A Study on Fixed Point Theorem in Cone Metric Space

M. Uma[1], P. Thirunavukarasu[2]
[1]Assistant Professor, Department of mathematics
Government arts and science college kadayanallur
(Affiliated to Manonmaniamsundaranar university)
e-mail: umaleelu@gmail.com
[2]Assistant Professor, PG & Research Department of Mathematics
Thanthai Periyar Government Arts and Science College,
Thiruchirapalli, Tamil Nadu, India
(Affiliated to Bharathidasan University)
e-mail : ptavinash1967@gmail.com

Article Info
Page Number: 6954 - 6963
Publication Issue:
Vol 71 No. 4 (2022)

Article History
***Article Received**: 25 March 2022*
***Revised**: 30 April 2022*
***Accepted**: 15 June 2022*
***Publication** 19 August 2022*

Abstract

In this paper, we look at the existence of coincidence points and the unique common fixed point for the four self maps in cone metric spaces under contractive conditions for non-continuous mappings, and also the relaxation of completeness in the space. We also demonstrated a fixed point theorem on cone metric spaces without ever using commutativity in this chapter.

Keywords: - Fixed point theorem, Cone metric space, non-continuous mapping, Commutativity

1 Introduction

Huang and Zhang [4] introduced the concept of a cone metric space in 2007, which is also a refinement of a metric space in which the set of real numbers is substituted by an ordered Banach space as well as several fixed point theorems for mapping achieving specific contractive requirements are established. Abbas and Jungck[1,2], as well as Ziaoyan Sun, Guang Xing Song, Yian Zhao, Guotao Wang [8, S. Rezapour and Halbarani [5], have investigated Huang and Zhang's [4] fixed point theorems in cone metric spaces.

The study of common fixed points of mappings satisfying specific contractive conditions has been a hot research area, with the several applications in disciplines such as differential equations, game theory, operator theory, computer science, and economics, amongst many others.

2 Preliminaries

Definition 1 [4]

If A be the real Banach space and let B be a subset of E. Now set B is called a cone :

(a) B is closed and non-empty, $B \neq \{0\}$;

(b) r,s$\in R$, r,s ≥ 0, a, b $\in$ B infersra + sb $\in$ B;

(c) a$\in P$ and –a$\in P$infers a = 0.

Definition 2 [4]

If B is considered to be a cone in a Banach space A, then express the partial ordering '$\leq$' where respect to B by a $\leq$ b if and only if b - a $\in$ B. Let a < b to indicate a $\leq$ b but a $\neq$ b, while a << b will stand for b - a $\in$ int B, here $int\ B$represents the internalof B. Now the cone Bwill be order cone.

Definition 3 [4]

If A is considered to be theBanach space and B $\subset$ A. The cone Bwill bestandard if there occurs d > 0 for each$a, b \ \in \ A$, 0 $\leq$ a $\leq$ b denotes $\|a\| \leq d\|b\|$. The minimumconstructive number T sustaining the directly above is termed the standardpersistentof B.

Definition 4 [4]

If T is the nonempty set of A. Supposingifρ: T × T → A contents:

(ρ 1). 0 <ρ (a,b) for each$a, b \ \in \ T$ and $\rho(a, b) \ = \ 0$ if and only if $a = b$;

(ρ 2). ρ (a, b) = ρ (b,a) for eacha,b$\in$ T;

(ρ 3). ρ (a,b) $\leq$ ρ (a,c) + ρ (b,c) for each$a, b, c$$\in$ T.

Nowρ is said to be the cone metric in T, (T, ρ) is said to be the cone metric space.

Definition 5 [4]

If (T, ρ) be the cone metric space. If $\{a_p\}$ be thearrangement in T and a $\in$ T. For each r $\in$ A with $0 \ << \ r$ then P will be for each p > P, ρ (a_n, a) << r, then $\{a_p\}$ will be convergent where $\{a_p\}$ congregates to a, and a will be abound of $\{a_p\}$.

This is denoted by

$$\lim_{p \to \infty} a_p = a \text{ or } a_n \to a \ (p \to \infty).$$

Lemma 1[4]

If (T, ρ) is considered as cone metric space, B will be thestandard cone with standard constant d. If $\{a_n\}$ is considered to be asarrangement in T. Now $\{a_p\}$ congregates to a if and only if ρ (a_p, a) →0 (p→∞).

