Numerical Integration Method for a Class of Singularly Perturbed Differential-Difference Equations

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Abstract
In this paper a class of singularly perturbed differential-difference equation having boundary layer at one end is analysed to get its solution by numerical integration method. Taylor's series expansion is applied on negative and positive shifts to get singularly perturbed differential equation. An asymptotically equivalent first order differential equation is obtained from SPDE using Taylor's transformation. To integrate resulting equation, composite Simpson's 1/3 rule is used to get three term recurrence relation. Thomas algorithm is used to get the solution of tridiagonal system of equations. Numerical solution obtained from this method approximates the available/exact solution very well.

Keywords- Singular perturbation, boundary layer, numerical integration.

1.INTRODUCTION

Due to the availability of supercomputing and cloud computing, now Mathematicians are seriously concentrating on developing the robust numerical methods for solving most challenging problems like Boundary Layer Problems. In general, a region in which the solution of the problem changes rapidly is called Boundary Layer. In fact the solution changes rapidly to satisfy the given conditions in the problem. Any ordinary differential equation in which the highest order derivative is multiplied by a small positive parameter which is popularly known as singularly perturbation problem always exhibits the boundary layer phenomenon. Also, any differential equation which contains at least one delay/advance parameters which is popularly called as delay/differential-difference equation also exhibits the boundary layer phenomenon. Solving these problems is very difficult due to the boundary layer phenomenon. These problems arise in the modelling of various practical phenomena in bioscience, engineering, control theory, such as in variational problems in control theory, in describing the human pupil-light reflex, in a variety of models for physiological processes or diseases and first exit time problems in the modelling of the determination of expected time for the generation of action potential in nerve cells by random synaptic inputs in dendrites. To solve these problems, perturbation methods such as Matched Asymptotic Expansions, WKB method are used extensively. These asymptotic expansions of solutions require skill, insight, and experimentation. Further, the Matching Principle: matching of the coefficients of the inner and outer regions solution expansions is also a demanding process. Hence, researchers started developing numerical methods. If we use the existing numerical methods with the step size more than the perturbation parameters, for solving these problems we get oscillatory solutions due to the presence of the boundary layer. Existing numerical methods will produce good results only when we take step size less than the perturbation parameters. This is very costly and time-consuming process. Hence, the researchers are concentrating on developing robust numerical methods, which can work with a reasonable step size. In fact, these robust numerical methods should be independent of the parameters. The efficiency of such numerical method is determined by its accuracy, simplicity in computing the solution and its sensitivity to the parameters of the given problem. For a detailed theoretical and numerical treatment, one can see the books and papers: [1-31]. With this motivation, we, present here, in this paper**,** a class of singularly perturbed differential-difference equation having boundary layer at one end is analysed to get its solution by numerical integration method. Taylor's series expansion is applied on negative and positive shifts to get singularly perturbed differential equation. An asymptotically equivalent first order differential equation is obtained from SPDE using Taylor's transformation. To integrate resulting equation, composite Simpson's 1/3 rule is used to get three term recurrence relation. Thomas algorithm is used to get the solution of tri-diagonal system of equations. Several model examples are solved to demonstrate the applicability of these methods. The solutions are tabulated and compared with the available/exact solutions. It is observed that our methods approximate the exact solution very well.

2.0 DESCRIPTION OF THE FITTED METHOD

2.1 TYPE-I: DELAY DIFFERENTIAL EQUATION HAVING BOUNDARY LAYER

Consider the delay differential equation of the form

$$\varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = f(x), \quad 0 \le x \le 1, \tag{1}$$

with boundary conditions

$$y(x) = \varphi(x), \quad -\delta \le x \le 0, \tag{2}$$

and
$$y(1) = \beta \tag{3}$$

where $0 < \varepsilon \ll 1$ is the perturbation parameter, $0 < \delta = O(\varepsilon)$ is the small delay parameter, $a(x)$, $b(x)$and $f(x)$are sufficiently differentiable functions in $(0,1)$.$\varphi(x)$ is also bounded continuous function on $[0,1]$ and β is a finite constant.

From the Taylor's series expansion $y'(x-\delta) \approx y'(x) - \delta y''(x)$ (4)

Substitute equation (4) into equation (1), we get

$$\varepsilon' y''(x) + A(x)y'(x) + B(x)y(x) = f(x), 0 \le x \le 1 \tag{5}$$

with boundary conditions

$$y(0) = \alpha \tag{6}$$

$$y(1) = \beta \tag{7}$$

where $\varepsilon' = \varepsilon - a(x)\delta$, $A(x) = a(x)$, $B(x) = b(x)$ and α is a finite constant. Further it is established that, if$a(x) \ge M > 0$ in [0, 1], then equation (1) has unique solution and a boundary layer at $x = 0$ and if$a(x) \le M < 0$ in [0, 1], then equation (1) has unique solution and a boundary layer at $x = 1$, where M is some positive number. Here we assume that $a(x) = a$ and $b(x) = b$ are constants for computational point of view.

2.2 TYPE-II: DIFFERENTIAL-DIFFERENCE EQUATION HAVING BOUNDARY LAYER

Consider the differential-difference equation of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x-\delta) + c(x)y(x) + d(x)y(x+\eta) = f(x), \quad (8)$$

$0 \leq x \leq 1$ with boundary conditions

$$y(x) = \varphi(x), \text{ on } -\delta \leq x \leq 0, \quad (9)$$

$$y(x) = \gamma(x), \text{ on } 1 \leq x \leq 1 + \eta, \quad (10)$$

with the constant coefficients (i.e.,$a(x) = a, b(x) = b, c(x) = c$ and $d(x) = d$ are constants) and $f(x)$, $\varphi(x)$ and $\gamma(x)$ are smooth functions. $0 < \varepsilon \ll 1$ is the perturbation parameter, $0 < \delta = O(\varepsilon)$ and $0 < \eta = O(\varepsilon)$ are the delay and advanced parameters respectively.

From Taylor's series expansion

$$y(x-\delta) \approx y(x) - \delta y'(x) + \frac{\delta^2}{2}y''(x) \quad (11)$$

$$y(x+\eta) \approx y(x) + \eta y'(x) + \frac{\eta^2}{2}y''(x) \quad (12)$$

Substitute equations (11) and (12) into equation (8), we get

$$\varepsilon' y''(x) + A(x)y'(x) + B(x)y(x) = f(x), 0 \leq x \leq 1 \quad (13)$$

with boundary conditions

$$y(0) = \alpha \quad (14)$$

$$y(1) = \beta \quad (15)$$

where

$$\varepsilon' = \varepsilon + b(x)\frac{\delta^2}{2} + d(x)\frac{\eta^2}{2} \quad (16)$$

$$A(x) = a(x) - \delta b(x) + \eta d(x) \quad (17)$$

$$B(x) = b(x) + c(x) + d(x) \quad (18)$$

Since $0 < \delta \ll 1$ and $0 < \eta \ll 1$, the transition fromequation (1) to equation (5) orequation (8) to equation (13) is admitted. For more details on the validity of this transition, one can refer El'sgolt's and Norkin [11]. The behaviour of the boundary layer is given by the sign of $A(x)$ and $B(x)$. Further it is established that, if $B(x) \leq 0$, $A(x) \geq M > 0$ in [0, 1] then equation (8) has unique solution and a boundary layer at $x = 0$and if $B(x) \leq 0$, $A(x) \leq M < 0$ in [0, 1] then equation (8) has unique solution and a boundary layer at $x = 1$, where M is a positive number.

2.3. CASE (I): FOR LEFT-END BOUNDARY LAYER

Consider equation (5) or (13) with their boundary conditions

$$\varepsilon' y''(x) + A(x)y'(x) + B(x)y(x) = f(x), 0 \leq x \leq 1 \quad (19)$$

$$y(0) = \alpha \quad (20)$$

$$y(1) = \beta \tag{21}$$

From Taylor's series expansion about the deviating argument $\sqrt{\varepsilon'}$ in the neighbourhood of the point$x$, we have

$$y(x-\sqrt{\varepsilon'}) \approx y(x) - \sqrt{\varepsilon'}y'(x) + \frac{\varepsilon'}{2}y''(x) \tag{22}$$

From equation (19) and (22), we have

$$y'(x) = p(x)y(x-\sqrt{\varepsilon'}) + q(x)y(x) + r(x) \tag{23}$$

where

$$p(x) = \frac{-2}{2\sqrt{\varepsilon'} + A(x)} \tag{24}$$

$$q(x) = \frac{2 - B(x)}{2\sqrt{\varepsilon'} + A(x)} \tag{25}$$

$$r(x) = \frac{f(x)}{2\sqrt{\varepsilon'} + A(x)} \tag{26}$$

The transition from equation (19) to (23) is valid, because of the condition that $\sqrt{\varepsilon'}$ is small. For more details on the validity of this transition, one can refer El'sgolt's and Norkin [11].

Now, we divide the interval [0, 1] into n equal parts with constant mesh length $h = 1/n$.

