Scattering State Solutions of a Rationally Extended PT Symmetric Complex Potentials Using Potential Algebra Approach

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Article Info
Page Number: 8340 - 8350
Publication Issue:
Vol 71 No. 4 (2022)

Article History
***Article Received**: 25 March 2022*
***Revised**: 30 April 2022*
***Accepted**: 15 June 2022*
***Publication**: 19 August 2022*

Abstract

In this work, we extend the idea of the potential algebra approach (also known as group theoretic approach) to the exactly solvable rationally extended (RE) potentials and obtain their scattering state solutions. As an example we consider the RE Scarf-II potential, which is complex and PT symmetric and obtain its transmission and reflection amplitudes by considering the asymptotic behavior of the generators corresponding to the above algebra.

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1 Introduction

Recently, after the discovery of two new families of orthogonal polynomials namely the X_l Laguerre and X_m Jacobi exceptional orthogonal polynomials (EOPs) [1, 2], a list of exactly solvable (ES) potentials is extended corresponding to the known usual potentials [3]. The bound state solutions of these extended potentials are already obtained by different authors using different approaches [4-15] and shown that the energy eigenvalues of these extended potentials are same as that of the corresponding usual potentials but the wavefuctions are completely different and obtained in terms of EOPs. These extended potentials also satisfy the usual supersymmetric shape invariant (SI) property [3]. Later on the scattering state solutions to these potentials are obtained using the conventional approach (i.e., by assuming asymptotic behaviors of the wavefunctions) [16, 17, 18] and shown that in particular case of $l = 0$ these corresponds to the conventional one and for $l = 1$ the obtained potentials are the rationally extended translationally SI potentials whose solutions are in terms of X_1 Laguerre or X_1 Jacobi polynomials [19, 20].

Since the last few decades the idea of the combined parity (P) and time reversal (T) symmetry in complex systems yields a large family of ES potentials [21] which have ample of applications in theoretical as well as experimental physics. After the discovery of EOPs, the conventional PT symmetric complex potentials have also been extended rationally [20] and obtained their solutions in the form of X_l EOPs.

Thereafter, following the ideas of Alhassid and others i.e., the idea of potential algebra [22, 23, 24, 25], some of the extended real as well as PT symmetric potentials are obtained by modifying the generators of the associated so(2;1) or sl(2;C) algebras through the introduction of a new operator [26, 27]. Hence the bound states corresponding to the extended potentials are also obtained in an elegant fashion. But it is not clear about the generators while finding the scattering sate solutions through this potential algebra approach. So, it will be interesting to find the scattering amplitudes of the extended potentials using the potential algebra approach, which shows that this approach is not only limited to obtain bound states only, it is also suitable to obtain the scattering states of the rationally extended potentials.

In the present work, first we discuss in brief the sl(2;C) algebra suitable to the extended potentials. Then, we consider an example of a rationally extended PT symmetric complex Scarf II potential and obtain the scattering amplitudes (reflection and transmission amplitudes) using the asymptotic behavior of the generators associated with the algebra.

The plan of the paper is as follows: In section 2, we briefly review the works of Yadav et al [26] and discuss the sl(2;C) algebra. The expression for the RE complex Scarf II potential and the corresponding bound states are also written in a closed form. In section 3, we obtain the scattering amplitudes of this potential in a completely algebraic way. Finally we summarize the results in section 4.

2 The sl(2;C) algebra

In this section, to make this manuscript self consistent, we follow the works of Yadav et al [26] and discuss the sl(2;C) algebra and its unitary representations. For an illustration, we use this algebra to the RE PT symmetric complex Scarf II potential associated with X_1 EOPs and discuss its bound states.

