Dynamical Mathematical Behavior and Influence of Nutrients, Dissolved Oxygen on Survival of Fish Population

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Abstract

A new mathematical model is proposed to study the effects of increasing nutrients and algae with decreasing dissolved oxygen on survival of fish population. The equations are modelled by considering four interacting variables (concentration of nutrients, density of algal population, concentration of dissolved oxygen and fish population) using Hollings type II interaction. It is assumed that the nutrients are constantly release into the waterbodies from various sources (agricultural runoffs, domestic sewage, industrial effluents etc.), due to which the algae grow abundantly depleting the dissolved oxygen. Unfortunately, the marine life is facing the adverse effects of this. In this view, the mathematical model is formulated and the equilibrium points of the model are obtained to study the dynamical mathematical behavior of the system. The stability of equilibrium points is studied and analyzed in detail using the Routh-Hurwitz criteria. Lastly, the numerical simulation is performed to illustrate the analytical findings of the proposed model.

Keywords: -Mathematical modelling; water pollution; stability analysis; Routh-Hurwitz criteria.

Introduction

Water, which plays an important role for sustenance of life is being polluted every single day. Access to clean water will become a major issue in the near future. It is not only deteriorating the health of human but also harming the marine organisms residing in it. Over the past 20 years, it is closely observed that there is enormous amount of death in fish population in numerous lakes [1]. The pollutants are being discharged into water bodies from various sources such as domestic sewage, industrial effluents, anthropogenic activities etc. One of the major sources of pollutants are from agricultural fertilizers which reaches the nearly lakes, ponds etc. through water off [2].

The fertilizers are mainly composed of nutrients such as nitrogen, phosphorous etc. When the water body becomes enriched with nutrients such that the productivity of the system decreases is called as eutrophication [3]. Nutrients help in growth of aquatic plants but ample quantity of nutrientspromote overgrowth of algae in a short period of time [4]. Also, the overabundant algae cover the surface of the water and prevents transfer of oxygen into water. This in turn reduces the dissolved oxygen (DO) in water. Dissolved oxygen plays a vital role to determine the quality of water and to support the survival of aquatic organisms, too low or too high levels disturbs the balance of aquatic ecosystem [5]. Further, fish population gets severely affected with low levels of DO with reduces its growth rate. When the water body becomes enriched with nutrients such that the productivity of the system decreases is called as eutrophication.

In this alarming view, several investigators have proposed and analyzed mathematical models earlier but managing water pollution still remains a major challenge. In [6] authors have proposed a model to study the effects on nutrients on algal blooms by considering the parameters such as nitrogen, phosphorous, algae and detritus. The study mainly focuses on reducing the algal blooms in lakes. In[7], authors have shown the repercussions of depleting dissolved oxygen on the algae and zooplankton. The results shows that the parameters make a stable relationship. Shikha Chaturvedi and Prabha S. Rastogi have modified the model by considering fish population [8]. The paper mainly focuses on survival of fish in the presence of nutrients. In [9], a new parameter has been considered i.e., measure of awareness among the farmers which plays a very important role in reducing the inflow of nutrients into a water body. This directly improves the water quality. In [10], authors have considered Ulsoor lake in Bangalore. The study shows increasing human population effects on the survival of fish in Ulsoor lake directly by fishing and indirectly by human activities in polluting the lake. The paper suggests to control the additional pollutants entering the lake in order to maintain the DO levels and health of fish population. O. P. Misra and Divya Chaturvedi have proposed a model to understand the fate of DO and fish survival with nutrient loading [11]. The survival and extinction of fish population is discussed under different equilibrium points. In [12], authors have modelled both direct and indirect effects of pollution on fish population. Along with organic and inorganic pollutants the parameters also include density of bacteria, DO and density of fish. The paper suggests to limit the release of organic and inorganic pollutants in order to save fish and to maintain the quality of water. In [13], authors have proposed a three-dimensional plankton-nutrient interaction model considering Holing's type II functional response. Authors have also modelled [14,15] the role on dissolved oxygen on organic pollutants along with bacteria as one of the parameters. Several other models have been proposed to study the effects of pollutants and the interacting parameters include detritus, protozoa, phytoplankton, zooplankton, macrophytes, fungi etc., [16,17,18,19,20,21] to name a few. Hence, the study of water pollution and survival of fish becomes is very crucial.

