Properties of Topological Concepts Associated with Weak Systems

Arafa A. Nasef(1), and M. Kamel El-Sayed(2)

(1) Department of Physics and Engineering Mathematics, Faculty of Engineering, Kafrelsheikh University, Kafrelsheikh, Egypt, e-mail: arafa.nasef@eng.kfs.edu.eg

(2) Department of Basic Science, Higher Institute of Engineering and Technology Kafrelsheikh, Egypt e-mail: dr.mkamel123@gmail.com

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Abstract

The main objective of the research work is to presents the notions of w-λ-open sets and W-λ-continuous functions on spaces with **weak systems** and studied some of their properties. We describe the W-λ-continuous function and the relation between W-continuity and W-λ-continuity.

Keywords: Weak systems, w-λ-open sets, W-λ-continuous functions, W-continuous functions.

1. Introduction

The **weak system was introduced in [2, 3, 9].** Which is nonempty set. And the notion of W-continuous function was defined as a function between **weak systems**. The study of objects definable in weak system [5,6,7,11,8,12] Topological spaces contain continuous functions have the same properties as W-continuous functions have been proved by them. Also, the first author and Roy [1] used **weak system** to define the concept of open sets w-b, W-β-continuous functions. The w-λ-open sets, w-λ-interior W-λ-closure operators on a space with a weak system have been introduced. The notion of W-λ-continuous function has been introduced and the characterizations for the W-λ-continuous function have been studied. The relationship between W-continuity and W-λ-continuity have been discussed.

2. Preliminaries

Definition 2.1. ([9, 10]). w_S is called weak system of a non-empty set *X* which is denoted as a subfamily of the power set $P(S)$ if $\phi \in w_S$ and $S \in w_S$. By (S, w_S).Simply we call (S, w_S) a space with a **weak system** w_S on *S*. $W(s) = \{U \in w_S : s \in U\}$.

Definition 2.2. [9]. Let the space (S, w_S) be a **weak system** w_S on *S*. *Which contains* η , the interior and closure of η are defined as the following:

(1) $w\ \mathrm{Int}(\eta) = \bigcup\{U : \eta \supseteq U,\ U \in w_S\}$.

(2) $w\ Cl(\eta) = \bigcap\{Q : Q \supseteq \eta,\ S \setminus Q \in w_S\}$.

Theorem 2.1. [9]. Let the space (S, w_S) with a **weak system** *S* on *S* and $\eta \subseteq S$.

(1) $S = w\ \mathrm{Int}(S)$, $\phi = w\ Cl(\phi)$.

(2) $\eta \supseteq w\ \mathrm{Int}(\eta)$ and $Cl(\eta) \supseteq \eta$.

(3) If $\eta \in w_S$, then $w\ \mathrm{Int}(\eta) = \eta$, $S \setminus Q \in w_S$, then $w\ Cl(Q) = Q$.

(4) If $\psi \supseteq \eta$, then $w\ \mathrm{Int}(\psi) \supseteq w\ \mathrm{Int}(\eta)$, $w\ Cl(\psi) \supseteq w\ Cl(\eta)$.

(5) $w\ \mathrm{Int}(w\ \mathrm{Int}(\eta)) = w\ \mathrm{Int}(\eta)$, $w\ Cl(w\ Cl(\eta)) = w\ Cl(\eta)$.

(6) $w\ Cl(S \setminus \eta) = S \setminus w\ \mathrm{Int}(\eta)$, $w\ \mathrm{Int}(S \setminus \eta) = S \setminus w\ Cl(\eta)$.

Definition 2.3. [9]. Let (S, w_S) , (Y, w_Y) two spaces with **weak systems** w_S , w_Y . Then $q : S \to Y$ is *W*-continuous if $s \in S$, $V \in W(q(s))$, $U \in W(s)$ that $\sigma \supseteq q(U)$.