Let $0 = x_0,\ x_1, \dots, x_n = 1$ be the mesh points, then we have $x_i = ih,\ i = 0, 1, 2, \dots, n$. From our earlier assumptions, $A(x)$ and $B(x)$ are constants. Therefore, $p(x)$ and $q(x)$ are constants. Equation (23) can be written as

$$y'(x) - qy(x) = py(x-\sqrt{\varepsilon'}) + r(x) \tag{27}$$

We take an integrating factor e^{-qx} to equation (27) and producing (as in McCartin [23])

$$\frac{d}{dx}[e^{-qx}y(x)] = e^{-qx}[py(x-\sqrt{\varepsilon'}) + r(x)] \tag{28}$$

On integrating equation (28) from x_i to x_{i+1}, we get

$$e^{-qx_{i+1}}y_{i+1} - e^{-qx_i}y_i = \int_{x_i}^{x_{i+1}} e^{-qx}\, py(x-\sqrt{\varepsilon'})dx + \int_{x_i}^{x_{i+1}} e^{-qx}\, r(x)dx \tag{29}$$

We evaluate integrals present in right hand side of equation (16) by Composite Simpson's 1/3 rule on $[x_i x_{i+1}]$.

$$e^{-qx_{i+1}}y_{i+1} = e^{-qx_i}y_i + \frac{ph}{12}\left[e^{-qx_i}y(x_i-\sqrt{\varepsilon'}) + e^{-qx_{i+1}}y(x_{i+1}-\sqrt{\varepsilon'}) + 2e^{-qx_{i+\frac{1}{2}}}y\left(x_{i+\frac{1}{2}} - \sqrt{\varepsilon'}\right) + 4\left\{e^{-qx_{i+\frac{1}{4}}}y\left(x_{i+\frac{1}{4}} - \sqrt{\varepsilon'}\right) + e^{-qx_{i+\frac{3}{4}}}y\left(x_{i+\frac{3}{4}} - \sqrt{\varepsilon'}\right)\right\}\right] + \frac{h}{12}\left[e^{-qx_i}r_i + e^{-qx_{i+1}}r_{i+1} + 2e^{-qx_{i+\frac{1}{2}}}r_{i+\frac{1}{2}} + 4\left\{e^{-qx_{i+\frac{1}{4}}}r_{i+\frac{1}{4}} + e^{-qx_{i+\frac{3}{4}}}r_{i+\frac{3}{4}}\right\}\right] \tag{30}$$

$$y_{i+1} = e^{qh}y_i + \frac{ph}{12}\left[e^{qh}y(x_i-\sqrt{\varepsilon'}) + y(x_{i+1}-\sqrt{\varepsilon'}) + 2e^{\frac{qh}{2}}y\left(x_{i+\frac{1}{2}} - \sqrt{\varepsilon'}\right) + 4\left\{e^{\frac{3qh}{4}}y\left(x_{i+\frac{1}{4}} - \sqrt{\varepsilon'}\right) + e^{\frac{qh}{4}}y\left(x_{i+\frac{3}{4}} - \sqrt{\varepsilon'}\right)\right\}\right] + \frac{h}{12}\left[e^{qh}r_i + r_{i+1} + 2e^{\frac{qh}{2}}r_{i+\frac{1}{2}} + 4\left\{e^{\frac{3qh}{4}}r_{i+\frac{1}{4}} + e^{\frac{qh}{4}}r_{i+\frac{3}{4}}\right\}\right] \quad (31)$$

We approximate the terms involve in equation (31), from Taylor's series expansion and finite difference approximation,

$$y\left(x_{i+\frac{1}{2}}-\sqrt{\varepsilon'}\right)\approx\left(\frac{3}{2}-\frac{\sqrt{\varepsilon'}}{h}\right)y_i-\left(\frac{1}{2}-\frac{\sqrt{\varepsilon'}}{h}\right)y_{i-1} \tag{32}$$

$$y\left(x_{i+\frac{1}{4}}-\sqrt{\varepsilon'}\right)\approx\left(\frac{5}{4}-\frac{\sqrt{\varepsilon'}}{h}\right)y_i-\left(\frac{1}{4}-\frac{\sqrt{\varepsilon'}}{h}\right)y_{i-1} \tag{33}$$

$$y\left(x_{i+\frac{3}{4}}-\sqrt{\varepsilon'}\right)\approx\left(\frac{7}{4}-\frac{\sqrt{\varepsilon'}}{h}\right)y_i-\left(\frac{3}{4}-\frac{\sqrt{\varepsilon'}}{h}\right)y_{i-1} \tag{34}$$

$$y\left(x_i-\sqrt{\varepsilon'}\right)\approx\left(1-\frac{\sqrt{\varepsilon'}}{h}\right)y_i+\frac{\sqrt{\varepsilon'}}{h}y_{i-1} \tag{35}$$

$$y\left(x_{i+1}-\sqrt{\varepsilon'}\right)\approx\left(1-\frac{\sqrt{\varepsilon'}}{h}\right)y_{i+1}+\frac{\sqrt{\varepsilon'}}{h}y_i \tag{36}$$

$$r_{i+\frac{1}{2}}\approx\frac{3}{2}r_i-\frac{1}{2}r_{i-1} \tag{37}$$

$$r_{i+\frac{1}{4}}\approx\frac{5}{4}r_i-\frac{1}{4}r_{i-1} \tag{38}$$

$$r_{i+\frac{3}{4}}\approx\frac{7}{4}r_i-\frac{3}{4}r_{i-1} \tag{39}$$

Substitute above equations into equation (31), we get

$$E_i y_{i-1}-F_i y_i+G_i y_{i+1}=H_i,\quad i=1,2,\dots,n-1 \tag{40}$$

where

$$E_i=\frac{ph}{12}\left[-\frac{\sqrt{\varepsilon'}}{h}e^{qh}+2e^{\frac{qh}{2}}\left(\frac{1}{2}-\frac{\sqrt{\varepsilon'}}{h}\right)+4e^{\frac{3qh}{4}}\left(\frac{1}{4}-\frac{\sqrt{\varepsilon'}}{h}\right)+4e^{\frac{qh}{4}}\left(\frac{3}{4}-\frac{\sqrt{\varepsilon'}}{h}\right)\right]$$

$$F_i=e^{qh}+\frac{ph}{12}\left[e^{qh}\left(1-\frac{\sqrt{\varepsilon'}}{h}\right)+\frac{\sqrt{\varepsilon'}}{h}+2e^{\frac{qh}{2}}\left(\frac{3}{2}-\frac{\sqrt{\varepsilon'}}{h}\right)+4e^{\frac{3qh}{4}}\left(\frac{5}{4}-\frac{\sqrt{\varepsilon'}}{h}\right)\right]$$

$$G_i=1-\frac{ph}{12}\left(1-\frac{\sqrt{\varepsilon'}}{h}\right)$$

$$H_i=\frac{h}{12}\left[e^{qh}r_i+\ r_{i+1}+2e^{\frac{qh}{2}}\left\{\frac{3}{2}r_i-\frac{1}{2}r_{i-1}\right\}+4e^{\frac{3qh}{4}}\left\{\frac{5}{4}r_i-\frac{1}{4}r_{i-1}\right\}+4e^{\frac{qh}{4}}\left\{\frac{7}{4}r_i-\frac{3}{4}r_{i-1}\right\}\right]$$

This is a tridiagonal system of $n-1$ equations. We solve this tridiagonal system with given two boundary conditions by Thomas algorithm.

2.4. CASE (II): FOR RIGHT-END BOUNDARY LAYER

Consider equation (5) or (13) with their boundary conditions

$$\varepsilon' y''(x)+A(x)y'(x)+B(x)y(x)=f(x),0\le x\le 1 \tag{41}$$

$$y(0)=\alpha \tag{42}$$

$$y(1)=\beta \tag{43}$$

From Taylor's series expansion about the deviating argument $\sqrt{\varepsilon'}$ in the neighbourhood of the point x, we have

$$y\left(x+\sqrt{\varepsilon'}\right) \approx y(x) + \sqrt{\varepsilon'}y'(x) + \frac{\varepsilon'}{2}y''(x) \tag{44}$$

From equation (41) and (44), we have

$$y'(x) = p(x)y\left(x+\sqrt{\varepsilon'}\right) + q(x)y(x) + r(x) \tag{45}$$

where

$$p(x) = \frac{-2}{-2\sqrt{\varepsilon'}+A(x)} \tag{46}$$

$$q(x) = \frac{2-B(x)}{-2\sqrt{\varepsilon'}+A(x)} \tag{47}$$

$$r(x) = \frac{f(x)}{-2\sqrt{\varepsilon'}+A(x)} \tag{48}$$

Let $0 = x_0,\ x_1, \dots, x_n = 1$ be the mesh points, then we have $x_i = ih,\ i = 0, 1, 2, \dots, n$. From our earlier assumptions, $A(x)$ and $B(x)$ are constants. Therefore, $p(x)$ and $q(x)$ are constants. Equation (45) can be written as

$$y'(x) - qy(x) = py\left(x+\sqrt{\varepsilon'}\right) + r(x) \tag{49}$$

We take an integrating factor e^{-qx} to equation (49) and producing (as in Mc Cartin[23])

$$\frac{d}{dx}\left[e^{-qx}y(x)\right] = e^{-qx}\left[py\left(x+\sqrt{\varepsilon'}\right) + r(x)\right] \tag{50}$$

On integrating equation (50) from x_{i-1} to x_i, we get

$$e^{-qx_i}y_i - e^{-qx_{i-1}}y_{i-1} = \int_{x_{i-1}}^{x_i} e^{-qx}\, py\left(x+\sqrt{\varepsilon'}\right)dx + \int_{x_{i-1}}^{x_i} e^{-qx}\, r(x)dx \tag{51}$$

We evaluate integrals present in right hand side of equation (51) by Composite Simpson's 1/3 rule on $[x_{i-1}x_i]$.