Definition 6 [4]

If (T, ρ) is considered as the cone metric space, $\{a_p\}$ is considered asarrangement in T. If for any r $\in$ A with 0 <<r , there is P in such a way for eachp,q> P, ρ (a_p, a_q) << r, then $\{a_p\}$is termed a Cauchy arrangement in T.

Definition 7 [4]

If (T, ρ) is considered as the cone metric space, if each Cauchy arrangement is convergent in T, then T is said to bethe complete cone metric space.

Lemma 2 [4]

If (T, ρ) is considered as the cone metric space, B is considered to be thestandard cone with normal constant d. Let $\{a_n\}$ be a arrangement in T. Here $\{a_p\}$ will be the Cauchy arrangement if and only if ρ $(a_p, a_q)\to 0$ $(p,q\to\infty)$.

Lemma 3 [4]

If (T, ρ) is considered as the cone metric space, B considered to be thestandard cone throughstandard constant d. If $\{a_p\}$ and $\{b_p\}$ are two arrangements in T and $a_p \to a$, $b_p \to b$ ($p\to\infty$).

Then ρ $(a_p, b_p) \to \rho$ (a,b) ($p \to\infty$).

Definition 8

Let l and m be own maps of a set T. If w= la =ma for particulara in T, then a is named a accident point of l and m, and w is named a argument of accident of l and m.

Definition 9 [6]

If D, l: T → T. Then the pair (D, l) is (IT)- commuting at c ∈ T if Dlc = lDc. Here (IT)-commuting on T (Jungck and Rhoades[27]) if Dlc = lDc for each c ∈ T such that Dc = lc.

Definition 10

Ifl,m and u be a function on T throughstandards in a cone metric space (T, ρ). The duo (l,m) is asymptotically systematic with reverence to u at a_0∈ T if there happens a arrangement $\{a_p\}$ in T such a way that

$ua_{2p+1} = la_{2p}$,

$ua_{2p+2} = ma_{2p+1}$, p =0, 1, 2… and

$\lim_{p \to \infty} \rho\, (ua_p, ua_{p+1}) = 0$..

3 Main Results

For non-continuous mappings and relaxation of completeness in the space, we prove the existence of coincidence points and the unique common fixed point for the four own maps under contractive conditions in cone metric spaces.

This result extends and improves the results of StojanRadenovic [7].

In 2009, StojanRadenovic [7] proved the subsequentdeduction.

Theorem 1 [7]

If (T, ρ) be a complete cone metric space, and B a standard cone with normal constant d.

Assume if the commuting mappings l,m : T→T are such that for some constant □∈ (0,1) and for every a,b∈ T,

$\|\rho\,(la, lb)\| \le □ \|\rho\,(ma, mb)\|$.

If g's range includes l's range and m's range is continuous, then l and m have a single common fixed point.

We extend the above result for four self maps.

Theorem 2

If (T, ρ) is considered as cone metric space and B a standard cone throughstandard constant S. Assume that the functions I,J,U and V are four own maps on T in such a way that
$J(T) \subset U(T)$ and $I(T) \subset V(T)$
and satisfy the condition

$$\| \rho (Ia, Jb) \| \leq d \| \rho (Ua, Vb) \| \qquad (1)$$

for all a,b$\in$ T, where d $\in$ (0,1) is a constant.

If one of I(T), J(T), U(T), V(T) is a comprehensive subspace of T, then {I, U} and {J,V} obligate a accident point in T. Furthermore, if{I,U} and {J,V} are (IT)-commuting then, I,J,U and V have a exclusive common fixed point in T.