3. *w*-λ-Open Sets , *W*-λ-Continuity

Definition 3.1. Let the space (S, w_S) be **weak system** w_S on *S* and $S \supseteq \eta$. Then a subset η of *S* is *w*-λ-open set if $w\ Cl(w\ \mathrm{Int}(\eta)) \supseteq \eta \bigcup\ w\ \mathrm{Int}(w\ Cl(\eta))$, *w*-$\lambda$-open set its complement is an *w*-λ-closed set. All of *w*-λ-open sets in *S* are denoted by $W\ \lambda O(s)$.

Remark 3.1. If a nonempty set given a **weak system** w_S on it is a topology, then *w*-λ-open set is λ-open [4].

Lemma 3.1. Let the space (S, w_S) be a **weak system** w_S on *S*. Then η is an *w*-λ-closed set if $\eta \supseteq w \operatorname{Int}(w\,Cl(\eta)) \cup (w\,Cl(w \operatorname{Int}(\eta)))$.

For (*w*-open set and *w*-closed) set is (*w*-λ-open , *w*-λ-closed) but for (*w*-λ-open and *w*-λ-closed) set is not *w* -open and *w* -closed).

Example 3.1. Let $S = \{a, b, c\}$ and let $w_S = \{\phi, \{a\}, \{b\}, S\}$ be a **weak system** on *S*. Assume $\eta = \{a, c\}$ .So $w\,Cl(w \operatorname{Int}(\eta)) \cup w \operatorname{Int}(w\,Cl(\eta)) =$ $\{a, c\}$. Then η is not w -open but *w*-λ-open.

Theorem 3.1. Let the space (S, w_S) with a **weak system** w_S on *S*. The *w*-λ-open sets its union is always *w*-λ-open.

Proof. Let η_i be an *w*-λ-open sets for $i \in J$. Then Definition (3.1) , Theorem (2.1) (4), will be:

$$\eta_i \subseteq w\,Cl(w \operatorname{Int}(\eta_i)) \cup w \operatorname{Int}(w\,Cl(\eta_i)) \subseteq w\,Cl(\cup\eta_i)) \cup w \operatorname{Int}(w\,Cl(\cup\eta_i)\,) \,.$$

This implies $\cup\eta_i \subseteq w\,Cl(w\operatorname{Int}(\cup\eta_i) \cup w \operatorname{Int}(w\,Cl(\cup\eta_i))$ and so $\cup\eta_i$ is *w*-λ-open.

Remark 3.2. Let the space (S, w_S) with a **weak system** w_S on *S*. For any two *w*-λ-open sets its intersection may not be *w*-λ-open.

Example 3.2. Let $S = \{a, b, c\}$ and $w_S = \{\phi, \{a, b\}, \{a, c\}, S\}$ a **weak system** in *S*. Then $\{a, b\}$, $\{a, c\}$ are *w*-λ-open sets. And $\{a\}$ is not *w*-λ-open that $w\,Cl \operatorname{Int}(\{\eta\})) \cup w \operatorname{Int}(w\,Cl(\eta)) = \phi$. For two *w*-$\lambda$-open sets its intersection is not *w*-*g*-open.

Theorem 3.2. Let the space (S, w_S) with a **weak system** w_S on *S*, then:

(1) For *w*-λ-closed sets its intersection is always *w*-λ-closed.

(2) For *w*-λ-closed sets its union fail to be *w*-λ-closed

Proof. (1) from Theorem (3.1).

(2) In Example (3.1), if $w_S = \{\phi, \{a\}, \{a, b\{, \{b, c\}, S\}$. $\{a\}$ and $\{b\}$ are *w*-λ-closed sets, their union $\{a, b\}$ not *w*-λ-closed.

Definition 3.2. Let the space (S, w_S) **weak system** w_S on *S*. For a subset $\eta \subset S$, (*w*-λ-closure of η, the *w*-λ-interior of η), is $w\ \lambda Cl(\eta)$ and $w\ \lambda \text{Int}(\eta)$, are defined :

$$w\ \lambda Cl(\eta) = \bigcap\{Q : \eta \subseteq Q,\ Q \text{ is } w\text{-}\lambda\text{-closed in } S\};$$

$$w\ \lambda \text{Int}(\eta) = \bigcup\{U : U \subseteq \eta,\ z \text{ if } U \mid \text{is } w\text{-}\lambda\text{-open in } S\}.$$

Theorem 3.3. Let the space (S, w_S) a **weak system** w_S on *S* and $S \subseteq \eta$. Then

(1) $\eta \supseteq w\ \lambda \text{Int}(\eta)$ and $w\ \lambda Cl(\eta) \supseteq \eta$.