In the above view, a mathematical model is formulated to understand the dropping dissolved oxygen levels and dying conditions of fish in the presence of nutrients. The paper is categorized as follow: In section 2, a mathematical model is formulated. Section 3 shows the equilibrium points of the system of equations. In section 4, stability analysis is conducted in detail. Section 5, demonstrates the numerical simulation of the formulated model and in the last section, the paper is concluded.

Mathematical Model

In this section, a new mathematical model is formulated to know the gravity of pollutants in water bodies and to study its influence on fish population.

Here, N is considered to be the concentration of nutrients A is considered to be the density of algae, C considered to be the concentration of dissolved oxygen and F is considered to be the fish population. It is assumed that there is cumulative rate of discharge of nutrients into the water body at constant Q. Nutrients are taken up by algae with a Hollings type II interaction, i.e., $\frac{\beta_1 AN}{K+A}$. Nutrients deplete naturally at the rate of v_0. It is assumed that the algal density is wholly dependent on nutrients for their growth [1] and algal density decreases due to natural mortality at rate v_1 and by consumption of fish at the rate β_2. Further, the input rate of dissolved oxygen is q_c. The natural depletion of DO is at the rate v_2 and depletion of DO due to fish is at the rate β_3. Fish population uses dissolved oxygen and algae for its growth at the rate of $\theta_3\beta_3$ and $\theta_2\beta_2$ respectively. Natural mortality of fish population is at the rate v_3.

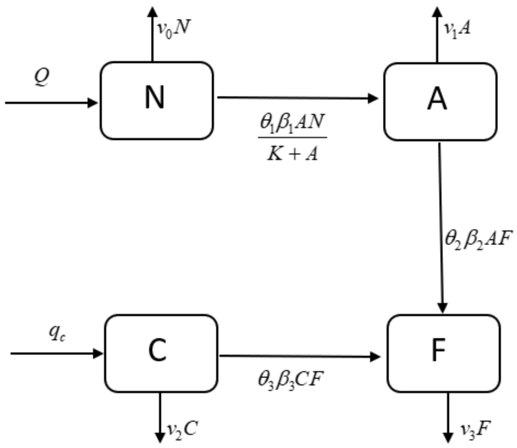

Fig. 1. Compartmental diagram for the mathematical model.

Based on the assumptions stated above, the following mathematical model is formulated.

$$\begin{aligned}
\frac{dN}{dt} &= Q - \frac{\beta_1 AN}{K+A} - v_0 N \\
\frac{dA}{dt} &= \frac{\theta_1\beta_1 AN}{K+A} - v_1 A - \beta_2 AF \\
\frac{dC}{dt} &= q_c - v_2 C - \beta_3 CF \\
\frac{dF}{dt} &= \theta_2\beta_2 AF + \theta_3\beta_3 CF - v_3 F
\end{aligned} \tag{1}$$

Where, $N(0) > 0, A(0) > 0, C(0) > 0, F(0) > 0$

In the above model, K is half saturation constant. v_0, v_1, v_2, v_3 are depletion coefficients and all the parameters are taken to be positive constants.

The equilibria of model (1) are derived by solving the following system of algebraic equations:

$$\begin{aligned}
&Q - \frac{\beta_1 AN}{K+A} - v_0 N = 0 \\
&\frac{\theta_1\beta_1 AN}{K+A} - v_1 A - \beta_2 AF = 0 \\
&q_c - v_2 C - \beta_3 CF = 0 \\
&\theta_2\beta_2 AF + \theta_3\beta_3 CF - v_3 F = 0
\end{aligned}$$

(2)

Table 1. Parameter values of the system

Parameter	**Value**
Q – Constant cumulative discharge rate of nutrients into the water body	3
q_c – input rate of dissolved oxygen due to various sources	9.04
v_0 – rate of nutrient loss	0.5
v_1 – algae mortality rate	0.09
v_2 – Natural depletion of oxygen	2
v_3 – Fish mortality rate	0.01
θ_1 – growth rate of algal population due to nutrients	1.0
θ_2 – growth rate of fish population due to algae	1.0
θ_3 – growth rate of fish population due to dissolved oxygen	1.0
K – Half saturation constant	0.1
β_1 – depletion rate of Nutrients by Algae	1.0
β_2 – depletion rate of algae by fish	1.0
β_3 – depletion rate of dissolved oxygen by fish	3

Analysis of Equilibrium

Equilibrium points of the model (1) can be obtained by setting $\frac{dN}{dt}=0, \frac{dA}{dt}=0, \frac{dC}{dt}=0$ and $\frac{dF}{dt}=0$. The system produces four dynamic equilibrium points where $E_i=(N,A,C,F)$ and $i=1,2,3$ these are listed below,

- $E_1 = (\frac{Q}{v_0}, 0, \frac{q_c}{v_2}, 0)$ always exists.

