Mathematical Formulation and Numerical Methods for Partial Differential Equations: A Review

Mr. Yogesh Tulashidas Khawal

Department of Mathematics

Dr. Shoyeb Ali Sayyed

College Name - Malwanchal University, Indore (M.P.)-452016

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Abstract

Mathematical models are helpful test devices for structure and testing hypotheses, evaluating quantitative guesses, responding to explicit inquiries, deciding sensitivities to change in parameter esteems and assessing key parameters from information. Displaying is a basic and indistinguishable piece of all logical action, and numerous logical orders have their very own thoughts regarding explicit sorts of demonstrating. A model is a rearranged unique perspective on the perplexing reality. A logical model speaks to exact articles, wonders, and physical procedures in a coherent manner. A PDE defines a relation existing between unfamiliar performance and partial derivatives. PDEs show up often in all aspects of engineering and physics. Additionally, in current years it has seen a remarkable rise in usage of PDEs in various areas like chemistry, biology, computer sciences and in economics. In every area in which there's an interaction among a selection of independent variables, we make an effort in order to explain features in such variables as well as to write/model a range of tasks by forming equations for such features.

INTRODUCTION

Today mathematics plays an always basic job in the physical and biological sciences, inciting an obscuring of limits between logical orders and a resurgence of enthusiasm for the cutting edge and the established systems of connected mathematics. In present occasions, NL DEs have a ton of consideration because numerous physical issues in science and building are depicted scientifically by NL DEs in at least one than one ward/free variables. Change is the law of nature. Most things advance with time and are likewise differing and non-uniform in space. Most regular wonders are

NL. Linear models are those measurable models in which a progression of parameters is masterminded as a linear blend. That is, inside the model, no parameter shows up as a multiplier, divisor, or example to some other parameter. Critically, the term 'linear' in this setting does not relate to the idea of the connection between the reaction variable and the indicator variable(s), and consequently linear models are not limited to 'linear' (straight-line) connections.

A PDE is described as relation among unfamiliar performance and the partial derivatives. PDEs are come across often in all aspects of engineering and physics. Additionally, in current years, there has seen remarkable rise in usage of PDEs in all fields especially sciences as well as in economics & finance. Actually, in each area in which there's an inter-relation among a selection of independent variables, we make an effort in order to explain features in these variables as well as to represent through model a range of tasks by deriving equations for such features. When worth of unknown relation/function(s) at a particular point is based just on what happens in neighborhood of the point, in general, we represent it by a PDE. Generally, a PDE for function u $(x_1, x_2, \ldots, x_n)$ is written as

$$F(x_1, x_2, \ldots, x_n, u, u_{x1}, u_{x2}, \ldots, u_{x11}, \ldots) = 0$$

Where $x_1, x_2, \ldots, x_n$ are the independent variables, u is an unknown function, and u_{xi} denotes partial derivative $\partial u/\partial xi$, which is, supplemented by additional circumstances e.g., for instance first factors (as we've typically observed in concept of ODEs) or maybe BCs. The evaluation of PDEs may be of different types. The classical strategy which dominated during nineteenth century for developing strategies for identifying the explicit solutions. Due to enormous applications of PDEs in various branches of physics, each mathematical development which allowed a particular formula of brand-new category of PDEs is accompanied by substantial improvement in physics.

CLASSIFICATION OF DES

Among the various types of differential equations, and wide combination of course of action techniques, despite for conditions of a comparative sort, also remarkable creates. Presently, some phrases which aides in arrangement of equations & by increase, assurance of arrangement procedures.

A normal DE or ODE is a condition which depends upon in any event or auxiliary of components of single variable. DEs presented with primary models are generally conventional DEs and consider such equations exclusively in this course.

• An incomplete DE or PDE is a condition which depends upon in any event one halfway subordinate of components of a couple of factors. Generally, speaking PDE is perceived by the diminishing of various ODES.

Ex. Consider heat equation,

$\frac{\partial U}{\partial T} = K^2 \frac{\partial^2 U}{\partial X^2}$(1.1)

here K is heating constant and a fractional differential condition, its solution u (x, t) is an element of two autonomous factors, and condition consolidates halfway subordinates with respect to the two factors.

• The solicitation of a differential condition is the solicitation of the most raised auxiliary of any dark capacity in condition.