(2) If $\psi \supseteq \psi$, then $w\ \lambda \text{Int}(\psi) \supseteq w\ \lambda \text{Int}(\eta)$, $w\ \lambda Cl(\psi) \supseteq w\ \lambda Cl(\eta)$.

(3) η is w-λ-open iff $w\ \lambda \text{Int}(\eta) = \eta$.

(4) Q is w-λ-closed iff $w\ \lambda Cl(Q) = \eta$.

(5) $w\ \lambda \text{Int}(w\ \lambda \text{Int}(\eta)) = w\ \lambda \text{Int}(\eta)$, $w\ \lambda Cl(w\ \lambda Cl(\eta)) = w\ \lambda Cl(\eta)$.

(6) $w\ \lambda Cl(X \setminus \eta) = X \setminus w\ \lambda \text{Int}(\eta)$, $w\ \lambda \text{Int}(S \setminus \eta) = S \setminus w\ \lambda Cl(\eta)$.

Proof. (1) and (2) clear.

(3) and (4), from Theorem (3.1).

(5) From (3) and (4).

(6) For $\eta \subset S$, we have

$$\begin{aligned} S \setminus w\ \lambda \text{Int}(\eta) &= S \setminus \bigcup\{U : U \subseteq \eta,\ U \text{ is } w\text{-}\lambda\text{-open}\} \\ &= \bigcap\{S \setminus U : U \subseteq \eta,\ U \text{ is } w\text{-}\lambda\text{-open}\} \\ &= \bigcap\{S \setminus U : S \setminus \eta \subseteq S \setminus U,\ U \text{ is } w\text{-}\lambda\text{-open}\} \\ &= w\ \lambda Cl(S \setminus \eta) \end{aligned}$$

We have $w\ \lambda \text{Int}(S \setminus \eta) = S \setminus w\ \lambda Cl(\eta)$

Theorem 3.4. Let the space (S, w_S) with a **weak system** w_S on S,

(1) $w\ \lambda \text{Int}(\eta \bigcup \psi) \supseteq w - \lambda \text{Int}(\eta) \bigcup w\ \lambda \text{Int}(\psi)$,

(2) $w\ \lambda Cl(\eta) \bigcap w\ \lambda Cl(\psi) \supseteq w\ \lambda Cl(\eta \bigcap \psi)$

Theorem 3.5. Let the space (S, w_S) with a **weak system** w_S on S and $S \supseteq \eta$. Then $s \in w\ \lambda Cl(\eta)$ iff $\eta \bigcap \sigma \neq \phi$ for *any* *w*-λ-open set σ containing *s*.

Proof. If *w*-λ-open set σ containing s, $\eta \bigcap \sigma = \phi$. Then $S \setminus \sigma$ is *w*-λ-closed set that $S \setminus \sigma \supseteq \eta$ and $s \in S \setminus \sigma$. Then $s \notin w\ \lambda Cl\,(\eta)$.

Theorem 3.6. Let the space (S, w_S) be a **weak system** w_S on *S* and $\eta \subseteq S$. Then $s \in w\ \lambda\,\mathrm{Int}(\eta)$ iff there exists an *w*-λ-open set *U* that $\eta \supseteq U$.

Theorem 3.7. Every w-semi-open set is *w*-λ-open.

Proof. Let η be an w-semi-open set in (S, w_S). Then $\eta \subset w\ Cl\,(w\ \mathrm{Int}(\eta))$. Then $\eta \subseteq w\ Cl\,(w\ Cl\,(w\ \mathrm{Int}(\eta))) \bigcup w\ \mathrm{Int}(w\ Cl\,(\eta))$ and η is *w*-λ-open in (S, w_S).