Initially, it has been considered that the rate of flow of nutrients and concentration of dissolved oxygen does not change and also the fish population is negligible small, thus $\frac{dN}{dt}=\frac{Q}{v_0}$ and $\frac{dC}{dt}=\frac{q_c}{v_2}$

- $E_2=(N',0,C',F')$ where,

$$N=\frac{Q}{v_0}, \quad A=0, \; C=\frac{v_3}{\beta_3\theta_3}, \; F=\frac{q_c\beta_3\theta_3-v_3v_2}{\beta_3 v_3}$$

- $E_3 = (N^*, A^*, C^*, F^*)$

Stability Analysis

In this section, the local stability behaviour is analysed and studied in detail to obtain the eigen values for Jacobian matrix corresponding to each equilibrium point E_1, E_2 and E_3.

Case (i) Now, corresponding to the equilibrium point $E_1 = (\frac{Q}{v_0}, 0, \frac{q_c}{v_2}, 0)$, the Jacobian matrix J_1 obtained is,

$$J_1 = \begin{bmatrix} -v_0 & -\frac{\beta_1 Q}{v_0 K} & 0 & 0 \\ 0 & \frac{\theta_1 \beta_1 Q}{v_0 K} - v_1 & 0 & 0 \\ 0 & 0 & -v_2 & -\frac{q_c \beta_3}{v_2} \\ 0 & 0 & 0 & \frac{q_c \beta_3 \theta_3}{v_2} - v_3 \end{bmatrix}$$

Eigen values of the above matrix is given by, $\lambda_1 = -v_2, \lambda_2 = -v_0, \lambda_3 = \frac{q_c \beta_3 \theta_3 - v_2 v_3}{v_2}, \lambda_4 = -\frac{K v_0 v_1 - \theta_1 \beta_1 Q}{v_0 K}$

The characteristic polynomial is given by, $\lambda^4 + A_1 \lambda^3 + A_2 \lambda^2 + A_3 \lambda + A_4 = 0$ (3)

Where,

$$A_1 = v_2 - \frac{q_c \beta_3 \theta_3}{v_2} + v_3 + v_0 - \frac{\theta_1 \beta_1 Q}{v_0 K} + v_1$$

$$A_2 = \left(-\frac{q_c \beta_3 \theta_3}{v_2} + v_3\right) v_2 + \left(v_0 - \frac{\theta_1 \beta_1 Q}{v_0 K} + v_1\right)\left({}_2 - \frac{q_c \beta_3 \theta_3}{v_2} + v_3\right) + \left(-\frac{\theta_1 \beta_1 Q}{v_0 K} + v_1\right) v_0$$

$$A_3 = \left(v_0 - \frac{\theta_1 \beta_1 Q}{v_0 K} + v_1\right)\left(-\frac{q_c \beta_3 \theta_3}{v_2} + v_3\right) v_2 + \left(-\frac{\theta_1 \beta_1 Q}{v_0 K} + v_1\right) v_0 \left(v_2 - \frac{q_c \beta_3 \theta_3}{v_2} + v_3\right)$$

$$A_4 = \left(-\frac{\theta_1 \beta_1 Q}{v_0 K} + v_1\right) v_0 \left(-\frac{q_c \beta_3 \theta_3}{v_2} + v_3\right) v_2$$

Based on Routh-Hurwitz criteria, all the eigen values obtained from the matrix J_1 must be negative. Here, it is noted that the three eigen values λ_1, λ_2 and λ_4 are negative. Therefore, E_1 will be stable if and only if λ_3 is also negative. Then the critical point E_1 will be stable if the $\lambda_3 < 0$.