LExample. Consider linear differential equation,

$\frac{dy}{dt}$ = cy – d, (1.2)

where c and d are constants, as principal derivative of y(t) shows up in condition. Then again, for ODE

y'' + $3y'$ + 2y = 0 (1.3)

is a second-order DE, whereas PDE identified as beam comparison,

$u_t = u_{xxxxx}$(1.4)

is 4th-order DE as have 4^{th} order partial derivatives.

A DE is straight if any direct mix of courses of action of condition is in like manner a solution of condition. A DE that isn't straight is supposed to be NL.

NL conditions are considered, incredibly difficult to comprehend, so all around one approximates a NL condition by a straight-line equation, called a linearization, which is even more immediately comprehended.

Example: Consider ODE

$y' + 3t^2y = e^t$, $y'' + (sint)y'$ + ty = 0. (1.5)

are models of direct differential conditions. Here, coefficients of these conditions are elements of free factor t, yet not of dependent variable y. On other hand, PDE called as Burger's condition,

$u_t + uu_x = 0$. (1.6)

is a NL differential condition. A linearization can be obtained by displacing coefficient u of u_x with some constant or a component of simply x or t.

Benefits to the society

The complex phenomena are encountered in various fields of science and technology such as physics, engineering, and geometry. Most of the mathematical modeling of these phenomena is governed by nonlinear partial differential equations. Finding solutions of such equations is an arduous task and only in certain special cases one can write down the solutions explicitly. However, exact solutions to nonlinear partial differential equations play an important role in the understanding of many phenomena and processes throughout the natural sciences.

Scope of the study

PDEs are ordinarily used to clarify complex wonders in various areas of science and innovation like physical science, calculation, and designing. The numerical demonstrating of these techniques is by and large administered by a solitary differential condition or perhaps a strategy for differential conditions. Subsequently, tackling such

Literature Review

Alfredsson, P. &Verrijdt, J. (2015), in his paper found numerical explanation of Korteweg-De Vriesequation. The term "soliton" is defined, discussed and determination of Adomian's Special Polynomials given. We consider a two-echelon stock framework for administration parts. To acquire high assistance levels for a minimal price we permit for typical inventory of parts as well as for crisis supply choices as far as horizontal parcels and direct conveyances. Subsequent to introducing the system we use for fulfilling client interest, we build an insightful model that we use to work out important execution measures. Recreation shows that our model produces precise evaluations, and that the presentation of the stock framework is obtuse toward the lead-time dispersion. Subsequent to presenting an expense structure we show that the system we propose can bring about impressive reserve funds when contrasted with utilizing just ordinary inventory. Examination of the aftereffects of our model with the consequences of different models demonstrates that the joined utilization of sidelong parcels and direct conveyances can prompt tremendous expense investment funds.

Kilbas, A. A., and Trujillo, J. J. (2007) the paper manages the alleged differential conditions of fragmentary request wherein an obscure capacity is limited under activity of a subordinate of partial request. A study of strategies and results in hypothesis of such standard fragmentary differential conditions is given. Specifically, strategy dependent on decrease of Cauchy-type issue for the fragmentary differential conditions to the Volterra indispensable conditions is talked about, and the Laplace change, operational calculus compositional strategies for the arrangement of direct differential conditions of partial request are introduced. Issues and recent fads of exploration are talked about.

SuriyaGharib (2015) This paper direct conditions are inspected exhaustively close by removal technique. This paper contains lattice show, and the prompt methodologies for direct conditions. The target of this assessment was to examine unmistakable removal strategies of direct conditions and measure execution of Guassian end to find their general importance and ideal situation in field of agent and numeric computation.

Wati et al. (2018)A direct condition is a variable based mathematical material that exists in center school to school. It's anything but an indispensable material for understudies in order to adjust additionally created science subjects. Thusly, straight condition material is essential to be aced. In any case, the delayed consequence of 2016 public assessment in Indonesia exhibited that understudies' accomplishment in handling straight condition issue was low. This reality transformed into an establishment to investigate understudies' difficulties in dealing with direct condition issues.

This examination used abstract realistic technique. An individual created test on straight condition endeavors was controlled, followed by gatherings. 21 model understudies of survey VIII of SMPIT InsanKamilKaranganyar did formed test, and six of these were met in this manner. The result exhibited that understudies with high math accomplishment do not experience issues, understudies with medium arithmetic accomplishment experience bona fide difficulties, and understudies with low science accomplishment have honest, hypothetical, operational, and rule difficulties. Taking into account the result there is an urgent need of reality instructing approach to guide understudies with conquering difficulties in dealing with direct condition issues.