Definition 3.3. Let a function $f : (S, w_S) \to (Y, w_Y)$ between two spaces with **weak systems** w_S and w_Y. Then *q* is *W*-λ-continuous if a point *s* and w-open set σ containing $q(s)$, there *w*-λ-open set *U* containing *x* that $q(U) \subseteq \sigma$.

W-continuity $\Rightarrow$ *W*-λ-continuity

The converse is not true

Example 3.3. Let $S = \{a, b, c\}$. Assume two **weak systems** defined :

$$w_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, S\}, \qquad w_2 = \{\alpha, \{a, b\}, \{a, c\}, S\}.$$

Let the identity function $q : (S, w_1) \to (Y, w_2)$. Then *q* is *W*-$\lambda$ continuous but not W-continuous.

Remark 3.3. Let the function $q : (S, w_S) \to (Y, w_Y)$ on two spaces with **weak systems** w_S and w_Y. If the **weak systems** w_S and w_Y are topologies on *S* and *Y*, , then q is λ-continuous.

Theorem 3.8. Let the function $q : (S, w_S) \to (Y, w_Y)$ on two spaces with **weak systems** w_S and w_Y. The following are equivalent:

(1) q is W-gg-continuous.

(2) Every w-open set σ in *Y*, $q^{-1}(\sigma)$ is *w*-λ-open in *S*.

(3) Each w-closed set ψ in *Y*, $q^{-1}(\psi)$ is $w\ \lambda$-closed in *S*.

(4) $w\ Cl\,(q(\eta)) \supseteq q(w\ Cl\,(\eta))$ for $S \supseteq \eta$.

(5) $q^{-1}(w\ Cl\,(\psi)) \supseteq w\ \lambda Cl\,(q^{-1}(\psi))$ for $Y \supseteq \psi$.

(6) $w\ \lambda\,\mathrm{Int}(q^{-1}(\psi)) \supseteq q^{-1}(w\ \mathrm{Int}(\psi))$ for $Y \supseteq \psi$.

Proof. (1) $\Rightarrow$ (2): Let w -open set denoted by σ in Y and $s \in q^{-1}(\sigma)$. $\exists$ an w-λ-open set U containing s by the hypothesis, $f(U) \subseteq \sigma$. So, we have $s \in U \subseteq q^{-1}(\sigma)$ for all $s \in q^{-1}(\sigma)$. In Theorem (3.1), $q^{-1}(\sigma)$ is w-λ-open.

(2) $\Rightarrow$ (3): It is clear.

(3) $\Rightarrow$ (4): For $\eta \subseteq S$,

$$q^{-1}(w\,Cl\,(q(\eta))) = Q \supseteq q^{-1}(\bigcap(Q \subseteq Q : q(\eta) \text{ and } w\text{-closed}\}$$
$$= \bigcap\{q^{-1}(Q) \subseteq S : q^{-1}(Q) \supseteq \eta \text{ and } Q \text{ is } w\text{-}\lambda\text{-closed}\}$$
$$\supseteq \bigcap\{K \subseteq S : K \supseteq \eta \text{ and } K \text{ is } w\text{-}\lambda\text{-closed}\}$$
$$= w\,\lambda Cl\,(A).$$

Hence $w\,Cl\,(q(\eta)) \supseteq q(w\,\lambda Cl\,(\eta))$.

(4) $\Rightarrow$ (5): For $Y \supseteq \psi$, from (4),

$$q(w\,\lambda Cl\,(q^{-1}(\psi))) \subseteq w\,Cl\,(q(q^{-1}(\psi))) \subseteq w\,Cl\,(\psi).$$

Then we get $w\,\lambda Cl\,(q^{-1}(\psi))) \subseteq w\,Cl\,(\psi)$.