Case (ii) Now, corresponding to the equilibrium point $E_2 = \left(\frac{Q}{v_0}, 0, \frac{v_3}{\beta_3\theta_3}, \frac{q_c\beta_3\theta_3 - v_3 v_2}{\beta_3 v_3}\right)$, the Jacobian matrix J_2 is obtained below,

$$J_2 = \begin{bmatrix} -v_0 & -\frac{\beta_1 Q}{v_0 K} & 0 & 0 \\ 0 & \frac{\theta_1\beta_1 Q}{v_0 K} - v_1 - \frac{q_c\beta_3\theta_3 - v_2 v_3}{v_3} & 0 & 0 \\ 0 & 0 & -\frac{(q_c\beta_3\theta_3 - v_2 v_3)\beta_3}{\beta_2 v_3} - v_2 & -\frac{v_3}{\theta_3} \\ 0 & \frac{(q_c\beta_3\theta_3 - v_2 v_3)\theta_2}{v_3} & \frac{(q_c\beta_3\theta_3 - v_2 v_3)\beta_3\theta_3}{\beta_2 v_3} & 0 \end{bmatrix}$$

Eigen values of the above matrix is given by,

$$\lambda_1 = -v_0$$
$$\lambda_2 = \frac{-\beta_3^2 q_c\theta_3 - v_2\beta_2 v_3 + \beta_3 v_2 v_3 + \sqrt{x}}{2\beta_2 v_3}$$
$$\lambda_3 = -\frac{\beta_3^2 q_c\theta_3 + v_2\beta_2 v_3 - \beta_3 v_2 v_3 + \sqrt{x}}{2\beta_2 v_3}$$
$$\lambda_4 = -\frac{K\beta_3 q_c\theta_3 v_0 + K v_0 v_1 v_3 - K v_0 v_1 v_3 - Q\beta_1\theta_1 v_3}{v_0 K v_3}$$

Where,

$$x = \beta_3^4 q_c^2\theta_3^2 + 2\beta_2\beta_3^2 q_c\theta_3 v_2 v_3 - 4\beta_2\beta_3^2 q_c\theta_3 v_3^2 - 2\beta_3^3 q_c\theta_3 v_2 v_3 + \beta_2^2 v_2^2 v_3^2 - 2\beta_2\beta_3 v_2^2 v_3^2 + 4\beta_2\beta_3 v_2 v_3^3 + \beta_3^2 v_2^2 v_3^2$$

The characteristic polynomial is given by, $\lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 = 0$ (4)

Where,

$$A_1 = \frac{(q_c\beta_3\theta_3 - v_2v_3)\beta_3}{\beta_2 v_3} + v_2 + v_0 - \frac{\theta_1\beta_1 Q}{v_0 K} + v_1 + \frac{q_c\beta_3\theta_3 - v_2v_3}{v_3}$$

$$A_2 = \left(\frac{(q_c\beta_3\theta_3 - v_2v_3)\beta_3}{\beta_2} + \left(v_0 - \frac{\theta_1\beta_1 Q}{v_0 K} + v_1 + \frac{q_c\beta_3\theta_3 - v_2v_3}{v_3}\right)\left(\frac{(q_c\beta_3\theta_3 - v_2v_3)\beta_3}{\beta_2 v_3} + v_2\right) + \left(-\frac{\theta_1\beta_1 Q}{v_0 K} + v_1 + \frac{q_c\beta_3\theta_3 - v_2v_3}{v_3}\right)v_0\right)$$

$$A_3 = \left(\frac{\left(v_0 - \frac{\theta_1\beta_1 Q}{v_0 K} + v_1 + \frac{q_c\beta_3\theta_3 - v_2v_3}{v_3}\right)(q_c\beta_3\theta_3 - v_2v_3)\beta_3}{\beta_2} + \left(-\frac{\theta_1\beta_1 Q}{v_0 K} + v_1 + \frac{q_c\beta_3\theta_3 - v_2v_3}{v_3}\right)v_0\left(\frac{(q_c\beta_3\theta_3 - v_2v_3)\beta_3}{\beta_2 v_3} + v_2\right)\right)$$

$$A_4 = \frac{\left(-\frac{\theta_1\beta_1 Q}{v_0 K} + v_1 + \frac{q_c\beta_3\theta_3 - v_2v_3}{v_3}\right)v_0(q_c\beta_3\theta_3 - v_2v_3)\beta_3}{\beta_2}$$

By Routh–Hurwitz criteria, all the eigenvalues with respect to E_2 are negative. Hence, the equilibrium point E_2 is locally asymptotically stable [23,24].

Theorem: E_3 is locally asymptotically stable with some conditions, otherwise it is unstable.