$$\begin{aligned} y_i = {} & e^{qh}y_{i-1} + \frac{ph}{12}\left[e^{qh}y\left(x_{i-1}+\sqrt{\varepsilon'}\right) + y\left(x_i+\sqrt{\varepsilon'}\right) + 2e^{\frac{qh}{2}}y\left(x_{i-\frac{1}{2}}+\sqrt{\varepsilon'}\right)\right. \\ & \left. + \ 4\left\{e^{\frac{3qh}{4}}y\left(x_{i-\frac{3}{4}}+\sqrt{\varepsilon'}\right) + e^{\frac{qh}{4}}y\left(x_{i-\frac{1}{4}}+\sqrt{\varepsilon'}\right)\right\}\right] \\ & + \frac{h}{12}\left[e^{qh}r_{i-1} + r_i + 2e^{\frac{qh}{2}}r_{i-\frac{1}{2}} + 4\left\{e^{\frac{3qh}{4}}r_{i-\frac{3}{4}} + e^{\frac{qh}{4}}r_{i-\frac{1}{4}}\right\}\right] \end{aligned} \tag{52}$$

We approximate the terms involve in equation (52), from Taylor's series expansion and finite difference approximation,

$$y\left(x_{i-\frac{1}{2}}+\sqrt{\varepsilon'}\right) \approx \left(\frac{3}{2}-\frac{\sqrt{\varepsilon'}}{h}\right)y_i - \left(\frac{1}{2}-\frac{\sqrt{\varepsilon'}}{h}\right)y_{i+1} \tag{53}$$

$$y\left(x_{i-\frac{1}{4}}+\sqrt{\varepsilon'}\right) \approx \left(\frac{5}{4}-\frac{\sqrt{\varepsilon'}}{h}\right)y_i - \left(\frac{1}{4}-\frac{\sqrt{\varepsilon'}}{h}\right)y_{i+1} \tag{54}$$

$$y\left(x_{i-\frac{3}{4}}+\sqrt{\varepsilon'}\right) \approx \left(\frac{7}{4}-\frac{\sqrt{\varepsilon'}}{h}\right)y_i - \left(\frac{3}{4}-\frac{\sqrt{\varepsilon'}}{h}\right)y_{i+1} \tag{55}$$

$$y\left(x_i+\sqrt{\varepsilon'}\right) \approx \left(1-\frac{\sqrt{\varepsilon'}}{h}\right)y_i + \frac{\sqrt{\varepsilon'}}{h}y_{i+1} \tag{56}$$

$$y\left(x_{i-1}+\sqrt{\varepsilon'}\right) \approx \left(1-\frac{\sqrt{\varepsilon'}}{h}\right)y_{i-1} + \frac{\sqrt{\varepsilon'}}{h}y_i \tag{57}$$

$$r_{i-\frac{1}{2}} \approx \frac{1}{2} r_{i-1} + \frac{1}{2} r_i \tag{58}$$

$$r_{i-\frac{1}{4}} \approx \frac{1}{4} r_{i-1} + \frac{3}{4} r_i \tag{59}$$

$$r_{i-\frac{3}{4}} \approx \frac{3}{4} r_{i-1} + \frac{1}{4} r_i \tag{60}$$

Substitute above equations into equation (52), we get

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 1, 2, \ldots, n-1 \tag{61}$$

where $E_i = -e^{qh} - \frac{ph}{12} e^{qh} \left(1 - \frac{\sqrt{\varepsilon'}}{h}\right)$

$$F_i = -1 + \frac{ph}{12}\left[\frac{\sqrt{\varepsilon'}}{h} e^{qh} + \left(1 - \frac{\sqrt{\varepsilon'}}{h}\right) + 2e^{\frac{qh}{2}}\left(\frac{3}{2} - \frac{\sqrt{\varepsilon'}}{h}\right) + 4e^{\frac{3qh}{4}}\left(\frac{7}{4} - \frac{\sqrt{\varepsilon'}}{h}\right) + 4e^{\frac{qh}{4}}\left(\frac{5}{4} - \frac{\sqrt{\varepsilon'}}{h}\right)\right]$$

$$G_i = -\frac{ph}{12}\left[\frac{\sqrt{\varepsilon'}}{h} - 2e^{\frac{qh}{2}}\left(\frac{1}{2} - \frac{\sqrt{\varepsilon'}}{h}\right) - 4e^{\frac{3qh}{4}}\left(\frac{3}{4} - \frac{\sqrt{\varepsilon'}}{h}\right) - 4e^{\frac{qh}{4}}\left(\frac{1}{4} - \frac{\sqrt{\varepsilon'}}{h}\right)\right]$$

$$H_i = \frac{h}{12}\left[e^{qh} r_{i-1} + r_i + 2e^{\frac{qh}{2}}\left\{\frac{1}{2} r_{i-1} + \frac{1}{2} r_i\right\} + 4e^{\frac{3qh}{4}}\left\{\frac{3}{4} r_{i-1} + \frac{1}{4} r_i\right\} + 4e^{\frac{qh}{4}}\left\{\frac{1}{4} r_{i-1} + \frac{3}{4} r_i\right\}\right]$$

This is a tridiagonal system of $n-1$ equations. We solve this tridiagonal system with given two boundary conditions by Thomas algorithm.

1. THOMAS ALGORITHM

Consider a recurrence relation (see [3])

$$y_i = \mathcal{W}_i y_{i+1} + \mathcal{T}_i \tag{62}$$

where $\mathcal{W}_i = \mathcal{W}(x_i)$ and $\mathcal{T}_i = \mathcal{T}(x_i)$ are to be found. From equation (62), we have

$$y_{i-1} = \mathcal{W}_{i-1} y_i + \mathcal{T}_{i-1} \tag{63}$$

From equation (40) or (61) and equation (63), we get

$$y_i = \frac{G_i}{F_i - E_i \mathcal{W}_{i-1}} y_{i+1} + \frac{E_i \mathcal{T}_{i-1} - H_i}{F_i - E_i \mathcal{W}_{i-1}} \tag{64}$$

On comparison of equations (62) and (64), we get

$$\mathcal{W}_i = \frac{G_i}{F_i - E_i \mathcal{W}_{i-1}} \tag{65}$$

and

$$\mathcal{T}_i = \frac{E_i \mathcal{T}_{i-1} - H_i}{F_i - E_i \mathcal{W}_{i-1}} \tag{66}$$

To find $\mathcal{W}_i$ and $\mathcal{T}_i$ for $1 \le i \le n-1$, we require initial conditions for $\mathcal{W}_0$ and $\mathcal{T}_0$. This can be done by considering equation (20) or (42), we get

$$y_0 = \alpha = \mathcal{W}_0 y_1 + \mathcal{T}_0 \tag{67}$$

If we take $\mathcal{W}_0 = 0$ then $\mathcal{T}_0 = \alpha$. Now, we determine $\mathcal{W}_i$ and $\mathcal{T}_i$ sequentially from equation (65) and (66). Hence, we compute y_i in the reverse process from equation (62) and equation (21). Under following assumptions

$$E_i > 0, \qquad G_i > 0, \qquad F_i \ge E_i + G_i \quad \text{and} |E_i| \le |G_i| \tag{68}$$

the above scheme is stable (see [24]). Under the assumptions, $\varepsilon' > 0, A(x) > 0$ and $B(x) < 0$, it is easy to show that, the above equation (68) holds. Therefore, the invariant imbedding scheme is stable.

4. ANALYSIS FOR STABILITY AND CONVERGENCE

Theorem 4.1. With the assumptions$A(x) \geq \mathcal{M} > 0$, $\varepsilon' > 0$, and $B(x) < 0$, $\forall 0 \leq x \leq 1$, the solution of equation (40) with conditions given at boundary exists and it's unique and also satisfies

$$||y||_{h,\infty} \leq \mathcal{M}^{-1}||f||_{h,\infty} + |\alpha| + |\beta| \tag{69}$$

where

$||x||_{h,\infty} = \underbrace{\max|x_i|}_{0\leq i\leq n}$ is the discrete l_∞ norm.

Proof. To prove the above theorem, one can refer (Kadalbajoo and Sharma [16]). Theorem 4.1. states that the solution from the above numerical method is uniformly bounded and independent from h and ε. Therefore, method is stable.