Proof

Consider anuninformed point a_0 in T, hypothesisarrangements $\{a_p\}$ and $\{b_p\}$ in T such that
$b_{2p} = Ia_{2p} = Ja_{2p+1}$ and $b_{2p+1} = Ja_{2p+1} = Ua_{2p+2}$, for all p=0, 1, 2, . . . By (1),
we have

$$\begin{aligned}\| \rho (b_{2p}, b_{2p+1}) \| &= \| \rho (Ia_{2p}, Ja_{2p+1}) \|,\\ &\leq d \| \rho (Ua_{2p}, Va_{2p+1}) \|\\ &\leq d \| \rho (b_{2p-1}, b_{2p}) \|.\end{aligned}$$

Similarly, it can be shown that

$$\rho (b_{2p+1}, b_{2p+2}) \leq d\rho (b_{2p}, b_{2p+1}).$$

Therefore, for all p,

$$\| \rho (b_{p+1}, b_{p+2}) \| \leq d \| \rho (b_p, b_{p+1}) \| \leq \ldots \leq d\, p+1 \| \rho (b_0, b_1) \|.$$

Now, for any q > p,

$$\begin{aligned}\| \rho (b_p, b_q) \| &\leq \| \rho (b_p, b_{p+1}) \| + \rho (b_{p+1}, b_{p+2}) \| + \ldots + \| \rho (b_{q-1}, b_q) \|\\ &\leq [d^p + d^{p+1} + \ldots + d^{q-1}] \| \rho (b_1, b_0) \|\\ &\leq d^p / 1-d \ \| \rho (b_1, b_0) \|.\end{aligned}$$

From (Definition 3), we have

$$\| \rho (b_p, b_q) \| \leq d^p / 1-d\, L \| \rho (b_1, b_0) \|,$$

which implies that $\| \rho (b_p, b_q) \| \to 0$ as $p, q \to \infty$, since $0 < d < 1$.

Hence $\{b_p\}$ is a Cauchy sequence.

It is assumed thatI(T) is complete subspace of T.

Completeness on I(T) inferspresence of $c \in I(T)$

As $\lim_{p\to\infty} b_{2p} = Ia_{2p} = c$.

$\lim_{n\to\infty} Va_{2p+1} = \lim_{n\to\infty} Ia_{2p} = \lim_{n\to\infty} Ua_{2p} = \lim_{n\to\infty} Ja_{2p+1} = c$.

Here for $0 << r$, for sufficiently huge p, we obligate$\rho\ (b_p , c) << r$.

Subsequently $c \in J(T) \subseteq U(T)$,

then there occurs a point $w \in X$ such that $c = Uw$.

Toverify that $c = Iw$.

By the three-way relationship inequality, we have

$$\|\rho\ (Iw, c)\| \le \|\rho\ (Iw,Ja_{2p+1})\| + \|\rho\ (Ja_{2p+1}, c)\|$$
$$\le d(\|\rho\ (Uw, Va_{2p+1})\|)+\|\rho\ (Ja_{2p+1}, c)\|.$$

Letting $p\to\infty$, we get

$$\|d(Su, z)\| \le k\ \|d(z, z)\|+\|d(z, z)\|, \le k(0)+0\ , = 0.$$

Thus, $Iw= c$.

Therefore, $c = Iw = Uw$. (2)

That is, p is aaccidentopinion of I and U.

Since, $c \in I(T) \subseteq U(X)$,

then there exists a point $z \in T$ such that $c = Vz$.

To proveJz = c.

Here $\|\rho\ (Jz, c)\| \le \|\rho\ (Iw,Jz)\|$

$$\le d\ \|\rho\ (Uw, Vz)\|$$
$$\le d\ \|\rho\ (c,c)\|$$
$$= 0$$

Implies $Jz = c$. (3)

Therefore, $c = Jz = Vz$.

That is, w is aaccident opinion of J and V.

From (2) and (3) it follows
Iw = Uw = Jz = Vz (= c).

Since (I,U) and (J,V) are (IT)–commuting

$$
\begin{aligned}
\| \rho\,(\mathrm{IIw}, \mathrm{Iw}) \| &= \| \rho\,(\mathrm{IIw}, \mathrm{Uw}) \| \\
&= \| \rho\,(\mathrm{IIw}, \mathrm{Jz}) \| \\
&\le d\, \| \rho\,(\mathrm{UIw}, \mathrm{Vz}) \| \\
&= d\, \| \rho\,(\mathrm{IUw}, \mathrm{Iw}) \| \\
&= d\, \| \rho\,(\mathrm{IIw}, \mathrm{Iw}) \|
\end{aligned}
$$

a contradiction (since, $d< 1$).
$\Rightarrow \rho$ (IIw, Iw) = 0.