Proof: corresponding to the coexistence equilibrium point $E_3 = (N^*, \mathrm{A}^*, \mathrm{C}^*, \mathrm{F}^*)$,

the Jacobian matrix $J_3(N^*, \mathrm{A}^*, \mathrm{C}^*, \mathrm{F}^*)$ is obtained below,

$$J_3 = \begin{bmatrix} -\frac{\beta_1 A^*}{K + A^*} - v_0 & -\frac{\beta_1 N^*}{K + A^*} + \frac{\beta_1 A^* N^*}{(K + A^*)^2} & 0 & 0 \\ \frac{\theta_1\beta_1 A^*}{K + A^*} & \frac{\theta_1\beta_1 A^*}{K + A^*} - \frac{\theta_1\beta_1 A^* N^*}{(K + A^*)^2} - v_1 - \beta_2 F^* & 0 & -\beta_2 A^* \\ 0 & 0 & -F^*\beta_3 - v_2 & -C^*\beta_3 \\ 0 & F^*\beta_2\theta_2 & F^*\beta_3\theta_3 & A^*\beta_2\theta + C^*\beta_3\theta_3 - v_3 \end{bmatrix}$$

Where,

$$N^* = \frac{Q}{-\frac{\beta_1 A^*}{K + A^*} - v_0}, A^* = \frac{\theta_1\beta_1 A^*}{K + A^*} - \frac{\theta_1\beta_1 A^* N^*}{(K + A^*)^2} - v_1 - \beta_2 F^*$$

$$C^* = \frac{q_c}{-F^*\beta_3 - v_2}, F^* = A^*\beta_2\theta + C^*\beta_3\theta_3 - v_3$$

The characteristic equation of system at the coexistence equilibrium point $E_3 = (N^*, \mathrm{A}^*, \mathrm{C}^*, \mathrm{F}^*)$ is
$\lambda^4 + \lambda^3 A_0 + A_1\lambda^2 + A_2\lambda + A_3 = 0$
(5)

From Routh–Hurwitz criterion, the roots of the above characteristic equation should be negative or should have negative real parts provided the condition given below holds:

$$A_1 > 0, \begin{vmatrix} A_1 & A_0 \\ A_3 & A_2 \end{vmatrix} = A_1A_2 - A_0A_3 > 0, \begin{vmatrix} A_1 & A_0 & 0 \\ A_3 & A_2 & A_1 \\ 0 & A_4 & A_3 \end{vmatrix} = A_1A_2A_3 - A_1^2A_4 - A_0A_3^2 > 0$$

By using the Routh–Hurwitz criteria, the stability at the interior equilibrium point can be investigated, where the roots of the equation are negative or has a negative real part if and only if for A_i, $i = 1, 2, 3, 4$, $A_1A_2 - A_3 > 0$ and $A_1A_2A_3 - A_1^2A_4 - A_0A_3^2 > 0$. Hence, the coexistence equilibrium point E_3 is locally asymptotically stable under the conditions.

Results and Discussions

In this section, the mathematical results are interpreted using MATLAB solver. To visualize the effect of interacting parameters numerical simulation is performed on the model. The parameters values are considered from the published papers [11], [22] and the remaining set of values have been assumed according to the model. The following set of parameter values are considered,

$$Q = 3; \beta_1 = 1.0; \beta_2 = 1.0; \beta_3 = 3; q_c = 9.04; \theta_1 = 1.0; \theta_2 = 1.0$$
$$\theta_3 = 1.0; v_0 = 0.5; v_1 = 0.09; v_2 = 2; v_3 = 0.01; K = 0.1$$

In Fig. 2, the contour 3D plots for the three equilibrium points are shown. Contour 3D plots are used to find the relationships between the variables which is clearly visible from the figure. Fig. 3. Shows the variation of Nutrient, Algae, Dissolved oxygen and Fish with time. Fig. 4. displays the variation in algae with respect to nutrient concentration, also displays the phase plane plot between density of algae and fish population i.e., whenever there is a decrease in algae the fish population also decreases. Phase plane plot between dissolved oxygen concentration and fish population, it is further displayed that the fish population decreases whenever the dissolved oxygen decreases due to overabundance of algae. The last sub phase plane plot is shown between nutrients, algae and fish population.

