# Some New Results for Certain Subclasses of Normalized Analytic Functions

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***Abstract***
In this Paper we define new subclasses of normalized regular functions namely $\mathcal{M}$ (t, p,α). Some examples for the functions in $\mathcal{M}$(t, p,α) are also discussed. With certain condition we find the maximum radius in unit disc for which functions in the subclass $\mathcal{M}$(0, p,α) to be in Y $((\alpha+1+|p|)^2)$.

**Keywords**: regular functions, univalent functions

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## 1. INTRODUCTION

Let A denote the family of all functions that are regular in unit disc Ʊ:= $\{z\colon |z| < 1\}$ and satisfy g(0)=0, g'(0)=1. Let M= {w: w is analytic in $D, |w(z)| \le 1, |z| \le 1\}$. S is set of all functions $g \in A$ that are univalent in D. [2] has introduced set $\mathcal{M}$ (α ) of all f∈ $A$ which satisfies,

$$\left|\left(\frac{z}{g(z)}\right)^2 g'(z) - 1\right| < \alpha \qquad z \in D, \alpha \in \mathbb{R}^+ \tag{1.1}$$

It is observed that well known Koebefunction given by:

$$K(z) = z(1-z)^{-2}$$

Then K (z) is member of $\mathcal{M}$(1), but it is not the member of the class of all starlike function of order t, where $t > 0$. Also the holomorphic function $\frac{2z+z^2}{2}$ is member $\mathcal{M}$(1) but not belongs to class $S^*(t)$, $t > 0$.

$\Upsilon\ (\lambda)$ is the collection of all g∈ $A$ with $g(z)\ \neq 0$ in punctured unit disc $\mho$, satisfying :

$$|(\frac{z}{g(z)})''| \leq \lambda \tag{1.2}$$

According to result due to Obradovic [1] and Ponnusamy,

$Y\ (2\ \lambda) \subset \mathcal{M}(0,1,\alpha)\ \subset S$ for $0 \leq \alpha \leq 1$ and

In current years, the class $\mathcal{M}$ (0, 1, 1) were studied in detail by [2],[3],[4][5],[9]. In the next result we used the term $\Upsilon\ (\lambda)$ radius means the maximum value of the radius in unit disc for which the functions g (z) in $\mathcal{M}$ (0, p, α) to be in $\Upsilon\ (\lambda)$.

**Example1.1 Show that the function defined by**

$$\frac{z-t}{f(z)} = p + \frac{\propto}{16}(z-t)^3$$

**belongs to the class $\mathcal{M}$(t, p, ∝).**

$$\text{Let } g(z) = \frac{z-t}{f(z)} = p + \frac{\propto}{16}(z-t)^3$$

$$g'(z)= \frac{p-\frac{\propto}{8}(z-t)^3}{(p+\frac{\propto}{16}(z-t)^3)^2}$$

$$\left|\left(\frac{z-t}{g(z)}\right)^2 g'(z) - p\right| = \left|\frac{\left(p+\frac{\propto}{16}(z-t)^3\right)^2 (p-\frac{\propto}{8}(z-t)^3)}{\left(p+\frac{\propto}{16}(z-t)^3\right)^2} - p\right|$$

$$= \left|\frac{\left(p+\frac{\propto}{16}(z-t)^3\right)^2\left(p-\frac{\propto}{8}(z-t)^3\right)-p\left(p+\frac{\propto}{16}(z-t)^3\right)^2}{\left(p+\frac{\propto}{16}(z-t)^3\right)^2}\right|$$

$$= \left|\frac{\propto}{8}(z-t)^3\right|$$

$$< \frac{\propto}{8}|(z-t)^3|$$

$$= \frac{\propto}{8}.\ 8$$

$$= \propto$$

Hence $g(z) \in \mathcal{M}(t, p, \propto).z^3$

**Example 1.2 Show that the function defined by**

$$\frac{z}{g(z)} = p + \frac{\propto}{2}(z)^3$$

**belongs to the class** $\mathcal{M}$**(0,p,** $\propto$) **and belongs to class** $\mathrm{Y}(2\propto)$ **for** $|z| < \frac{2}{3}$.