Equation (40) in matrix form, we get

$$AY = C \tag{70}$$

where $A = [m_{ij}]$ is a "tridiagonal matrix" of order $n-1$, $1 \leq i,j \leq n-1$ with

$$m_{ii+1} = 1 - \frac{ph}{12}\left(1 - \frac{\sqrt{\varepsilon'}}{h}\right)$$

$$m_{ii} = -e^{qh} - \frac{ph}{12}\left[e^{qh}\left(1 - \frac{\sqrt{\varepsilon'}}{h}\right) + \frac{\sqrt{\varepsilon'}}{h} + 2e^{\frac{qh}{2}}\left(\frac{3}{2} - \frac{\sqrt{\varepsilon'}}{h}\right) + 4e^{\frac{3qh}{4}}\left(\frac{5}{4} - \frac{\sqrt{\varepsilon'}}{h}\right) + 4e^{\frac{qh}{4}}\left(\frac{7}{4} - \frac{\sqrt{\varepsilon'}}{h}\right)\right]$$

$$m_{ii-1} = \frac{ph}{12}\left[-\frac{\sqrt{\varepsilon'}}{h}e^{qh} + 2e^{\frac{qh}{2}}\left(\frac{1}{2} - \frac{\sqrt{\varepsilon'}}{h}\right) + 4e^{\frac{3qh}{4}}\left(\frac{1}{4} - \frac{\sqrt{\varepsilon'}}{h}\right) + 4e^{\frac{qh}{4}}\left(\frac{3}{4} - \frac{\sqrt{\varepsilon'}}{h}\right)\right]$$

$$Y = (y_1, y_2, \dots, y_{n-1})^t$$

and $C = (d_i)$ is a column vector with

$$d_i = \frac{h}{12}\left[e^{qh}r_i + r_{i+1} + 2e^{\frac{qh}{2}}\left\{\frac{3}{2}r_i - \frac{1}{2}r_{i-1}\right\} + 4e^{\frac{3qh}{4}}\left\{\frac{5}{4}r_i - \frac{1}{4}r_{i-1}\right\} + 4e^{\frac{qh}{4}}\left\{\frac{7}{4}r_i - \frac{3}{4}r_{i-1}\right\}\right]$$

where $i = 1,2,\dots,n-1$ with truncation error

$$T_i(h) = h^2\left[-\frac{1}{2}pqy_i - \frac{1}{2}py_i' + \frac{1}{2}pq\sqrt{\varepsilon'}y_i' - \frac{5}{12}p\sqrt{\varepsilon'}y_i'' + \frac{1}{2}y_i'' - \frac{1}{2}qr_i\right] + O(h^3) \tag{71}$$

$$A\bar{Y} - T(h) = C \tag{72}$$

where $\bar{Y} = (\overline{y_1}, \overline{y_2}, \dots, \overline{y_{n-1}})^t$ denotes exact solution,

$T(h) = (T_1(h), T_2(h), \dots, T_{n-1}(h))^t$ denotes truncation error column vector.

From equation (70) and (72), we obtain

$$A(\bar{Y} - Y) = T(h) \tag{73}$$

Therefore, error equation becomes $\qquad AE = T(h) \qquad (74)$

here, $E = \bar{Y} - Y = (e_1, e_2, \dots, e_{n-1})^t$.

Assume that S_i is the sum of i^{th} row elements of matrix A, we obtain

$S_i = \frac{11}{12}p\sqrt{\varepsilon'} + h\left[-q - \frac{17}{12}p + \frac{1}{2}pq\sqrt{\varepsilon'}\right] + O(h^2)$ for $i = 1$

$S_i = h[-p-q] + O(h^2)$ for $i = 2,3,\dots,n-2$

$S_i = -\frac{1}{12}p\sqrt{\varepsilon'} - 1 + h\left[-q - \frac{11}{12}p\right] + O(h^2)$ for $i = n-1$

Since$1 \gg \varepsilon > 0$, $O(\varepsilon) = \delta > 0$ and $O(\varepsilon) = \eta > 0$, A is irreducible and monotone for sufficiently small step size (see [27]). Then, A becomes invertible and elements of A^{-1} are nonnegative.

From equation (74), we obtain

$$E = A^{-1}T(h) \tag{75}$$

After taking norm on both sides, we get

$$||E|| \le ||A^{-1}||.||T(h)|| \tag{76}$$

Consider $A^{-1} = [u_{ki}]$, $1 \le k, i \le n-1$. As $u_{ki} \ge 0$, hence from matrix analysis we have

$$\sum_{i=1}^{n-1} u_{ki}S_i = 1, \quad k = 1,2,\dots,n-1 \tag{77}$$

From equation (77), we get

$$\sum_{i=1}^{n-1} u_{ki} \le \frac{1}{\underbrace{\min S_i}_{1\le i\le n-1}} = \frac{1}{h|\Re|} \tag{78}$$

where $\Re$ is a constant which is not dependent on h.

We define $||A^{-1}|| = \underbrace{\max \sum_{i=1}^{n-1} |u_{ki}|}_{1\le k\le n-1}$ and $||T(h)|| = \underbrace{\max|T_i(h)|}_{1\le i\le n-1} = h^2\Omega$

where Ω is a constant independent from h.

From equation (75), we get the error at k^{th} tuple

$$e_k = \sum_{i=1}^{n-1} u_{ki}\, T_i(h),\ k = 1,2,\dots,n-1 \tag{79}$$

From equation (68), we obtain

$$||E|| \le \underbrace{\max \sum_{i=1}^{n-1} |u_{ki}|}_{1\le k\le n-1} . \underbrace{\max|T_i(h)|}_{1\le i\le n-1} \tag{80}$$

Substitute equation (78) into equation (80), we get

$$||E|| = \underbrace{\max |e_i|}_{1\le i\le n-1} = |e_j| \le \frac{1}{h|\Re|}h^2\Omega = h\mathcal{K} \tag{81}$$

where $\mathcal{K}$ is a constant which is not dependent on h and

$|e_j| = \max(|e_1|, |e_2|, \dots, |e_{n-1}|)^t$.

Hence, $||E|| = O(h)$.

Hence, proposed method is of first order convergence for uniform h.

5. NUMERICAL EXPERIMENTS

In this section, six model examples are solved and the solutions are compared with the exact/available solutions. The exact solution of equation (8) is given by (with assumptions. $f(x) = f$, $\varphi(x) = \varphi$ and $\gamma(x)$= γ are constant)

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + f/c' \tag{82}$$

where

$$c' = b + c + d$$

$$m_1 = \left[-(a - \delta b + \eta d) + \sqrt{(a - \delta b + \eta d)^2 - 4\varepsilon c'}\right]/2\,\varepsilon$$

$$m_2 = \left[-(a - \delta b + \eta d) - \sqrt{(a - \delta b + \eta d)^2 - 4\varepsilon c'}\right]/2\,\varepsilon$$
$$c_1 = [-f + \gamma c' + e^{m_2}(f - \varphi c')]/[(e^{m_1} - e^{m_2})c']$$
$$c_2 = [f - \gamma c' + e^{m_1}(-f + \varphi c')]/[(e^{m_1} - e^{m_2})c']$$

Example 1. Consider the delay differential equation having left boundary layer:

$$\varepsilon y''(x) + y'(x - \delta) - y(x) = 0,\ 0 \le x \le 1; \text{with } y(0) = 1 \text{ and } y(1) = 1.$$

The exact solution is given by

$$y = ((1 - e^{m_2})e^{m_1 x} + (e^{m_1} - 1)e^{m_2 x})/(e^{m_1} - e^{m_2})$$

where

$$m_1 = \frac{-1 - \sqrt{1 + 4(\varepsilon - \delta)}}{2(\varepsilon - \delta)} \text{and} m_2 = \frac{-1 + \sqrt{1 + 4(\varepsilon - \delta)}}{2(\varepsilon - \delta)}$$

Numerical solution, exact solution, comparison solution and boundary layer action are shown in their respective tables and graphs.

Example 2. Consider the differential-differential equation having left boundary layer:

$$\varepsilon y''(x) + y'(x) - 2y(x - \delta) - 5y(x) + y(x + \eta) = 0, \qquad 0 \le x \le 1;$$
$$\text{with } y(0) = 1 \text{ and } y(1) = 1.$$

Numerical solution, exact solution, comparison solution and boundary layer action are shown in their respective tables and graphs.

Example 3. Consider the differential-differential equation having left boundary layer:

$$\varepsilon y''(x) + y'(x) - 3y(x) + 2y(x + \eta) = 0,\ 0 \le x \le 1; \text{with } y(0) = 1 \text{ and } y(1) = 1.$$

Numerical solution, exact solution, comparison solution and boundary layer action are shown in their respective tables and graphs.

Example 4. Now we consider the delay differential equation having right boundary layer:

$$\varepsilon y''(x) - y'(x - \delta) - y(x) = 0,\ 0 \le x \le 1;\ \text{with } y(0) = 1 \text{ and} y(1) = -1.$$

The exact solution is given by

$$y = ((1 + e^{m_2})e^{m_1 x} - (e^{m_1} + 1)e^{m_2 x})/(e^{m_2} - e^{m_1})$$

where

$$m_1 = \frac{1 - \sqrt{1 + 4(\varepsilon + \delta)}}{2(\varepsilon + \delta)} \text{and} m_2 = \frac{1 + \sqrt{1 + 4(\varepsilon + \delta)}}{2(\varepsilon + \delta)}$$

Numerical solution, exact solution, comparison solution and boundary layer action are shown in their respective tables and graphs.

Example 5. Consider the differential-differential equation having right boundary layer:

$$\varepsilon y''(x) - y'(x) - 2y(x - \delta) + y(x) - 2y(x + \eta) = 0, \qquad 0 \le x \le 1;$$
$$\text{with } y(0) = 1 \text{ and } y(1) = -1.$$

Numerical solution, exact solution, comparison solution and boundary layer action are shown in their respective tables and graphs.

Example 6. Consider the differential-differential equation having right boundary layer:

$$\varepsilon y''(x) - y'(x) + y(x) - 2y(x + \eta) = 0,\ 0 \le x \le 1; \text{with } y(0) = 1 \text{ and } y(1) = -1.$$

Numerical solution, exact solution, comparison solution and boundary layer action are shown in their respective tables and graphs.

6. Conclusion

A numerical integration method is presented to solve a class of singularly perturbed differential-difference equation having boundary layer at one end. Stability and convergence analysis of the method is also discussed. Method is simple and easy to implement on the class of singularly perturbed differential-difference equations having layer at one end. This method is implemented on six standard model examples and found that the numerical solutions are in agreement with available or exact solutions. Computational results, available or exact results and layer action are presented in their respective tables and graphs.