It is well known that that the sl(2;C) algebra consists of three generators $K_\pm$ and K_z and satisfying the commutation relations

$$[K_+, K_y] = -2K_z;\ \ [K_z, K_\pm] = \pm 2K_z \tag{1}$$

The differential realization of these generators corresponding to the RE potentials [26] are defined as

$$K_\pm = e^{\pm i\varphi}\left[\pm\frac{\partial}{\partial x} - \left(\left(-i\frac{\partial}{\partial\varphi} \pm \frac{1}{2}\right)P(x) - Q(x)\right) - T(x, -i\frac{\partial}{\partial\varphi} \pm \frac{1}{2})\right]$$

and

$$K_z = -i\frac{\partial}{\partial\varphi}. \tag{2}$$

Here P(x) and Q(x) are two functions and $T(x, -i\frac{\partial}{\partial\varphi} \pm \frac{1}{2})$ is a functional operator. This functional operator $T(x, -i\frac{\partial}{\partial\varphi} \pm \frac{1}{2})$act on a basis $|\tilde{k}, m_1\rangle$ to give a function $(x, m_1 \pm \frac{1}{2})$. These three functions will be different for the different potentials. Here at least one (or all) of the function(s) must be complex and hence $K_+ \neq K_-^\dagger$.

In order to satisfy the sl(2;C) algebra by these new generators $K_\pm, K_-$and K_z the commutation relations (1) have to be satisfied, which provides following restrictions on the functions P(x), Q(x) and $T(x, m_1 \pm \frac{1}{2})$

$$\frac{d}{dx}P(x) + P(x)^2 = 1; \qquad \frac{d}{dx}Q(x) + P(x)Q(x) = 0 \tag{3}$$

and

$$[T^2\left(x, m_1 - \frac{1}{2}\right) - \frac{d}{dx}T(x, m_1 - \frac{1}{2}) + 2T\left(x, m_1 - \frac{1}{2}\right)\left(P(x)\left(m_1 - \frac{1}{2}\right) - Q(x)\right)] - [T^2\left(x, m_1 + \frac{1}{2}\right) - \frac{d}{dx}T(x, m_1 + \frac{1}{2}) + 2T\left(x, m_1 + \frac{1}{2}\right)\left(P(x)\left(m_1 + \frac{1}{2}\right) - Q(x)\right)]=0. \tag{4}$$

The Casimir, for the sl(2;C) algebra in terms of the above generators is given by

$$K^2 = K_z^2 - \frac{1}{2}(K_\pm K_- + K_- K_+) = K_z^2 \mp K_z - K_\pm K_\mp\ . \tag{5}$$

For the bound states, the basis for an irreducible representation of extended algebra is characterized by

$K^2|\tilde{k},m_1\rangle=\tilde{k}(\tilde{k}+1)|\tilde{k},m_1\rangle$; $\quad K_z|\tilde{k},m_1\rangle=\tilde{k}|\tilde{k},m_1\rangle$ (6)

$$K_\pm|\tilde{k},m_1\rangle = \left[-\left(\tilde{k}\mp m_1\right)(\tilde{k}\pm m_1+1)\right]^{1/2}|\tilde{k},m_1\pm 1\rangle. \qquad (7)$$

Using (2), the differential realization of the Casimir operator in terms of P(x), Q(x) and T(x, K_z-1/2) is given by

$$K^2=\frac{d^2}{dx^2}+\left(1-P^2(x)\right)\left(K_z^2-\frac{1}{4}\right)-2\frac{dQ(x)}{dx}K_z-Q^2(x)-\frac{1}{4}-\left[T^2\left(x,K_z-\frac{1}{2}\right)+\left(\left(K_z-\right.\right.\right.$$

12Px−Q(x)Tx,Kz−12+Tx,Kz−12Kz−12Px−Q(x)−ddxTx,Kz−12, (8)

and the basis $|\tilde{k},m_1\rangle$ in the form of function is given as

$$|\tilde{k},m_1\rangle=\psi_{\tilde{k}m_1}(x,\varphi)=\psi_{\tilde{k}m_1}(x)=e^{im_1\varphi}. \qquad (9)$$

