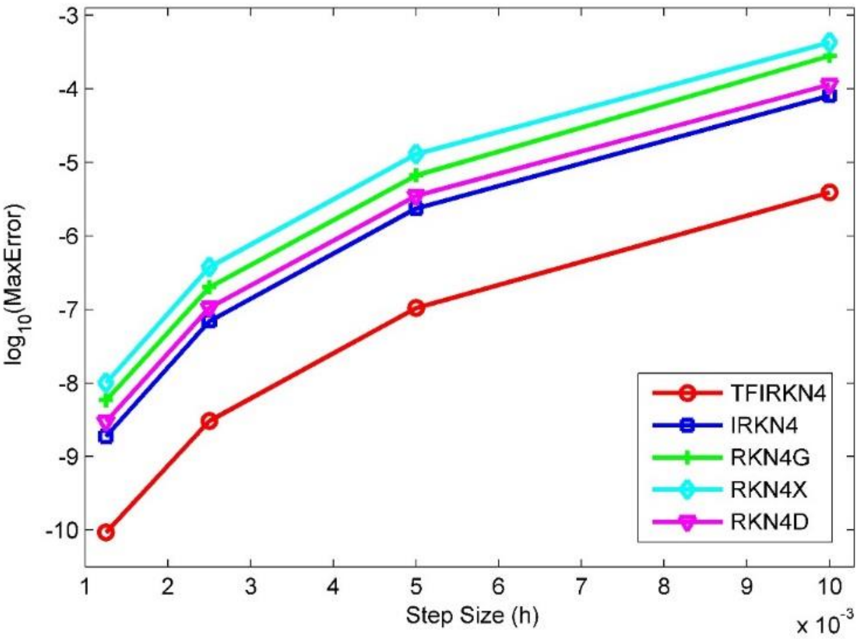

Fig. 2: Competence curves with step size $h = \frac{0.01}{2^r}, r = 0,1,2,3.$ for problem 2

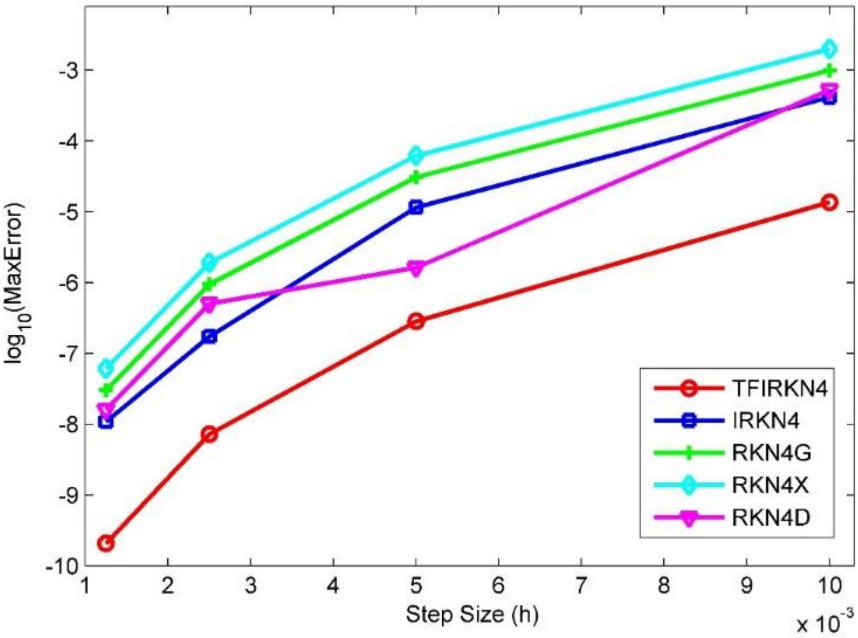

Fig. 3: Competence curves with step size $h = \frac{0.01}{2^r}, r = 0,1,2,3.$ for problem 3

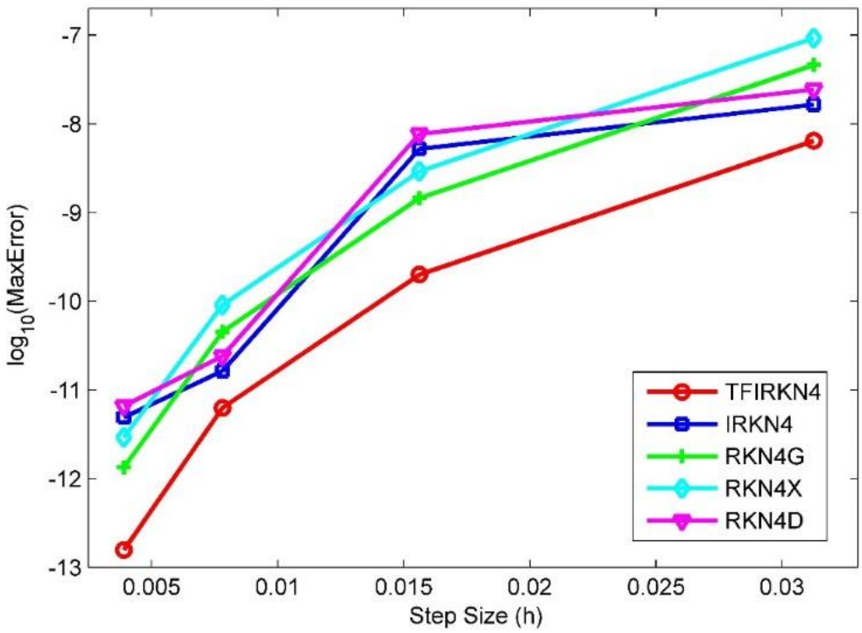

Fig. 4: Competence curves with step size $h = \frac{0.03125}{2^r}, r = 0,1,2,3.$ for problem 4

In Figs. 1-4, numerical results are displayed for solving second-order ODEs with integration interval [0, 1000], and the TFIRKN4 approach is compared to the existing IRKN4, RKN4G, RKN4X, and RKN4D methods. As the number of step sizes h decreases, it is obvious that the maximum global error reduces. In solving four different numerical tests, the suggested method, TFIRKN4, has the least maximum global error of the four methods. We show the effectiveness of the suggested method for the inhomogeneous problem in Fig. 1, and the methods of TFIRKN4, IRKN4, RKN4D,RKN4X, and RKN4G are shown in decreasing order of efficiency. We demonstrate the efficacy of the five techniques for the nonlinear nonhomogeneous system in Figs. 2 to 4. We note that the new TFIRKN4 approach is more effective than IRKN4, RKN4D, RKN4X, and RKN4G methods in terms of accuracy.

4. Conclusions

In this article, we derived trigonometrically-fitted conditions of the IRKN method for solving oscillatory problems. Consequently, we constructed a trigonometrically-fitted three-stage fourth-order IRKN method denoted as the TFIRKN4 method. Numerical results are presented which show that the TFIRKN4 method is more accurate and effective than current methods for solving second-order IVPs that have oscillatory solutions.

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