Saraswati et al. (2016) This exploration planned to depict how polynomial math tiles can bolster understudies' comprehension of LEs with single variable. This research is a piece of bigger research on learning innovative design of LEs with single variable utilizing polynomial math tiles joined to adjusting method. In this way, it simply examines one action concentrated on way how understudies utilize polynomial math tiles to figure out how to tackle LEs with single variable, designed examination is utilized as a methodology in this investigation. It comprises of three steps, to be specific (i) preliminary design (ii) teaching experiment and (iii) review analysis. Video enrollments, understudies' composed works, field notes, pre-test, and meeting methods to gather information. The information is dissected by looking at speculative learning direction and real learning procedure. The outcome demonstrates such polynomial math tiles which could bolster understudies' in understanding to locate formal solution of linear equation with single variable.

RESULTS ANDDISCUSSION

Example

Consider 4th order BV problem

$$\Gamma^4(\tau) = \tau(1 + e^\tau) + 3e^\tau + \Gamma(\tau) - \int_0^\tau \Gamma(\tau)d\tau, 0 < \tau < 1$$

Boundary conditions for problem (5.16) are:

$$\left.\begin{array}{c}\Gamma(0) = 1, \Gamma''(0) = 2,\\ \Gamma(1) = 1 + e, \Gamma''(1) = 3e,\end{array}\right\}$$

The exact solution of the problem is given by

$\Gamma = 1 + \tau e^\tau$.

Implementing Laplace transform (LT) of equation (5.16), we find

$$L(\Gamma^4)(\tau) = L\big(\tau(1 + e^\tau)\big) + 3L(e^\tau) + L\big(\Gamma(\tau)\big) - L\big(\int_0^\tau \Gamma(\tau)d\tau\big),$$

$$\Gamma(s) = \frac{1}{s} + \frac{A}{s^2} + \frac{2}{s^3} + \frac{B}{s^4} + \frac{1}{s^6} + \frac{1}{s^4(s-1)^2} + \frac{3}{s^4(s-1)} + \left\{\frac{L}{s^4}\left(\int_0^\tau \Gamma(\tau)d\tau\right)\right\}$$

where $A = \Gamma'(0)$ and $B = \Gamma''(0)$ are unfamiliar constants to be driven. Using inverse Laplace transform (ILT) on each side of (5.19), we find

$$\Gamma(\tau) = 1 + A\tau + \tau^2 + \frac{B\tau^3}{3i} + \frac{\tau^5}{5i} + L^{-1}\left\{\frac{1}{s^4(s-1)^2}\right\} + L^{-1}\left\{\frac{3}{s^4(s-1)}\right\} + L^{-1}\left\{\frac{L}{s^4}\left(\int_0^\tau \Gamma(\tau)d\tau\right)\right\}$$

$$= 2 + A\tau + \frac{\tau^2}{2} + \frac{(B-2)\tau^3}{3i} + \frac{\tau^5}{5i} - (1-\tau)e^\tau + L^{-1}\left\{\frac{L}{s^4}\left(\int_0^\tau \Gamma(\tau)d\tau\right)\right\}$$

Implementing HPM on both sides of (5.20), we find

$$\sum_{n=0}^{\infty} p^n \Gamma_n(\tau) = 2 + A\tau + \frac{\tau^2}{2} + \frac{(B-2)\tau^3}{3i} + \frac{\tau^5}{5i} - (1-\tau)e^\tau + pL^{-1}\left\{\frac{L}{s^4}\left(\sum_{n=0}^{\infty} p^n\left(\int_0^\tau \Gamma_n(\tau)d\tau\right)\right)\right\}$$

Comparing coefficients of same powers of p of equation (5.21), we find

$$\left.\begin{array}{r} p^0: \Gamma_0(\tau) = 2 + A\tau + \dfrac{\tau^2}{2} + \dfrac{(B-2)\tau^3}{3i} + \dfrac{\tau^5}{5i} - (1-\tau)e^\tau \\ p^1: \Gamma_1(\tau) = -1 - \tau - \dfrac{\tau^2}{2} - \dfrac{\tau^3}{3i} + \dfrac{(A-2)\tau^5}{5i} + \dfrac{(1-A)\tau^6}{6i} + \dfrac{(B-3)\tau^7}{7i} + \dfrac{(B-2)\tau^8}{8i} + \dfrac{\tau^9}{9i} - \dfrac{\tau^{10}}{10i} + e^\tau \end{array}\right\}$$

Imposing the boundary conditions (5.3.2) at $\tau = 1$, values of constants obtained for $\tau = 1$ are.