The functions (9) satisfy the Schrodinger equation

$$\left[\frac{d^2}{dx^2}+V_{m_1}(x)\right]\psi_{\tilde{k}m_1}(x)=E\psi_{\tilde{k}m_1}(x), \qquad (10)$$

where $V_{m_1}(x)$ is one parameter family of m_1-dependent potential given by

$$V_{m_1}(x)=(P^2(x)-1)\left({m_1}^2-\frac{1}{4}\right)+2\tilde{k}\frac{d}{dx}\,Q(x)+Q^2(x)+\left(m_1-\frac{1}{2}\right)^2+\left[T^2\left(x,m_1-\frac{1}{2}\right)+\right.$$

2k−12Px−QxTx,m1−12−ddxTx,m1−12, (11)

and the corresponding energy eigenvalues are given by

$$E_{\tilde{k}}=-\left(\tilde{k}+\frac{1}{2}\right)^2 \ . \qquad (12)$$

Thus the Hamiltonian in terms of the Casimir operator is given by

$$H=-\left(K^2+\frac{1}{4}\right). \qquad (13)$$

Unitary representation of sl(2;C) algebra: Here we discuss two classes of unitary representation of sl(2;C) algebra:

(a) In the case of discrete spectrum for which $\tilde{k} < 0$ i.e.,

$$m_1 = \tilde{k} + n; \;\; n = 0,1,2 \ldots, \tag{14}$$

Thus the energy eigenvalues (12) corresponding to this series will be

$$E_n = -\left(n - \left(m_1 - \tfrac{1}{2}\right)\right)^2. \tag{15}$$

(b) In the case of continuous spectra for which $\tilde{k}$ is complex i.e.,

$$\tilde{k} = -\tfrac{1}{2} + i\kappa; \quad (0 < \kappa < \infty); \quad m_1 = 0, \pm 1, \pm 2, \ldots, or\; m_1 = \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \ldots\ldots, \tag{16}$$

which describes the scattering state of (10) with energy

$$\mathrm{E_n} = \kappa^2 > 0. \tag{17}$$

2.1 Example of PT symmetric complex Scarf II potential

The bound states of the conventional PT symmetric complex Scarf II potential are obtained by defining [25]

$$P(x) = \tanh(x)\,; \qquad Q(x) = ibsech(x) \tag{18}$$

In addition to these functions for the rationally extended complex Scarf II potential the function $T(x, m_1 \pm 1/2)$ [26] is defined as

$$T\left(x, m_1 \pm \tfrac{1}{2}\right) = \left[\frac{2ibcos\,h(x)}{\left(-2ibsin\,h(x) + 2\left(m_1 \pm \frac{1}{2}\right) - 1\right)} - \frac{2ibcos\,h(x)}{\left(-2ibsin\,h(x) + 2\left(m_1 \pm \frac{1}{2}\right) + 1\right)}\right] \tag{19}$$

such that Eqs. (3) and (4) are satisfied.

Substituting these functions P(x), Q(x) and $T\left(x, m_1 \pm \frac{1}{2}\right)$ in (11), we get the rationally extended PT symmetric Scarf II potential which is defined on the full-line $-\infty < x < \infty$ i.e.,

$$V(x, m_1) = V_{Scarf}\,(x, m_1) + V_{rat}\,(x, m_1), \tag{20}$$

where

$$V_{Scarf}\,(x, m_1) = \left[(ib)^2 - \left(m_1 - \frac{1}{2}\right)\left(k + \frac{1}{2}\right)\right]\mathrm{sech}^2(x) + ib\left[2\left(m_1 - \frac{1}{2}\right) + 1\right] sech(x)\tanh(x) \tag{21}$$

is the conventional PT symmetric Scarf-II potential [24] and

$$V_{rat}(x, m_1) = \frac{-4m_1}{(-2ib\sin h(x)+2m_1)} + \frac{2[4(ib)^2+(2m_1)^2]}{(-2ib\sin h(x)+2m_1)^2} \quad (22)$$

is the rational part of the extended potential. The energy eigenvalues for this extended complex potential are real and isospectral to the conventional one given by

$$E_n = -\left(n - \left(m_1 - \frac{1}{2}\right)^2\right); n = 0,1, \ldots . n_{\{max\}}; \;\; n_{\{max\}} < \left(m_1 + \frac{1}{2}\right). \quad (23)$$