Therefore,
IIw = Iw (= c).
Iw = IIw = IUw = UIw.

That is,
IIw = UIw = Iw (= c).
Therefore, Iw (= c) is a mutual fixed point of I and U. (4)

Similarly, Jz = JJz= JVz = VJz.
Implies, JJz = VJz = Jz (= c).

Therefore,
Jz (= c) is a mutual fixed point of J and V (5)

From (4) and (5) it tracks I,J,U and V obligate a mutual fixed point specifically c.

Let c1 be alternativemutual fixed point of I,J,U and V.
Then

$$
\begin{aligned}
\| \rho\,(c, c1) \| &= \| \rho\,(\mathrm{Ic}, \mathrm{Jc1}) \| \\
&\le d\, \| \rho\,(\mathrm{Uc}, \mathrm{Vc1}) \| \\
&\le d\, \| \rho\,(c, c1) \|
\end{aligned}
$$

$< \| \rho\,(c, c1) \|$ (Since, $d < 1$)
which is a contradiction
$\Rightarrow c = c1$.
Therefore, I,J,U and V obligate a exclusivemutual fixed point.

Remark 1
If I = J and U = V then the statementdiminishes to the Statement1 of StojanRadenovic [7] with U(T) comprehensive, this is an enhancement of Statement 2 of [7].

SubsequentlyV(T) is comprehensive,this will be an super space of U(T).

4 Common Fixed Point Theorem in Cone Metric Spaces

We show a fixed point theorem in cone metric spaces that generalises Theorem 1 of [4] without relying on commutativity.

M. Abbas and G. Jungck [2] proved the following theorem in 2008.

Theorem 3

If (T, ρ) is considered to be the cone metric space, and B a standard cone with standardcontinual d. Suppose the mappings l.m : $T \rightarrow T$ satisfy

ρ (la, lb) $\leq d\rho$ (ma, mb), for all a,b$\in$ T where d $\in [0,1)$ is a perpetual.

If m's range includes l's range and m(T) is a full subspace of T, l and m have a single point of coincidence in T.

Furthermore, if l and m are weakly compatible, they share a single fixed point in common.

Theorem 4

If (Y, ρ) is considered to be a cone metric space and B a standard cone with standard constant d.

Assumeif the functionsl,m : $T \rightarrow T$ are such that for all a,b$\in$ T

$$\rho \text{ (la, lb)} \leq \square\, \rho \text{ (ma, mb)} \tag{6}$$

for some constant $\square\square \in [0,1)$

If the range of m contains the range of l and m(T) is a complete subspace of T, then l and m have a coincidence point in T, and l,m have a unique common fixed point in T.

Proof

If a_0 is considered to be theuninformed point in T, and let $a_1 \in$ T bepreferredin such a way

$b_0 = l(a_0) = m(a_1)$.

Since

$l(T) \subseteq m(T)$, $a_2 \in T$ can be chosen such that

$b_1 = l(a_1) = m(a_2)$.

Remaining this progression,

consumingpreferred $a_p \in T$,

we chose $a_{p+1} \in$ T such that $b_p = l(a_p) = m(a_{p+1})$.

We first show that

ρ (b_p, b_{p-1}) $\leq$ □ρ (b_{p-1}, b_{p-2}) for p = 2, 3 …

Indeed,

ρ (b_p, b_{p-1}) = ρ (la_p , la_{p-1})

$\leq$ □ρ (ma_p , ma_{p-1}) (by 6)

$\leq$ λd(yn-1 , yn-2).

Implies that

ρ (b_p, b_{p-1}) $\leq$ □ρ (b_{p-1}, b_{p-2})

$\leq \ldots \leq$ □$^{p-1}$ ρ (b_1, b_0). (7)

Now we shall show that {b_p} is a Cauchy sequence.

By the triangle inequality, for p > q we have

ρ (b_p, b_q) $\leq$ ρ (b_p, b_{p-1})+ ρ (b_{p-1}, b_{p-2}) + … + ρ (b_{q+1}, b_q).

Now by (7) ,ρ (b_p , b_q) $\leq$ (□$^{p-1}$ +□$^{p-2}$ +…+ □^{q}) ρ (b_1, b_0).