A = 0.999818465, B = 3.0012294.

Thus, approximate solution is given as follows:

$$\Gamma(\tau) = 2 + A\tau + \frac{\tau^2}{2} + \frac{(B-2)\tau^3}{3i} + \frac{\tau^5}{5i} - (1-\tau)e^\tau - 1 - \tau - \frac{\tau^2}{2} - \frac{\tau^3}{3!} + \frac{(A-2)\tau^5}{5i} + \frac{(1-A)\tau^6}{6i} + \frac{(B-3)\tau^7}{7i} - \frac{(B-2)\tau^8}{8i} + \frac{\tau^9}{9i} - \frac{\tau^{10}}{10i} + e^\tau \ldots$$

$$= 1 + (A-1)\tau + \frac{(B-3)\tau^3}{3i} + \tau e^{\tau} + \frac{(A-1)\tau^5}{5i} + \frac{(1-A)\tau^6}{6i} + \frac{(B-3)\tau^7}{7i} - \frac{(B-2)\tau^8}{8i} + \frac{\tau^9}{9i} - \frac{\tau^{10}}{10i} + \cdots$$

After applying the values of unknowns, A and B.

$$\Gamma(\tau) = 1 - 0.000181535\tau + 0.0002049055\tau^3$$
$$-0.000001512791667\tau^5 + 0.0000002521319444\tau^6$$
$$+0.000000243935119\tau^7 + 0.00002483207919\tau^8$$
$$+0.000002755731922\tau^9$$
$$-0.0000002755731922\tau^{10} + \tau e^{\tau} + \ldots$$

At this point, we obtained the rough option of with boundary situations after effectively applied the LT & HPM as well as by utilizing Mathematical that contrasts with that received by variational iterative method (Exact solution and VIM) as shown table 5.1 and figure 5.1. Optimum complete error is

$E(\tau) = |(1+\tau)e^{\tau} - \Gamma(\tau)|$.

Table 1: Comparison of LT-HPM, VIM and Exact solutions

τ	**LT-HPM** **$*10^{-5}$**	**VIM** **$*10^{-5}$**	**Exact** **$*10^{-5}$**	**M.A. Err (LT-HPM)** **$*10^{-5}$**	**M.A. Err (VIM)** **$*10^{-4}$**
0.2	124424	124411	124428	34668E-05	165737
0.4	159667	159645	159673	59497E-05	284804
0.6	209320	209296	209327	643182E- 05	312185
0.8	271828	278022	278043	36194E-05	213844
1.0	371828	371828	371828	5529769E-05	0.0

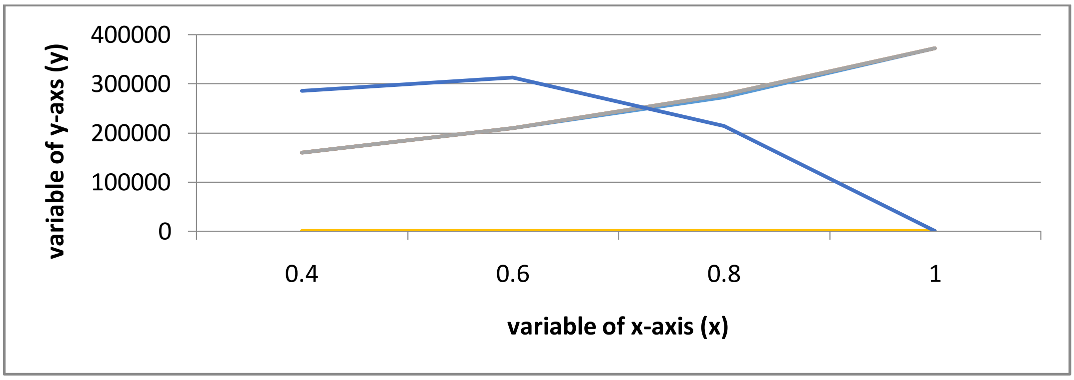

Figure 1: Graphical comparison of solution by LT-HPM solution with that by VIM solution and exact solution