3 Calculation of scattering amplitudes

In this section, we follow the works of Alhassid et al [22, 23] in which the continuous series representation function $\psi_{\tilde{k}m_1}(x)$ (16) satisfies the one-dimensional Schrodinger equation

$$\left(-\frac{d^2}{dx^2} + V(x, m_1)\right)\psi_{\tilde{k}m_1}(x) = \kappa^2\psi_{\tilde{k}m_1}(x). \quad (24)$$

Now we define the asymptotic scattering states by considering the $limit\; x \to \pm\infty$ i.e., $|\tilde{k}, m_1\rangle^{\pm\infty} = \lim_{x\to\pm\infty}|\tilde{k}, m_1\rangle$.The asymptotic generators $K_\pm^{\pm\infty}$, $K_z^{\pm\infty}$ are similarly related to $K_\pm; K_z$.

Thus from Eq. (2), we have

$$K_\pm^{\infty} = e^{\pm i\phi}[\pm\frac{\partial}{\partial x} - \left(-i\frac{\partial}{\partial\phi} \pm \frac{1}{2}\right)]$$

$$K_\pm^{-\infty} = e^{\pm i\phi}[\pm\frac{\partial}{\partial x} + \left(-i\frac{\partial}{\partial\phi} \pm \frac{1}{2}\right)]$$

and

$$K_z^{\pm\infty} = K_z = -i\frac{\partial}{\partial\phi}. \quad (25)$$

These asymptotic generators still form an sl(2;C) algebra,

$$[K_+^{\pm\infty}, K_-^{\pm\infty}] = -2K_z^{\pm\infty};$$

and

$$[K_z^{\infty}, K_{\pm}^{\infty}] = \pm K_{\pm}^{\infty}; \qquad [K_z^{-\infty}, K_{\pm}^{-\infty}] = \pm K_{\pm}^{-\infty} \tag{26}$$

The asymptotic states $|\tilde{k}, m_1\rangle^{\pm\infty}$ have the form

$$|\tilde{k}, m_1\rangle^{-\infty} = A_{m_1} e^{i\kappa x} e^{im_1\phi} + B_{m_1} e^{-i\kappa x} e^{im_1\phi}$$

$$|\tilde{k}, m_1\rangle^{+\infty} = C_{m_1} e^{i\kappa x} e^{im_1\phi} \tag{27}$$

This asymptotic states still form a standard basis for sl(2;C), i.e.,

$$K_{\pm}^{\infty}|\tilde{k}, m_1\rangle^{+\infty} = \left[(m_1 \mp \tilde{k})(m_1 \pm \tilde{k} \pm 1)\right]^{1/2} |\tilde{k}, m_1 \pm 1\rangle^{+\infty}$$

$$K_{\pm}^{\infty}|\tilde{k}, m_1\rangle^{-\infty} = \left[(m_1 \mp \tilde{k})(m_1 \pm \tilde{k} \pm 1)\right]^{1/2} |\tilde{k}, m_1 \pm 1\rangle^{-\infty}$$

$$K_z^{\infty}|\tilde{k}, m_1\rangle^{\pm\infty} = m_1 |\tilde{k}, m_1\rangle^{\pm\infty} \tag{28}$$