Table-1: **Example 1**: $h = 0.01, \varepsilon = 0.02$ and $\delta = 0.001$

x	Numerical Solution $y(x)$	Exact Solution $y_1(x)$	Result by [13] $y_c(x)$
0.0	1.00000000	1.00000000	1.00000000
0.02	0.40030832	0.59611217	0.37627796
0.04	0.38470877	0.46292256	0.38302924
0.06	0.39155769	0.42247446	0.39076295
0.08	0.39941373	0.41386729	0.39865395
0.1	0.40746127	0.41626092	0.40670430
0.2	0.45020470	0.45597316	0.44946123
0.3	0.49743206	0.50299126	0.49671321
0.4	0.54961366	0.55487431	0.54893279
0.5	0.60726922	0.61210911	0.60664224
0.6	0.67097295	0.67524764	0.67041870
0.7	0.74135933	0.74489885	0.74089999
0.8	0.81912938	0.82173453	0.81879100
0.9	0.90505767	0.90649574	0.90487070
1.0	1.00000000	1.00000000	1.00000000

Fig.1

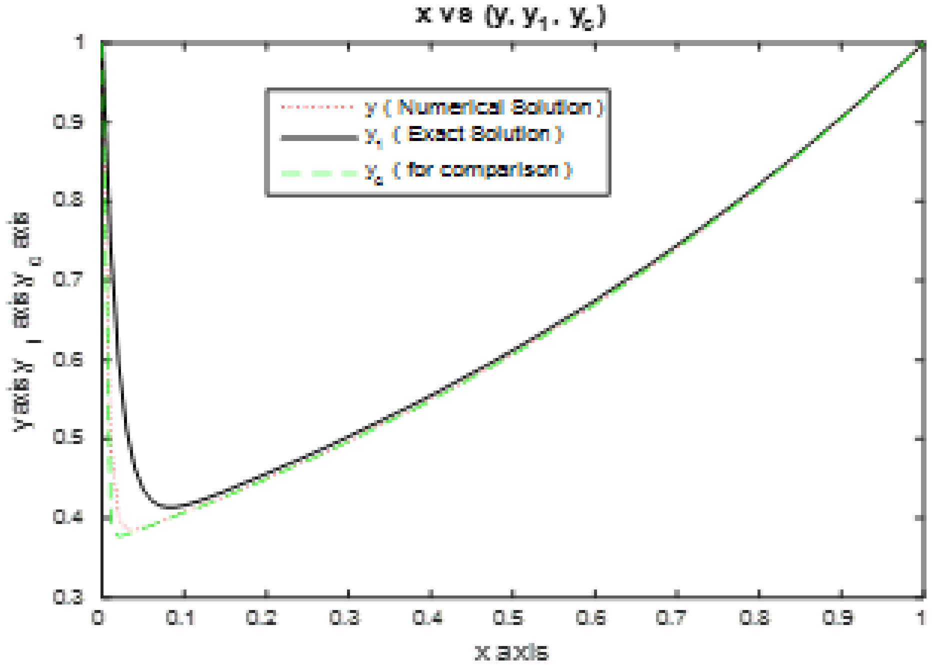

Table-2: **Example 1**: $h = 0.01, \varepsilon = 0.001$ and $\delta = 0.0002$

x	Numerical Solution $y(x)$	Exact Solution $y_1(x)$	Result by [13] $y_c(x)$
0.0	1.00000000	1.00000000	1.00000000
0.02	0.37656838	0.37560498	0.37538059
0.04	0.38305980	0.38318659	0.38292012
0.06	0.39079269	0.39092122	0.39065503
0.08	0.39868365	0.39881199	0.39854620
0.1	0.40673394	0.40686202	0.40659676
0.2	0.44949034	0.44961616	0.44935559
0.3	0.49674136	0.49686302	0.49661105
0.4	0.54895946	0.54907470	0.54883603
0.5	0.60666680	0.60677293	0.60655312
0.6	0.67044041	0.67053424	0.67033991
0.7	0.74091798	0.74099575	0.74083468
0.8	0.81880425	0.81886155	0.81874288
0.9	0.90487803	0.90490969	0.90484412
1.0	1.00000000	1.00000000	1.00000000

Fig. 2.

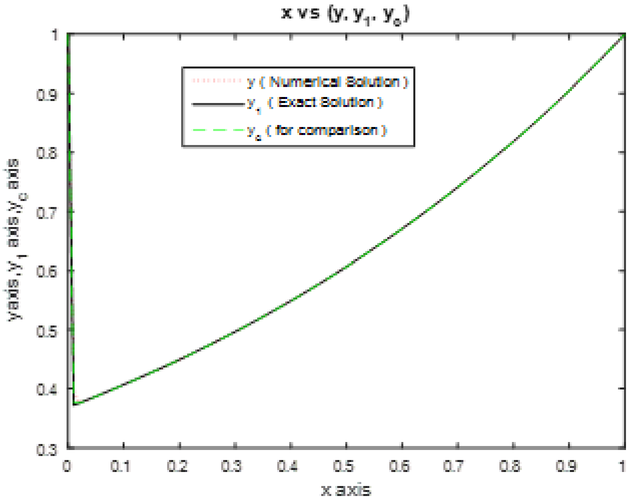

Table-3: **Example 2**: $h = 0.01, \varepsilon = 0.001, \delta = 0.0003$ and $\eta = 0.0004$

x	Numerical Solution $y(x)$	Exact Solution $y_1(x)$	Result by [18] $y_c(x)$
0.0	1.00000000	1.00000000	1.00000000
0.02	0.00497361	0.00291068	0.00427704
0.04	0.00322821	0.00327906	0.00209920
0.06	0.00363304	0.00369407	0.00238035
0.08	0.00409428	0.00416160	0.00270679
0.1	0.00461410	0.00468830	0.00307801
0.2	0.00838751	0.00850730	0.00585264
0.3	0.01524682	0.01543718	0.01112841
0.4	0.02771565	0.02801199	0.02115992
0.5	0.05038148	0.05082999	0.04023417
0.6	0.09158339	0.09223506	0.07650258
0.7	0.16648018	0.16736785	0.14546451
0.8	0.30262748	0.30370226	0.27659100
0.9	0.55011588	0.55109188	0.52591919
1.0	1.00000000	1.00000000	1.00000000

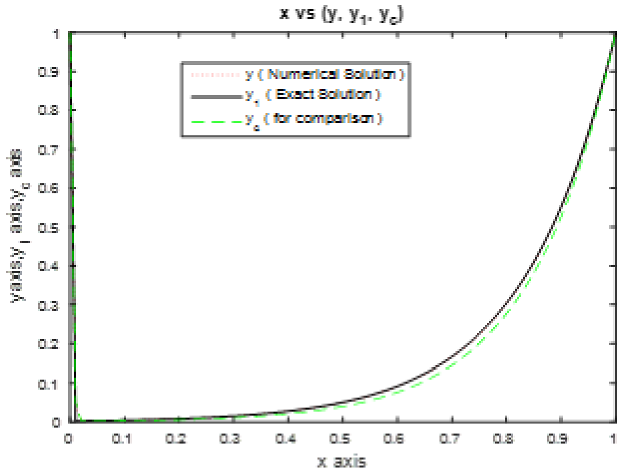

Fig.3.

Table-4: **Example 2**: $h = 0.01, \varepsilon = 0.0003, \delta = 0.0001$ and $\eta = 0.0002$

x	Numerical Solution $y(x)$	Exact Solution $y_1(x)$	Result by [18] $y_c(x)$
0.0	1.00000000	1.00000000	1.00000000
0.02	0.00332373	0.00283102	0.03402341
0.04	0.00318521	0.00319113	0.00144049
0.06	0.00359023	0.00359704	0.00039356
0.08	0.00404708	0.00405459	0.00042170
0.1	0.00456205	0.00457034	0.00049785
0.2	0.00830336	0.00831676	0.00115893
0.3	0.01511287	0.01513422	0.00269805
0.4	0.02750682	0.02754011	0.00628120
0.5	0.05006494	0.05011543	0.01462298
0.6	0.09112278	0.09119629	0.03404307
0.7	0.16585181	0.16595215	0.07925407
0.8	0.30186550	0.30198724	0.18450765
0.9	0.54942288	0.54953365	0.42954354
1.0	1.00000000	1.00000000	1.00000000

Fig. 4.

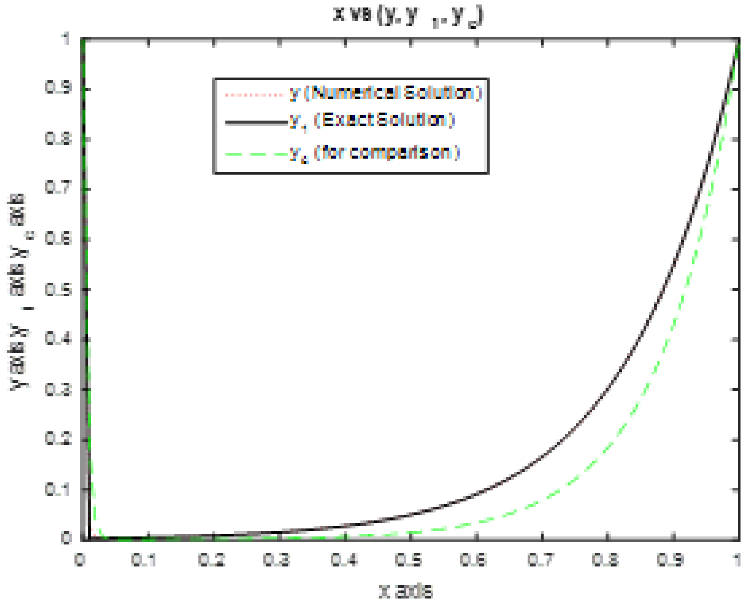

Table-5: **Example 3:**$h = 0.01, \varepsilon = 0.01, \delta = 0.002$ and $\eta = 0.002$

x	Numerical Solution $y(x)$	Exact Solution $y_1(x)$	Result by [18] $y_c(x)$
0.0	1.00000000	1.00000000	1.00000000
0.02	0.37836202	0.37714362	0.37255229
0.04	0.38454949	0.38472418	0.38009154
0.06	0.39227954	0.39245711	0.38782922
0.08	0.40016818	0.40034547	0.39572443
0.1	0.40821546	0.40839239	0.40378036
0.2	0.45094539	0.45111912	0.44658778
0.3	0.49814808	0.49831600	0.49393350
0.4	0.55029171	0.55045070	0.54629865
0.5	0.60789346	0.60803982	0.60421537
0.6	0.67152468	0.67165402	0.66827223
0.7	0.74181649	0.74192364	0.73912018
0.8	0.81946609	0.81954500	0.81747919
0.9	0.90524366	0.90528725	0.90414556
1.0	1.00000000	1.00000000	1.00000000

Fig.5.