(5) $\Rightarrow$ (6); For $\psi \subseteq Y$, from $w\ \mathrm{Int}(\psi) = Y \setminus Cl\,(Y \setminus \psi)$ and (5),

$$q^{-1}(w\ \mathrm{Int}(\psi)) = q^{-1}(Y \setminus w\,Cl\,(Y \setminus \psi))$$
$$= s \setminus q^{-1}(w\,Cl\,(Y \setminus \psi))$$
$$\subseteq s \setminus w\,\lambda Cl\,((q^{-1})Y \setminus \psi))$$
$$= w\,\lambda\,\mathrm{Int}(q^{-1}(\psi)).$$

Then $q^{-1}(w\ \mathrm{Int}(\psi)) \subseteq w\,\lambda\,\mathrm{Int}(q^{-1}(\psi))$.

(6) $\Rightarrow$ (1): Let w-open set (σ) containing $q(s)$ and $s \in S$. Then from Theorem (2.1), $s \in q^{-1}(\sigma) = q^{-1}(w\ \mathrm{Int}(\sigma)) \subseteq w\,\lambda\,\mathrm{Int}(q^{-1}(\sigma))$. Theorem (3.6), there is w-λ-open set U containing s, $s \in U \subseteq q^{-1}(\sigma)$. Then q is W-λ-continuous.

Lemma 3.2. Let the space (S, w_S) with a **weak system**s w_S on S and $S \supseteq \eta$,

(1) $w\ \mathrm{Int}(w\,Cl\,(w\,\lambda Cl\,(\eta))) \bigcup w\,Cl\,(w\ \mathrm{Int}(w\,\lambda Cl\,(\eta))) \subseteq w\,\lambda Cl\,(\eta)$,

(2) $w\,\lambda\,\mathrm{Int}(\eta) \subseteq w\ \mathrm{Int}(w\,Cl\,(w\,\lambda\,\mathrm{Int}(\eta))) \bigcup w\,Cl\,(w\ \mathrm{Int}(w\,\lambda\,\mathrm{Int}(\eta))) \subseteq$
$w\ \mathrm{Int}(w\,Cl\,(\eta)) \bigcup w\,Cl\,(w\ \mathrm{Int}\,(\eta))$.

Proof. (1) For $\eta \subseteq S$, Theorem (3.3) $w\ \lambda Cl\ (\eta)$ is *w*- λ -closed set. We have from Lemma (3.1),

$$w\ \text{Int}(w\ Cl\ (\eta)) \subseteq w\ \text{Int}(w\ Cl\ (\eta))) \subseteq w\ \lambda Cl\ (\eta).$$

Theorem 3.9. let the two spaces function $q:(S, w_S) \to (Y, w_Y)$ with **weak systems** w_S and w_Y , The following are equivalent:

(1) Q is *W*- λ -continuous.

(2) $w\ \text{Int}(w\ Cl\ (q^{-1}(\sigma)) \bigcup w\ Cl\ (w\ \text{Int}\ (q^{-1}(\sigma))) \supseteq q^{-1}(\sigma)$ for *w*-open set σ in *Y*.

(3) $q^{-1}(Q) \supseteq w\ \text{Int}(w\ Cl\ (Q))) \bigcup w\ Cl\ (w\ \text{Int}(q^{-1}(Q))$ for *w*-closed set Q in *Y*.

(4) $q\ (w\ \text{I}q^{-1}(w\ Cl\ (\psi)) \supseteq \text{nt}(w\ Cl\ (q^{-1}(\psi))) \bigcup w\ Cl\ (w\ \text{Int}(q^{-1}(\psi)))$ for $\eta \subseteq S$.

(5) $w\ \text{Int}(w\ Cl\ (q^{-1}(\psi))) \bigcup w\ Cl\ (w\ \text{Int}(q^{-1}(\psi))) \subseteq q^{-1}(w\ Cl\ (\psi))$ for $\psi \subseteq Y$.

(6) $q^{-1}(w\ \text{Int}(\psi)) \subseteq w\ Cl\ (w\ \text{Int}(q^{-1}(\psi))) \bigcup w\ \text{Int}(w\ Cl\ (q^{-1}(\psi)))$ for $\psi \subseteq Y$.