We rewrite the asymptotic scattering states (27) in the form of ket vector as

$$|\tilde{k}, m_1\rangle^{-\infty} = A_{m_1}|+\kappa, m_1\rangle^{-\infty} + B_{m_1}|-\kappa, m_1\rangle^{-\infty}$$

$$|\tilde{k}, m_1\rangle^{+\infty} = C_{m_1}|-\kappa, m_1\rangle^{+\infty} \tag{29}$$

where

$$|\kappa, m_1\rangle^{\pm\infty} = e^{\pm i\kappa x} e^{im_1\phi}\ . \tag{30}$$

Now we define the Euclidean E(2) algebra which is composed of two translational generators u_1 and u_2 and of one rotational generator K_z and satisfy the relation

$$[u_+, u_-] = 0; [K_z, u_{\pm}] = \pm u_{\pm} \tag{31}$$

where$u\pm = u_1 \pm iu_2$. In terms of coordinates $x$and φ the asymptotic $E(2)$ generators have the form

$$u_{\pm}^{\infty} = u_{\pm}^{-\infty} = e^{\pm i\Phi}\left(-i\frac{\partial}{\partial x}\right)$$

$K_z^{\pm\infty} = K_z = -i\frac{\partial}{\partial\Phi}$, (32)

The Casimir invariant is thus given by

$u^2 = u_1^2 + u_2^2$. (33)

The irreducible representations of $E(2)$ are labeled by $\pm\kappa$ and the action of $u_{\pm}^{\pm\infty}, u_{-}^{\pm\infty}, K_z$ and u^2 in these representation is given by

$$u_{\mp}^{\infty}|\kappa, m_1\rangle = u_{\mp}^{-\infty}|\kappa, m_1\rangle = \kappa|\kappa, m_1 \mp 1\rangle;$$

$$u_{\mp}^{\infty}|-\kappa, m_1\rangle = u_{\mp}^{-\infty}|-\kappa, m_1\rangle = -\kappa|\kappa, m_1 \mp 1\rangle$$

$u^2|\pm\kappa, m_1\rangle = \kappa^2|\pm\kappa, m_1\rangle$, and $K_z|\pm\kappa, m_1\rangle = m_1|\pm\kappa, m_1\rangle$. (34)

Using Eqs. (32) and (34) the generators $K_{+}^{\pm\infty}$ in terms of $u_{\pm}^{\pm\infty}(= u_+)$ and $K_z^{\infty}(= K_z)$ are given by

$K_{+}^{\infty} = \frac{1}{(\pm\kappa)}[\left(\frac{1}{2} \pm i\kappa\right)u_+ - K_z u_+]$,

and

$K_{-}^{-\infty} = \frac{1}{(\pm\kappa)}[\left(\frac{1}{2} \pm i\kappa\right)u_+ + K_z u_+]$. (35)

For $x \to \infty$, on operating K_{+}^{∞} on Eq. (29), we get

$K_{+}^{\infty}|\mathrm{k}, m_1\rangle^{\infty} = \left(-m_1 + ik - \frac{1}{2}\right)C_{m_1}|k, m_1 + 1\rangle$ (36)

On the other hand, from Eqs. (28) and (29), we obtain

$K_{+}^{\infty}|\mathrm{k}, m_1\rangle^{\infty} = [(m_1 - \mathrm{k})(m_1 + \mathrm{k} + 1)]^{\frac{1}{2}}C_{m_1+1}|\kappa, m_1 + 1\rangle$ (37)

Comparing Eq. (36) with Eq. (37), we get

$$[(m_1 - \mathrm{k})(m_1 + \mathrm{k} + 1)]^{\frac{1}{2}} C_{m_1+1} = (-m_1 + ik - \tfrac{1}{2}) Cm_1. \quad (38)$$