**Let** $g(z) = \frac{z}{p+\frac{\propto}{2}z^3}$

$$g'(z) = \frac{p-\propto z^3}{\left(p+\frac{\propto}{2}(z)^3\right)^2}$$

$$\left|\left(\frac{z}{g(z)}\right)^2 g'(z) - p\right| = \left|\frac{\left(p+\frac{\propto}{2}(z)^3\right)^2 (p-\propto(z)^3)}{\left(p+\frac{\propto}{2}(z)^3\right)^2} - p\right|$$

$$= \left|\frac{\left(p+\frac{\propto}{2}(z)^3\right)^2 (p-\propto(z)^3) - p\left(p+\frac{\propto}{2}(z)^3\right)^2}{\left(p+\frac{\propto}{2}(z)^3\right)^2}\right|$$

$= |\propto z^3|$

$= \propto|z^3|$

$< \propto$

Hence, $g(z) \in \mathcal{M}(0, p, \propto)$.

Now, $\left(\frac{z}{g(z)}\right)' = \frac{3\propto}{2} z^2$

$$\left|\left(\frac{z}{g(z)}\right)''\right| = 3\propto z$$

$$\left|\left(\frac{z}{g(z)}\right)''\right| < 2\propto, \quad \text{for } |z| < \frac{2}{3}.$$

Hence, $\Upsilon(2\propto)$ $radius\ for\ the\ function\ g(z)\ to\ be\ in$ $\mathcal{M}$ (0, p, ∝) is $\frac{2}{3}$

**2.** $\Upsilon((|\propto| + p + 1)^2)$**- Radius For The Class $\mathcal{M}$(0 ,p, ∝) with Certain Condition**

Shaffer [7] gives following lemma.

**Lemma 2.1** Let h (z) be analytic for $|z| < 1$ and Re h $(z) > k, 0 \leq k < 1$. h (z) has following expansion,

$$h(z)=1+a_l z^l + a_{l+1} z^{l+1} + \text{--------},\quad l \geq 1, \text{ then}$$

(i) $$|h'(z)| \leq l\, |z|^{l-1} \frac{(|h(z)+c|)^2}{c+1} \quad \text{for } |z| \leq \frac{(1+l^2)^{\frac{1}{2}}-1}{l}$$

(ii) $$|h'(z)| \leq |z|^{l-2}(4|z|^2 + l^2(1-|z|^2)^2 \frac{(|h(z)+c|)^2}{4(1-|z|^2)(1+c)}$$

Where c=1-2k for $|z| > \frac{(1+l^2)^{\frac{1}{2}}-1}{l}$

**Theorem 2.2.** Let f $\in \mathcal{M}(0, p, \propto)$ with Re $\left(\left(\frac{z}{g(z)}\right)^2 g'(z)\right) > 0$, then $\left|\left(\frac{z}{g(z)}\right)''\right| \leq (\propto + |p| + 1)^2$ with $|p| - \propto \geq 0$

**Proof.** Suppose g∈ $\mathcal{M}$ (0, p, ∝)

$$\left|\left(\frac{z}{g(z)}\right)^2 g'(z) - p\right| < \propto$$

$$H_g(z) = \left(\frac{z}{g(z)}\right)^2 g'(z) - p$$

$$\left|\left(\frac{z}{g(z)}\right)^2 g'(z) - p\right| < \propto$$

If $g(z) = z + a_2 z^2 + a_3 z^3 + \text{------}$

$$t(z) = \frac{z}{g(z)} = b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \text{----}$$

z=t (z) g (z)

Computing and equating coefficient we get ,

$$\frac{z}{g(z)} = 1 - a_2 z + (a_2^2 - a_3)\, z^2 + ----$$

$$H_g(z) = 1 - p + (a_3 - a_2^2) z^2 + ------$$

$$H_g(z) + p = 1 + (a_3 - a_2^2) z^2 + ------$$

$$(H_g)'(z) = \left(\frac{z}{g(z)}\right)' - z\left(\frac{z}{g(z)}\right)'' - \left(\frac{z}{g(z)}\right)'$$

$$= -z\left(\frac{z}{g(z)}\right)''$$

$$|H_g(z) + p| \leq |H_g(z)| + |p| < \propto + |p|$$

$$-(\propto + |p|) < Re(H_g(z) + p) < \propto + |p|$$

$$0 \leq Re(H_g(z) + p) < \propto + |p|$$

By applying lemma 2.1(i) with $H_g(z) + p$, c=1-2k, k=0

$$|(H_g)'(z)| = |(H_g(z) + p)'|$$

$$< |z|\,(\propto + |p| + 1)^2$$

Hence by (2.2) implies

$$\left|\left(\frac{z}{g(z)}\right)''\right| \leq (\propto + |p| + 1)^2 \qquad \text{for } |z| < r_0 = \frac{\sqrt{5}-1}{2}$$

Thus, g has $P((|\propto| + p + 1)^2$ property for the disc for $|z| < r_0 = \frac{\sqrt{5}-1}{2}$.