Proof. (1) $\Leftrightarrow$ (2): definition of *w*- λ -open sets, theorem (3.8). It follows:
(1) $\Leftrightarrow$ (3): lemma (3.1), theorem (3.8). It follows:
(3) $\Rightarrow$ (4): Let $\eta \subseteq S$. Lemma (3.2), theorem (3.8), it follows:

$$w\ \text{Int}(w\ Cl\ (\eta)) \bigcup w\ Cl\ (w\ \text{Int}(\eta)) \subseteq w\ \lambda Cl\ (\eta) \subseteq q^{-1}(w\ Cl\ (q\ (\eta))).$$

Then $q\ (w\ \text{Int}(w\ Cl\ (\eta))) \bigcup q\ (w\ Cl\ (w\ \text{Int}(\eta))) \subseteq w\ Cl\ q\ (\eta))$.

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(5) $\Rightarrow$ (6): theorem (2.1), from (5) it follows:

$$\begin{aligned} q^{-1}(w\ \text{Int}(\psi)) &= q^{-1}(Y \setminus w\ Cl\ (Y \setminus \psi)) \\ &= S \setminus q^{-1}(w\ Cl\ (Y \setminus \psi)) \\ &\subseteq S \setminus w\ \text{Int}Cl\ (q^{-1}(Y \setminus \psi)) \bigcup w\ Cl\ (w\ \text{Int}(q^{-1}(Y \setminus \psi))) \\ &= w\ Cl\ (w\ \text{Int}(q^{-1}(\psi))) \bigcup w\ \text{Int}(w\ Cl\ (q^{-1}(\psi))). \end{aligned}$$

Then, its obtained (6) .

(6) $\Rightarrow$ (1): Let σ be an *w*-open set in *Y*. By (6) and Theorem (2.1), we have $q^{-1}(V) = q^{-1}(w\ \text{Int}(\sigma)) \subseteq w\ Cl\ (w\ \text{Int}(q^{-1}(\sigma))) \bigcup w\ \text{Int}(w\ Cl\ (q^{-1}(\sigma)))$. Then $q^{-1}(\sigma)$ is an *w*- λ -open set. by (2), *f* is *W*- λ -continuous.

Definition 3.4. Let the function $q:(S,w_S)\to(Y,w_Y)$ on spaces (S,w_S) and (Y,w_Y) with **weak systems** w_S and w_Y. Then q is W-λ-open if w-open set G in S, $q(G)$ is w-λ-open in Y.

Theorem 3.10. Let the function $q:(S,w_S)\to(Y,w_Y)$ on space (S,w_S) and (Y,w_Y) with **weak system** w_S and w_Y. The following are equivalent:

(1) q is W-λ-open.

(2) $q(w\ \mathrm{Int}(\eta))\subseteq w\ \lambda\mathrm{Int}(q(\eta))$ for each $A\subseteq S$.

(3) $w\ \mathrm{Int}(q^{-1}(\psi))\subseteq q^{-1}(w\ \lambda\mathrm{Int}(\psi))$ for each $\psi\subseteq Y$.

Proof. (1) $\Rightarrow$ (2): For $\eta\subseteq S$, from W-λ-openness of q

$$q(w\ \mathrm{Int}(\eta))=w\ \lambda\mathrm{Int}(q(w\ \mathrm{Int}(\eta)))\subseteq w\ \lambda\mathrm{Int}(q(\eta)).$$

Then, $q(w\ \mathrm{Int}(\eta))\subseteq w\ \lambda\mathrm{Int}(q(\eta))$.

(2) $\Rightarrow$ (3): For $\psi\subseteq Y$, $q(w\ \mathrm{Int}(q^{-1}(\psi)))\subseteq w\ \lambda\mathrm{Int}(q(q^{-1}(\psi)))\subseteq w\ \lambda\mathrm{Int}(\psi)$. Then $w\ \mathrm{Int}(q^{-1}(\psi))\subseteq q^{-1}(w\ \lambda\mathrm{Int}(\psi))$.

(3) $\Rightarrow$ (1): G is an w-open set in S. Then:

$$G=w\ \mathrm{Int}(G)\subseteq w\ \mathrm{Int}(q^{-1}(q(G)))\subseteq q^{-1}(w\ \lambda\mathrm{Int}(q(G))).$$

Then $q(G)\subseteq w\ \lambda\mathrm{Int}(q(G))$, so $f(G)$ is w-λ-open.