Similarly for $x \rightarrow -\infty$, operating $K_+^{-\infty}$on Eq. (29) we get

$$K_+^{-\infty}|\mathrm{k}, m_1\rangle^{-\infty} = A_{m_1}\left(m_1 + ik + \frac{1}{2}\right)|k, m_1 + 1\rangle + B_{m_1}\left(m_1 - ik + \frac{1}{2}\right)|-k, m_1 + 1\rangle \quad (39)$$

and Eqs. (28) and (29) provide

$$K_+^{-\infty}|\mathrm{k}, m_1\rangle^{-\infty} = [(m_1 - \mathrm{k})(m_1 + \mathrm{k} + 1)]^{\frac{1}{2}} A_{m_1+1}|\kappa, m_1 + 1\rangle + \left[(m_1 - \mathrm{k})(m_1 + \mathrm{k} +\right.$$

112Bm1+1−κ, m1+ 1. (40)

Again comparing Eq. (39) with Eq. (40), we get

$$\left[(m_1 - \mathrm{k})(m_1 + \mathrm{k} + 1)\right]^{\frac{1}{2}} A_{m_1+1} = \left(m_1 + i\kappa + \frac{1}{2}\right) A_{m_1} \quad (41)$$

and

$$\left[(m_1 - \mathrm{k})(m_1 + \mathrm{k} + 1)\right]^{\frac{1}{2}} B_{m_1+1} = \left(m_1 - i\kappa + \frac{1}{2}\right) B_{m_1} (42)$$

Thus from Eqs. (41) and (42) the recursion relation for reflection amplitude $r_{m_1+1,1}(\kappa) = A_{m_1+1}/B_{m_1+1}$is given by

$$r_{m_1+1,1}(\kappa) = \frac{\left(m_1 - i\kappa + \frac{1}{2}\right)}{\left(m_1 + i\kappa + \frac{1}{2}\right)} r_{m_1,1}(\kappa)(43)$$

and from Eqs. (38) and (41) the recursion relation for transmission amplitude $t_{m_1+1,1}(\kappa) = C_{m_1+1}/A_{m_1+1}$is given by

$$t_{m_1+1,1}(\kappa) = \frac{\left(-m_1 - i\kappa - \frac{1}{2}\right)}{\left(m_1 + i\kappa + \frac{1}{2}\right)} t_{m_1,1}(\kappa) \,. \quad (44)$$

After solving these two equations, we finally obtain the transmission and reflection amplitude in the forms of gamma functions as

$$r_{m_1+1,1}(\kappa) = \frac{\Gamma\left(m_1 - i\kappa - \frac{1}{2}\right)\Gamma\left(i\kappa + \frac{3}{2}\right)}{\Gamma\left(m_1 + i\kappa - \frac{1}{2}\right)\Gamma\left(-i\kappa + \frac{3}{2}\right)} r_{1,1}(\kappa) \,, \quad (45)$$

and

$$t_{m_1+1,1}(\kappa) = (-1)^{m_1-1} R_{m_1,1}(\kappa) \times \left(\frac{t_{1,1}(\kappa)}{r_{1,1}(\kappa)}\right), \qquad (46)$$

where $r_{1,1}(\kappa)$ and $t_{1,1}(\kappa)$ is independent of k. The poles of the scattering amplitudes (45) giving the correct bound states (15). In this way the same may be easily calculated for the potentials associated with X_l EOPs and the corresponding reflection $r_{m_1+1,l}(\kappa)$and transmission amplitudes $t_{m_1+1,l}(\kappa)$ can be obtained.

4 Summary and discussion

In this manuscript, we have discussed in brief the potential algebra approach suitable to the RE potentials. Using the asymptotic behavior of the generators associated with *sl*(2, C) algebra the transmission and reflection amplitudes for the RE *PT* -symmetric complex Scarf II potential whose bound states are associated with the X_1 exceptional Jacobi polynomials have been obtained. The poles of the scattering amplitudes giving the correct bound states. Thus we are now able to obtain the complete spectrum of the conventional as well as the rationally extended potentials through this approach. The approach can easily be extended to the X_l case of extended potentials.

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