Similarly by lemma 2.1(ii)

$$\left|\left(\frac{z}{g(z)}\right)''\right| \leq \frac{1 - |z|^2 + |z|^4}{|z|(1 - |z|^2)} \frac{(|\propto| + p + 1)^2}{2} = \varphi(z) \quad \text{for} \quad |z| > r_0 = \frac{\sqrt{5}-1}{2}$$

Where,

$$\varphi(t)=\frac{1-|t|^2+|t|^4}{|t|(1-|t|^2)}\frac{(\propto+|p|+1)^2}{2} \qquad r_0<t<1$$

Here $\varphi$ is increasing function with $\varphi(r_0)=2<\varphi(t)$ for $r_0< t < 1$.

Now we will prove that $r_0$ is the best possible value.
Consider the function defined by

$$\frac{z}{g_b(z)}= p- \propto z\int_0^z \frac{z+b}{1+bz}\,dz$$

Where b is real and $|b|< 1$.

$$\left|\frac{z+b}{1+bz}\right| < 1.$$

Let define the function $e : \mho \to \mho$ as $e(z) = \frac{z+b}{1+bz}$

It is observed that e (z) is regular function.

$$\left| \propto z\int_0^z \frac{z+b}{1+bz}\,dz \right| = \left|\propto\right| |z^2| \left|\int_0^1 e(zt)dt\right|$$

$$\leq |\propto| |z^2| \int_0^1 |e(zt)|dt$$

$$\leq |\propto| |z^2| \int_0^1 \left|\frac{z+b}{1+bz}\right|dt$$

$$= \propto |z^2|$$

$$<\propto$$

We will prove that $\propto z\int_0^z \frac{z+b}{1+bz}\,dz \neq p.$

Conversly suppose that,

$$\propto z\int_0^z \frac{z+b}{1+bz}\,dz= p$$

Which together with (2.4) gives us that $|p| <\propto$.

It contradicts to the assumption that $|p| - \propto\geq 0$

$\therefore \propto z \int_0^z \frac{z+b}{1+bz} \mathrm{dz} \neq p.$

Hence $\frac{z}{g_b(z)} \neq 0.$

$g_b$ is well defined. 2

Let $r_1$ be fixed number such that,

$r_0 < r_1 < 1$ and $b_1 = \frac{1-2r_1^2}{r_1^3}$

$\therefore r_1^2\ (1 - r_1) < 1 - r_1^2$ and $1 - r_1^2 < r_1^2\ (1 + r_1)$

$\therefore r_1^2\ (1 - r_1) < 1 - r_1^2 < r_1^2\ (1 + r_1)$

$r_1^2 - r_1^3 < 1 - r_1^2 < r_1^2 + r_1^3$

$- r_1^3 < 1 - 2\, r_1^2 < r_1^3$

$\therefore -1 < \frac{1-2r_1^2}{r_1^3} < 1$

$\therefore -1 < b_1 < 1.$

Hence $|b_1| < 1$

With some simplification we will get

$\left| \left( \frac{z}{g_{b_1}(z)} \right)'' \right| \leq \frac{1-{r_1}^2+{r_1}^4}{r1(1-{r_1}^2)} \frac{(\propto+|p|+1)^2}{2}$ for $|z| > r_0 = \frac{\sqrt{5}-1}{2}$

$= e\ (r_1) > 2\, \frac{(\propto+|p|+1)^2}{2}$

$$\therefore \left| \left( \frac{z}{g_{b_1}(z)} \right)'' \right| > (\propto + |p| + 1)^2$$

$\therefore$for $r_0 < r_1 < 1$ there exist function $g_{b_1} \in \mathcal{M}(0,\mathrm{p}, \propto)$ *such that*

$$\left|\left(\frac{z}{g_{b_1}(z)}\right)''\right| > (\propto + |p| + 1)^2$$

Hence $\mathcal{M}(0, p, \propto)$ has $(\propto + |p| + 1)^2$ property in disc $|z| < r_0 = \frac{\sqrt{5}-1}{2}$ but not in disc with longer radius.

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