Example:

We consider 4th order BVP

$\Gamma^4 (\tau) = \Gamma (\tau) + \Gamma'' (\tau) + e^{\tau} (\tau - 3)$

BCs of problem (5.25) are given as:

$$\left.\begin{matrix} \Gamma(0) = 1, \Gamma^{'}(0) = 0, \\ \Gamma(1) = 0, \Gamma^{'}(1) = -e, \end{matrix}\right\}$$

Exact solution of problem is as under:

$\Gamma (\tau) = (1 - \tau) e^{\tau}$

Taking Laplace transform of each side of (5.25), we find

$$L\left(\Gamma^4(\tau)\right) = L\big(\Gamma(\tau)\big) + L\left(\Gamma^{''}(\tau)\right) + L\big(e^{\tau}(\tau - 2)\big),$$

$$s^4 L\big(\Gamma(\tau)\big) - s^3\Gamma(0) - s^2\Gamma^{'}(0) - s\Gamma^{''}(0) - \Gamma^{'''}(0) = L\big(\Gamma(\tau)\big) + L\left(\Gamma^{''}(\tau)\right) + \frac{1}{(s-1)^2} - \frac{3}{(s-1)},$$

$$\Gamma(s) = \frac{1}{s} + \frac{A}{s^3} + \frac{B}{s^4} + \frac{L\big(\Gamma(\tau)\big)}{s^4} + \frac{L\left(\Gamma^{''}(\tau)\right)}{s^4} + \frac{1}{s^4(s-1)^2} - \frac{3}{s^4(s-1)}$$

In which $A = \Gamma''(0)$ & $B = \Gamma'''(0)$ are unknown constants to be evaluated.

After, implementing inverse Laplace transform to equation, we find

$$\Gamma(\tau) = 1 + \frac{A\tau^2}{2} + \frac{B\tau^3}{3!} + L^{-1}\left\{\frac{L}{s^4}\Big(\big(\Gamma(\tau)\big)\Big) + L\left(\Gamma^{''}(\tau)\right)\right\} + L^{-1}\left\{\frac{1}{s^4(s-1)^2}\right\} - 3L^{-1}\left\{\frac{1}{s^4(s-1)}\right\}$$

$$= 8 + 6\tau + \frac{(A+5)\tau^2}{2} + \frac{(B+4)\tau^3}{3!} + \tau e^{\tau} + 7e^{\tau} + L^{-1}\left\{\frac{L}{s^4}\Big(\big(\Gamma(\tau)\big)\Big) + L\left(\Gamma^{''}(\tau)\right)\right\}$$

After, implementing HPM on both sides of (5.30), we find

$$\sum_{n=0}^{\infty} p^n \Gamma(\tau) = 8 + 6\tau + \frac{(A+5)\tau^2}{2} + \frac{(B+4)\tau^3}{3!} + \tau e^{\tau} - 7e^{\tau} + pL^{-1}\left\{\frac{L}{s^4}\Big(\sum_{n=0}^{\infty} p^n \Big(\big(\Gamma(\tau)\big)\Big) +$$

$$L\Gamma''\tau$$

On both sides, equating coefficients of same powers of p of (5.31), we find

$$\left.\begin{matrix} p^0 : \Gamma_0(\tau) = 8 + 6\tau + \frac{(A+5)\tau^2}{2} + \frac{(B+4)\tau^3}{3!} + \tau e^{\tau} - 7e^{\tau}, \\ p^1 : \Gamma_1(\tau) = 20 + 2\tau e^{\tau} + 20e^{\tau} + 18\tau + 8\tau^2 + \frac{7}{2}\tau^3 + \frac{(A+13)\tau^4}{4!} + \frac{(B+10)\tau^5}{5!} + \dots, \\ p^2 : \Gamma_2(\tau) = 52 + 4\tau e^{\tau} - 52e^{\tau} + 48\tau + 22\tau^2 + \frac{20}{3}\tau^3 + \frac{3}{2}\tau^4 + \frac{4\tau^5}{15} + \cdots, \end{matrix}\right\}$$

where A and B denote the constant terms.