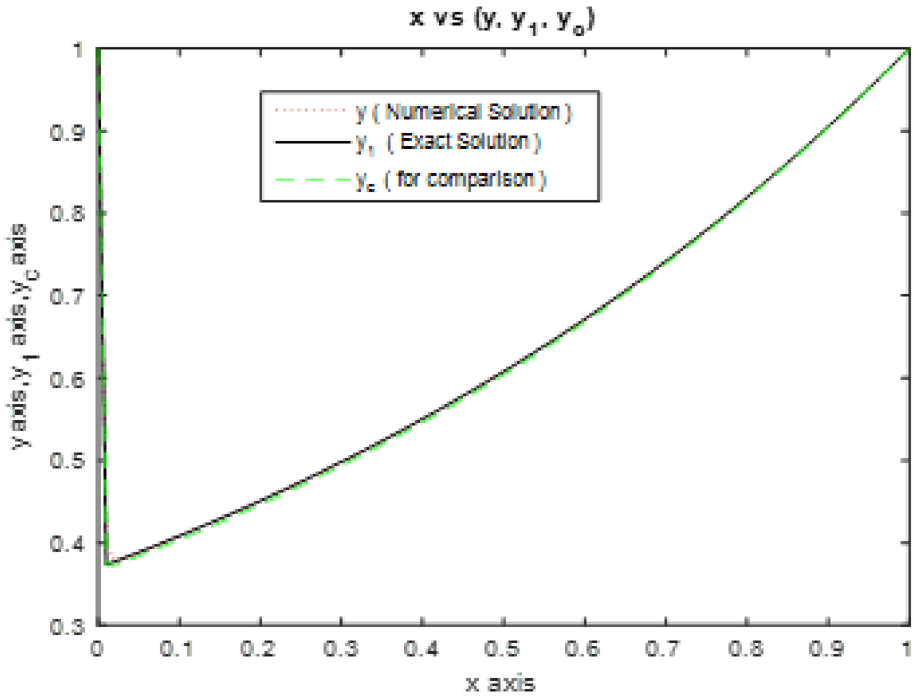

Table-6: **Example 3**: $h = 0.01, \varepsilon = 0.0001, \delta = 0.001$ and $\eta = 0.002$

x	Numerical Solution $y(x)$	Exact Solution $y_1(x)$	Result by [18] $y_c(x)$
0.0	1.00000000	1.00000000	1.00000000
0.02	0.37688878	0.37681580	0.32042216
0.04	0.38440381	0.38439659	0.32284657
0.06	0.39213710	0.39212989	0.33050324
0.08	0.40002597	0.40001878	0.33838077
0.1	0.40807355	0.40806637	0.34644634
0.2	0.45080604	0.45079899	0.38975127
0.3	0.49801338	0.49800657	0.43846920
0.4	0.55016416	0.55015771	0.49327676
0.5	0.60777604	0.60777011	0.55493512
0.6	0.67142091	0.67141566	0.62430062
0.7	0.74173051	0.74172617	0.70233664
0.8	0.81940277	0.81939957	0.79012696
0.9	0.90520869	0.90520692	0.88889086
1.0	1.00000000	1.00000000	1.00000000

Fig. 6.

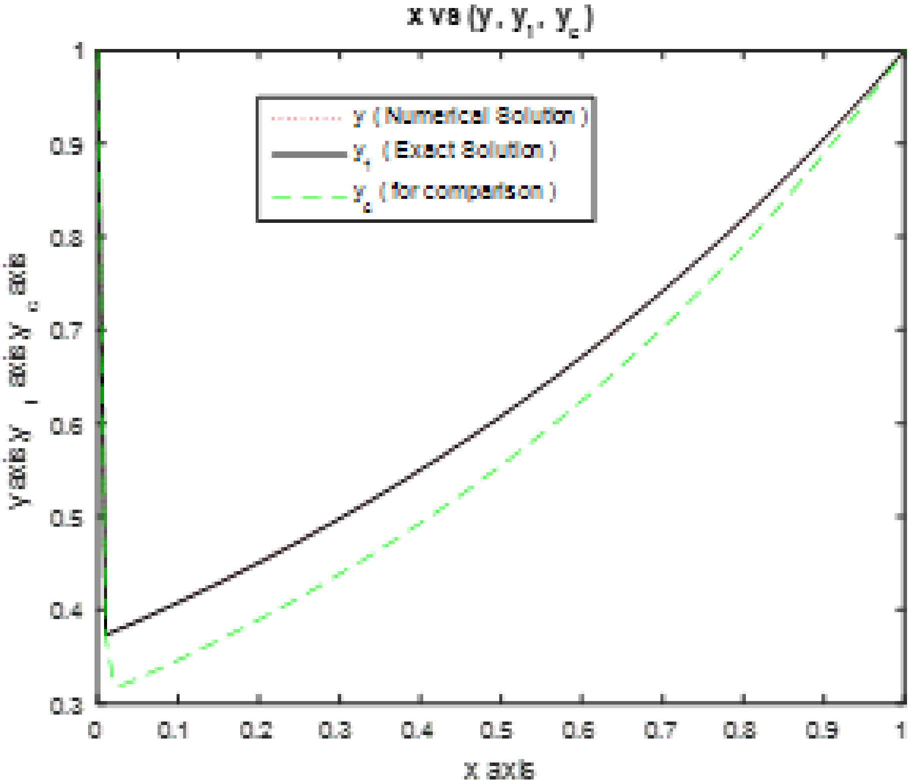

Table-7: **Example 4**: $h = 0.01, \varepsilon = 0.001$ and $\delta = 0.003$

x	Present Solution	Exact Solution	Result by [13]
0.0	1.00000000	1.00000000	1.00000000
0.1	0.90493550	0.90519655	0.89913994
0.2	0.81890826	0.81938080	0.80845264
0.3	0.74105916	0.74170068	0.72691207
0.4	0.67061074	0.67138490	0.65359568
0.5	0.60685947	0.60773530	0.58767399
0.6	0.54916868	0.55011990	0.52840116
0.7	0.49696223	0.49796664	0.47510659
0.8	0.44971877	0.45075769	0.42718732
0.9	0.40696648	0.40802431	0.38409753
0.92	0.39891664	0.39997663	0.37597278
0.94	0.39102493	0.39208728	0.36749433
0.96	0.38317158	0.38429459	0.35238288
0.98	0.36289935	0.36772893	0.24915769
1.0	-1.00000000	-1.00000000	-1.00000000

Fig. 7.

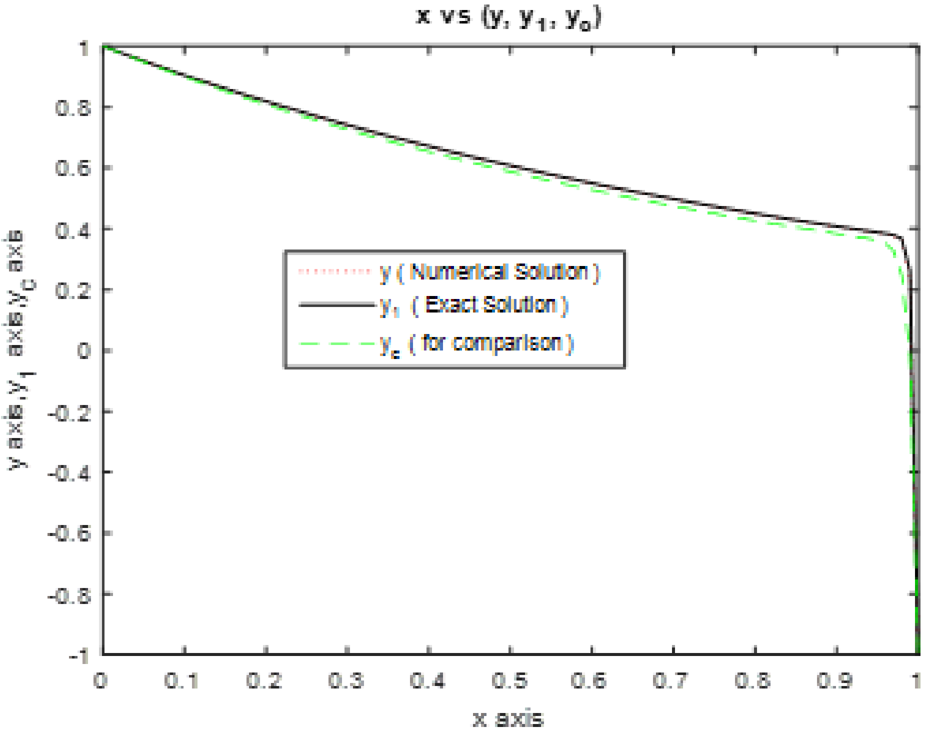

Table-8: **Example 4**: $h = 0.01, \varepsilon = 0.001$ and $\delta = 0.0007$

x	Present Solution	Exact Solution	Result by [13]
0.0	1.00000000	1.00000000	1.00000000
0.1	0.90489933	0.90499073	0.90317119
0.2	0.81884280	0.81900822	0.81571821
0.3	0.74097030	0.74119485	0.73673319
0.4	0.67050353	0.67077447	0.66539620
0.5	0.60673819	0.60704468	0.60096668
0.6	0.54903699	0.54936981	0.54277580
0.7	0.49682320	0.49717458	0.49021947
0.8	0.44957498	0.44993839	0.44275210
0.9	0.40682010	0.40719007	0.39988094
0.92	0.39876998	0.39914069	0.39181829
0.94	0.39087907	0.39125044	0.38391677
0.96	0.38312321	0.38351615	0.37603417
0.98	0.37016935	0.37592433	0.35462761
1.0	-1.00000000	-1.00000000	-1.00000000

Fig. 8.