From (Definition .3) implies

‖ ρ (b_p, b_q)‖ $\leq$ 1 q □ − □ d‖ ρ (b_1, b_0)‖→ 0 as q→∞.

It follows that {b_p} is a Cauchy sequence.

Ifm(T) is comprehensive, there happens a u in m(T) such that b_p → v as p → ∞.

Therefore, to obtain a u in T such a way that m(u)=v. to prove that l(u) = v.

From (6) ρ (ma_p, lu) = ρ (la_{p-1}, lu)

$\leq$ □ ρ (ma_{p-1}, mu),

⇒ρ (mu, lu)

$\leq$ □ ρ (mu, mu) = 0.

That is,

ρ (mu, lu) = 0.

Hence, mu = v= lu, u is a accident point of l and m.

From (6),

ρ (u,mu) $\leq$ ρ (u,b_p) + ρ (b_p,mu) (by the triangle inequality)

= ρ (u,b_p) + ρ (la_p, lu)(Since, lu = mu)

$\leq$ ρ (u,b_p) + □ ρ (ma_p, mu).

From (Definition 3),

‖ ρ (u, mu)‖ $\leq$ d (‖ρ (u, la_p)+ □ ρ (ma_p, mu)‖)

$\leq$ d (‖ρ (u, la_p)‖ + □‖ ρ (ma_p, mu)‖) as p→ ∞

we have

$\| \rho\ (u, mu) \| \leq d\ (\| \rho\ (u,v) \| + \square \| \rho\ (v, mu) \|)$
$\leq d\ (\| \rho\ (u, mu) \| + \square \| \rho\ (mu, mu) \|)$
$\leq d \| d(p, gp) \|$
$< \| \rho\ (u, mu) \|$
$\Rightarrow \| \rho\ (u, mu) \| = 0.$

Hence, u = mu.

Now, ρ (lu,u) = ρ (lu, mu)
= ρ (lu, lu) (since lu=mu)
$\leq \square\ \rho$ (mu, mu) ≤ 0.
$\Rightarrow \rho$ (lu, u) = 0.
That is, lu = u.
Since, lu = mu.
Therefore, lu = mu = u, l and m have a mutual fixed point in T.

Let u1 be additionalmutual fixed point of l and m, then
ρ (u, u1) = ρ (lu, mu1)
= ρ (lu, lu1)
$\leq \square\ \rho$ (mu, mu1)
$\leq \square\ \rho$ (u,u1)
$<\rho$ (u,u1), a contradiction.

Implies ρ (u, u1) = 0.
That is u = u1.
Therefore, l and m have a unique common fixed point in T.

5 Conclusion

The existence of coincidence points and a single common fixed point for the four self maps in cone metric spaces for non-continuous mappings and relaxation of completeness in the space are investigated under contractive conditions. We also established a fixed point theorem on cone metric spaces without using commutativity in this chapter.

6. Reference

1. M. Abbas and G. Jungck, Common fixed point results for non commuting mappings without continuity in cone metric spaces. J. Math Anal. Appl. 341(2008) 416-420.
2. M.Abbas and B.E.Rhoades, Fixed and periodic point results in cone metric spaces, Appl. Math. Lett. 22 (2009), 511-515.
3. Guang Xing Song ,Xiaoyan Sun , Yian Zhao, Guotao Wang. New common fixed point theorems for maps on cone metric spaces. Applied Mathematic Letters 32(2010) 1033-1037.

4. L.G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings,J.Math.Anal.Appl.332(2) (2007)1468-1476.
5. S.Rezapour and Halbarani, Some notes on the paper “cone metric spaces and fixed point theorem of contractive mappings “, J. Math. Anal. Appl. 345(2008), 719-724.
6. S.L. Singh, ApichaiHematulin and Rajendra Pant, New coincidence and Common fixed point theorems, Applied General Topology, Vol. 10, No.1 (2009), 121-130.
7. StojanRadenovic, Common fixed points under contractive conditions in cone metric spaces, Computers and Mathematics with Applications 58(2009)1273-1278.
8. Ziaoyan Sun, Guang Xing Song ,Yian Zhao, Guotao Wang. New common fixed point theorems for maps on cone metric spaces. Applied Mathematic Letters 32(2010) 1033-1037.