Thus, the approximate solution is given in form of constant terms of the problem (5.25) as follows

$$\Gamma(\tau) = 512 + 480\tau + \frac{(A+449)\tau^2}{2} + \frac{(B+418)\tau^3}{3!} + \frac{(385+A)\tau^4}{4!} + \frac{(B+354)\tau^5}{5!} + \frac{(2A+322)}{4!}\tau^6 + \frac{(B+146)\tau^7}{2520} + \frac{(254+3A)\tau^8}{8!} + \cdots$$

After, implementing boundary conditions (5.26) at τ = 1, the values of constants are obtained.

A = −0.999916277, B = −2.00299259.

Now, approximate solution (by putting values of constant terms) of problem is as under:

$$\Gamma(\tau) = 512 + 480\tau + 24.0000418615\tau^2 + 69.332834568333\tau^3 + 16.000003488458\tau^4 + 2.9333083950833\tau^5 + 0.44444467700833\tau^6 + 0.057141669607143\tau^7 + 0.0062252046420883\tau^8 + \ldots\ldots$$

Here, we obtained the approximate solution of after successfully applying the LT-HPM Comparison between which is precise solution. Most absolute error is

$E(\tau) = |(1-\tau)e^{\tau} - \Gamma(\tau)|$.

Table 2: Absolute Error and Comparison of LT-HPM and Exact solution

τ	**LT-HPM*10^{-9}**	**Exact *10^{-9}**	**M. A. Error** 469769*E-04
0.0	1000000000	1000000000	0
0.1	0994654749	09946538264	9228E-011
0.2	0977127897	09771222072	5690884E-10
0.3	0944918488	09449011666	17322704E-09
0.4	0895133584	08950948205	3876597E-09
0.5	0824433424	08243606378	52788749E-09
0.6	0728969053	07288475228	121530761E-09
0.7	0604311571	06041258147	185759376E-08
0.8	0445371527	04451081875	63341665E-08
0.9	0246307288	02459603111	3469769E-08
1.0	0	0	0

Conclusion

It has proposed VIM for solving non-linear PDE and in applied variation iteration method on various non-linear models like duffing equation, mathematical pendulum, vibrations of the eardrum and then compared the approximation obtained by the proposed method to the Adomian's method and conclude that VIM provides the solution faster than Adomian's method. Further, it has proposed Homotopy perturbation method (HPM) by using Homotopy concept used in topology and classical perturbation technique. It has applied HPM successfully on various non-linear differential equations. It made a comparison between HPM and HAM and concludes that HPM is a better option for non-linear problems than HAM.

It has applied HPM to fathom coupled systems of non-linear reaction-diffusion equation and compared its result with ADM. Moreover, they conclude that the result obtained from HPM is in good concurrence with those of ADM. It has used HPM for calculating the Adomianpolynomial. It gave a review of the VIM for solving some non-linear problems and he also listed useful iteration formula for some general non-linear problem. Further, he successfully implemented the variational iteration method on the integrodifferential equation, non-linear boundary value problem, oscillator, and wave equations. It has studied the solution of homogeneous and non-homogenous advection problem using VIM and ADM and presented the comparative study between these two methods. It has proposed some general guidelines to the researcher for choosing the Homotopy equation and then applied these guidelines for solving some time-dependent equation like Klein-Gordon (K-G) equation, Emden Fowler equation, Evolution equation, and Cauchy reaction-diffusion equation.

It has defined He's polynomial to solve the non-linear problem and conclude that it is an easy and effective technique for solution of NL problem than Adomian polynomial. Further, Ghorbani presented the comparative study of He's HPM with other methods like ADM, direct method, and series solution method on Integro-DEs and conclude that HPM is more reliable than other traditional methods. It has combined Laplace transformation with VIM to beat the trouble of figuring the Lagrange's multiplier and used for solving non-linear problems. Moreover, they conclude that the proposed technique is more efficient than the variational iteration method. It has pertained HPM for analytical solution of fractional NL Schrodinger equation, time, and space fractional advection-dispersion equation and fractional PDEs evolved in liquid mechanics like wave equation, Korteweg-de Vries, Zakharov-Kuznetsov equation, Burgers' equation, and Klein–Gordon (K-G) equation.

It has proposed a modified form of ADM, by this iterative method, the solution of a non-linear problem is obtained without calculating Adomian polynomial separately, this technique is implemented on NL PDEs and compared with ADM and VIM. Further, they conclude that this technique leads to the outcomes which are equivalent to those acquired by the variational iteration method It has solved time fractional diffusion equation having the external force and absorbent term whereas it has solved convection-diffusion fractional differential equation using HPM but having non-linear source term

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