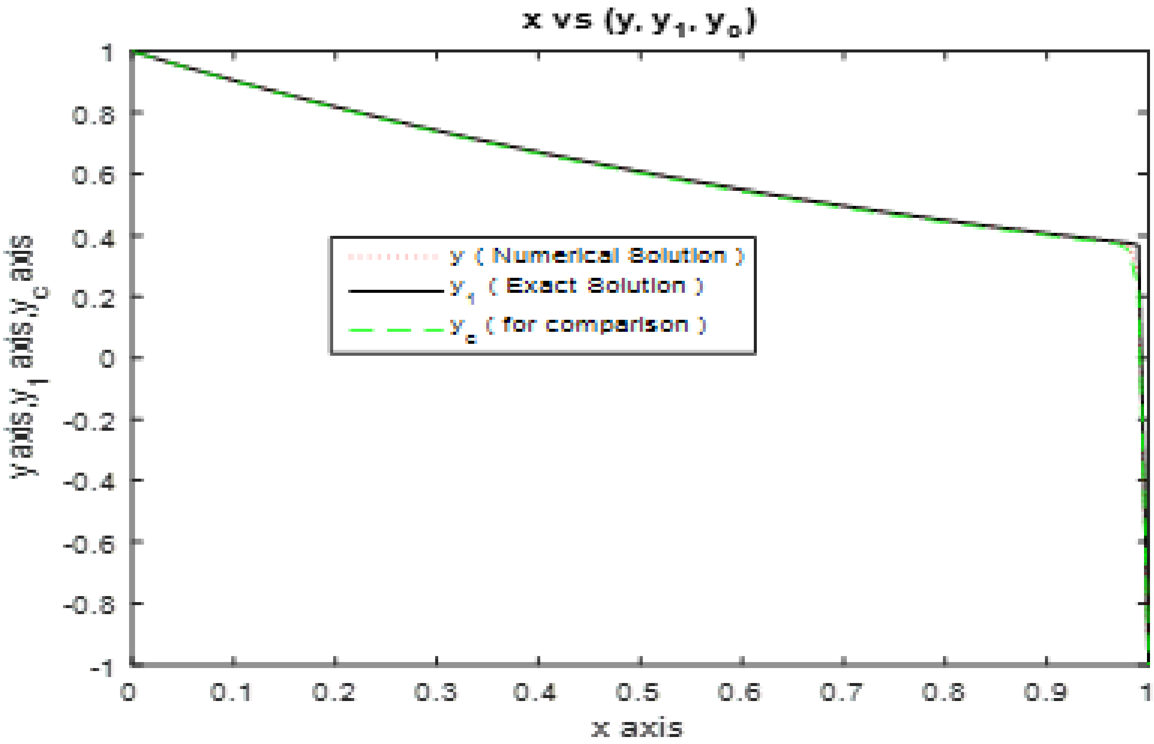

Table-9: **Example 5**: $h = 0.01, \varepsilon = 0.002, \delta = 0.001$ and $\eta = 0.003$

x	Present Solution	Exact Solution	Result by [18]
0.0	1.00000000	1.00000000	1.00000000
0.1	0.74218968	0.74300903	0.73953779
0.2	0.55084553	0.55206242	0.54691614
0.3	0.40883187	0.41018737	0.40446515
0.4	0.30343079	0.30477292	0.29911727
0.5	0.22520320	0.22644903	0.22120852
0.6	0.16714349	0.16825368	0.16359206
0.7	0.12405218	0.12501400	0.12098251
0.8	0.09207024	0.09288653	0.08947114
0.9	0.06833358	0.06901553	0.06616729
0.92	0.06437796	0.06503478	0.06229245
0.94	0.06065121	0.06128363	0.05864452
0.96	0.05711887	0.05774884	0.05521010
0.98	0.04907776	0.05437473	0.05161449
1.0	-1.00000000	-1.00000000	-1.00000000

Fig. 9.

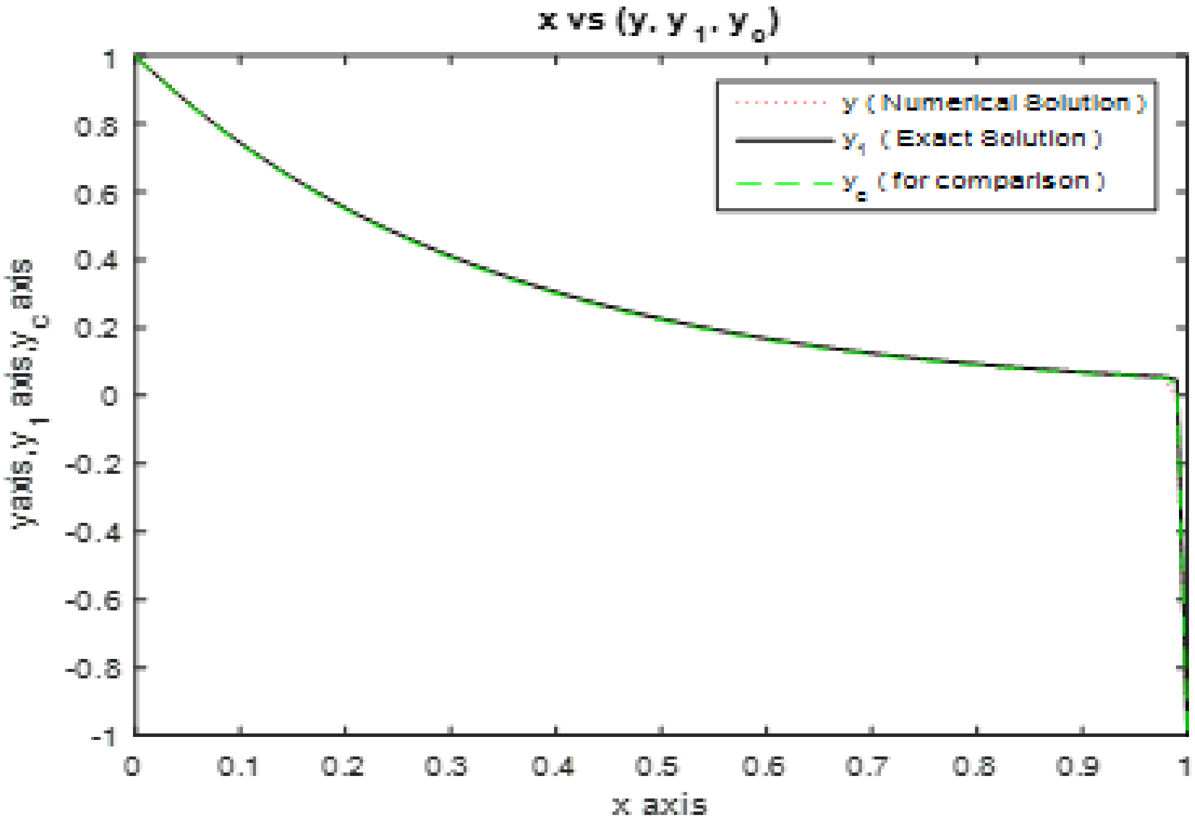

Table-10: **Example 5**: $h = 0.01, \varepsilon = 0.002, \delta = 0.0004$ and $\eta = 0.0006$

x	Present Solution	Exact Solution	Result by [18]
0.0	1.00000000	1.00000000	1.00000000
0.1	0.74139854	0.74222455	0.73877608
0.2	0.54967179	0.55089729	0.54579010
0.3	0.40752586	0.40888949	0.40321667
0.4	0.30213908	0.30348782	0.29788683
0.5	0.22400547	0.22525611	0.22007166
0.6	0.16607733	0.16719062	0.16258368
0.7	0.12312949	0.12409298	0.12011293
0.8	0.09128802	0.09210486	0.08873656
0.9	0.06768080	0.06836248	0.06555645
0.92	0.06374936	0.06440579	0.06170466
0.94	0.06004619	0.06067810	0.05807919
0.96	0.05653632	0.05716616	0.05466660
0.98	0.04846546	0.05381272	0.05108430
1.0	-1.00000000	-1.00000000	-1.00000000

Fig. 10.