4. Some Applications

Definition 4.1. Let the space (S,w_S) with a **weak system** w_S. Then (S,w_S) is an w-T_2 [4] (resp. w-λ-T_2) space if the pair points s, y of S $\exists$ disjoint w-open (resp. w-λ-open) U, V sets containing s,y

Remark 4.1. Every w-T_2 space is an w-λ-T_2 space. But the converse is not true as in the next example.

Example 4.1. Let $S=\{a,b,c\}$ and $w_S=\{\phi,\{a\},\{b\},S\}$ be a **weak system** on S. Then $\lambda(S)=\{\phi,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\}\}$. Thus (S,w_S) is w-λ-T_2 but not w-T_2 space.

Theorem 4.1. Let the two spaces (S, w_S) and (Y, w_Y) be S and Y with **weak systems** w_S and w_Y .If $\exists$ W-λ-continuous injective function $q:(S, w_S) \to (Y, w_Y)$ that (Y, w_Y) is w-λ-T_2 then (S, w_S) is w-T_2

Proof. Let two distinct points s_1 and s_2 of S. Then as q injective, $q(s_1) \neq q(s_2)$. Thus $\exists$ two disjoint w-λ-open sets σ_1 and σ_2 containing s_1 and s_2, respectively. Since q is W-λ-continuous, $\exists$ w-open sets U_1 and U_2 such that $s_1 \in U_1$ and $q(U_1) \subseteq \sigma_1$ in y subset of Y that $(U \times \sigma) \bigcap G(q) = \phi$.

Definition 4.2. A function $q:(S, w_S) \to (Y, w_Y)$ is W-λ-closed graph if $(s, y) \in (S \times Y) \backslash G(q)$, $\exists$ w_S-open set U containing s in S and an w_Y-open set σ in Y containing y that $(U \times V) \bigcap G(q) = \phi$.

Lemma 4.1. A function $q:(S, w_S) \to (Y, w_Y)$ has W-λ-closed graph iff for $(s, y) \in (S \times Y) \backslash G(q)$, $\exists$ an w_S-pen set U containing s in S and an w_Y-open set σ of Y containing y that $q(U) \bigcap \sigma = \phi$.

Theorem 4.2. A function $q:(S, w_S) \to (Y, w_Y)$ is W-λ-continuous and (Y, w_Y) is w-λ-T_2, then $G(q)$ is W-λ-closed

Proof. Let $(s, y) \in (S \times Y) \backslash G(q)$. Then $y \neq q(s)$. Since Y is w-λ-T_2, $\exists$ w-λ-open sets σ and Z in Y containing y and $q(s)$, that $\sigma \bigcap Z = \phi$. Since q is W-λ-continuous, $\exists$ an w-open set U in S containing s that $q(U) \subseteq Z$. Thus $q(U) \bigcap \sigma = \phi$. Then by Lemma (4.1) $G(q)$ is W-λ-closed.

Theorem 4.3. If the injective W-λ-continuous function denoted by $q:(S, w_S) \to (Y, w_Y)$ with a W-λ-closed graph, then (S, w_S) is w-T_2.

Proof. Let any two points s and y of S. then f is injective, $q(s) \neq q(y)$ and $(s, q(y)) \in (S \times Y) \backslash G(q)$. Since $G(q)$ by Lemma (4.1), is W-λ-closed, $\exists$ an w-open set U in S containing s and an w-λ-open set V containing $q(y)$ in Y that $q(U) \bigcap \sigma = \phi$. q is W-λ-

continuous, $\exists$ an *w*-open set G in S that $\sigma \supseteq q(G)$. Then $q(G) \bigcap q(U) = \phi$ that $G \bigcap U = \phi$. Then S is w-T_2.

5. Conclusions

The concept of w-λ-open sets , w-λ-continuity have been introduced and their properties are studies, we hope that this paper is inst a beginning of a new **weak system**. It will inspire many contribute to the cultivation of generalized topology under the name of **weak system** in the field of discrete mathematics.

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