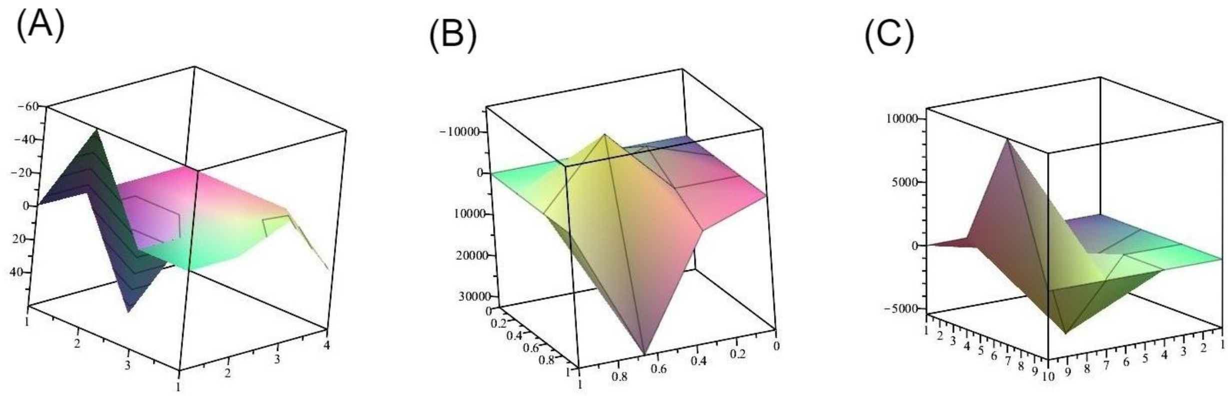

Fig. 2 (A) Contour 3D plot for the equilibrium point E_1, (B) Contour 3D plot for the equilibrium point E_2, (C) Contour 3D plot for the equilibrium point E_3.

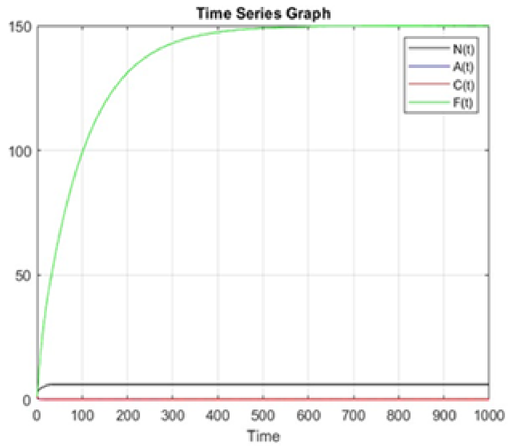

Fig. 3 The variation of Nutrient, Algae, Dissolved oxygen and Fish with time.

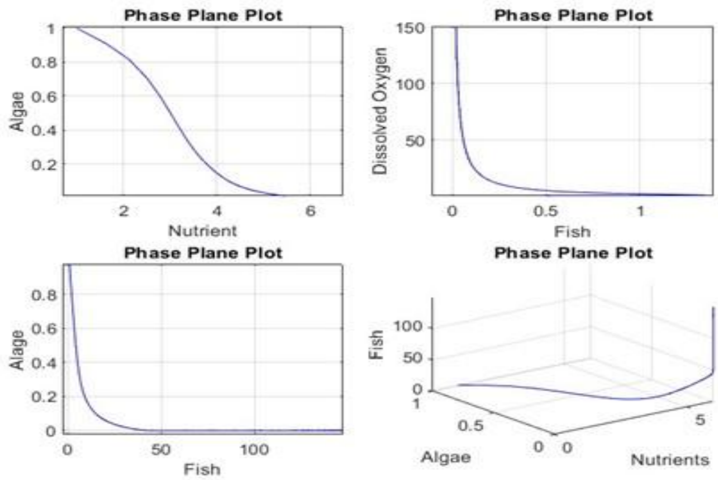

Fig. 4 The variation of Nutrient, Algae, Dissolved oxygen and Fish with time. Phase plane plot between concentration of nutrients (phosphorus and nitrogen) and Algae, density of algae and fish

population, concentration of dissolved oxygen and fish population and between concentration of nutrients, algae and fish population.

Conclusion

In this paper, a new mathematical model is formulated considering nutrients, algae, dissolved oxygen on fish survival. The stability of the equilibrium points is carried out in order to determine the relative importance of the model parameters. The equilibrium point E_1 is conditionally stable. The equilibrium points E_2 and E_3 are stable. This shows that the input of nutrients and growth of algae is directly proportional. However, algal growth and dissolved oxygen are inversely proportion. Similarly, dissolved oxygen and growth of fish population are directly proportional to each other. Numerical simulation is performed to support the analytical findings. The study suggests that the reduced input amount of nutrients will balance the aquatic ecosystem and all the species can co-exists.

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