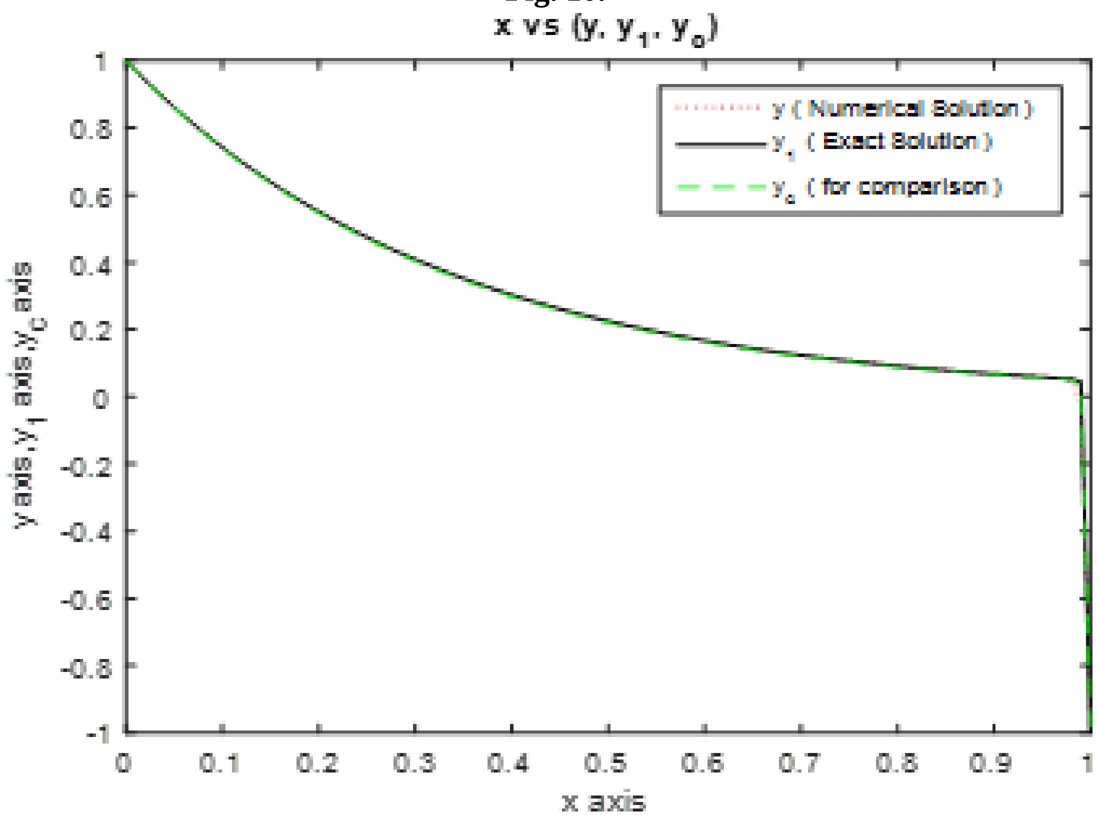

Table-11: **Example 6**: $h = 0.01, \varepsilon = 0.002, \delta = 0.001$ and $\eta = 0.0003$

x	Present Solution	Exact Solution	Result by [18]
0.0	1.00000000	1.00000000	1.00000000
0.1	0.90495921	0.90507163	0.90459812
0.2	0.81895117	0.81915465	0.81829776
0.3	0.74111740	0.74139363	0.74023062
0.4	0.67068102	0.67101434	0.66961123
0.5	0.60693897	0.60731605	0.60572906
0.6	0.54925501	0.54966452	0.54794137
0.7	0.49705338	0.49748577	0.49566674
0.8	0.44981303	0.45026025	0.44837920
0.9	0.40706245	0.40751778	0.40560299
0.92	0.39901281	0.39946906	0.39755048
0.94	0.39112222	0.39157932	0.38965784
0.96	0.38335815	0.38384539	0.38192187
0.98	0.36942212	0.37620367	0.37418034
1.0	-1.00000000	-1.00000000	-1.00000000

Fig. 11.

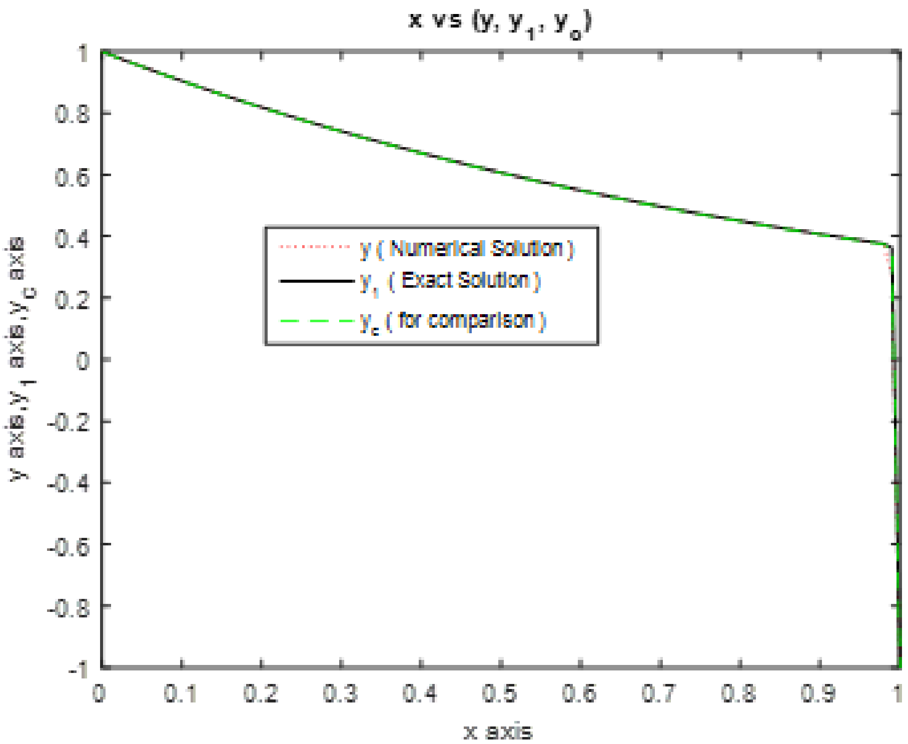

Table-12: **Example 6**: $h = 0.01, \varepsilon = 0.01, \delta = 0.002$ and $\eta = 0.0003$

x	Present Solution	Exact Solution	Result by [18]
0.0	1.00000000	1.00000000	1.00000000
0.1	0.90504975	0.90577777	0.90570405
0.2	0.81911506	0.82043336	0.82029984
0.3	0.74133989	0.74313030	0.74294889
0.4	0.67094949	0.67311091	0.67289182
0.5	0.60724267	0.60968890	0.60944085
0.6	0.54958483	0.55224265	0.55197305
0.7	0.49740162	0.50020912	0.49992423
0.8	0.45017322	0.45307829	0.45278341
0.9	0.40742916	0.41033218	0.41003061
0.92	0.39937995	0.40192249	0.40161483
0.94	0.39147589	0.39126793	0.39092957
0.96	0.38309658	0.36263908	0.36215452
0.98	0.34612408	0.19736527	0.19639572
1.0	-1.00000000	-1.00000000	-1.00000000

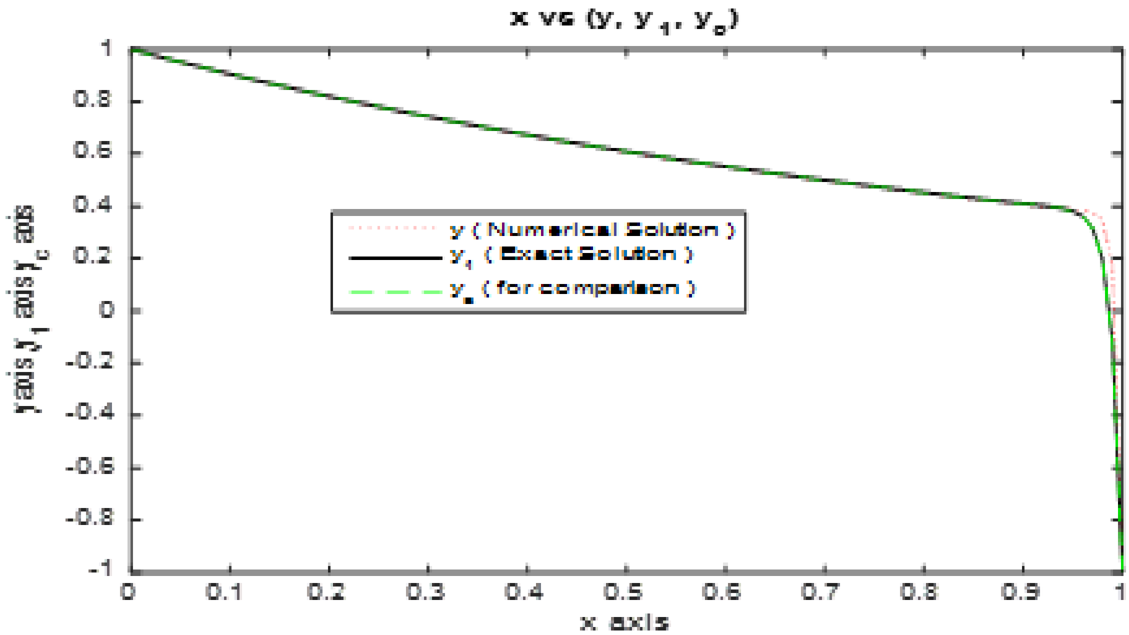

Fig. 12.

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