Some Finite Value Problem Applications of Wronskian Characteristics

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Abstract
In this essay, we've covered a few Wronskian characteristics and illustrated how they work using relevant instances. In the current work, we also created a few new Wronskian characteristics and used examples to validate the findings.
Keywords — Wronskian, Properties of Wronskian.

Introduction

Engineering, physics, chemistry, economics, and other fields all benefit greatly from the application of differential equations. The boundary value problem, which consists of a classical differential equation and a further set of restrictions known as the parameter, plays a significant role in many areas.

Properties of Wronskian

Definition 2.1. Let φ_1, φ_2 be any two differential functions of x

$$\text{Then, } W[\varphi_1, \varphi_2] = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi'_1 & \varphi'_2 \end{vmatrix}$$

which is called the Wronskian of φ_1, φ_2. It is a function, and its value at x isdenoted by

$W[\varphi_1, \varphi_2](x)$.

Lemma 2.2. The Wronskian defined in (2.1) satisfies the following prop-erties:

(a) $W[\varphi_1, \varphi_2] = -W[\varphi_2, \varphi_1]$.

(b) $W[\alpha\varphi_1, \beta\varphi_2] = \alpha\beta W[\varphi_1, \varphi_2]$.Here α and β are constants.

Proof. (a) From the definition of Wronskian, we get

$W[\varphi_2, \varphi_1] = \varphi_2\varphi' - \varphi'\varphi_1 = -W[\varphi_1, \varphi_2]$.

(b) By the definition of Wronskian with simple steps, our result is trivial.

Lemma 2.3. Let φ_1, φ_2 be any two differential functions of x, then

$$\mathbf{W[\varphi_1 + \alpha, \varphi_2 + \alpha] = W[\varphi_1, \varphi_2] + \alpha\ d/dx\ [\varphi_2 - \varphi_1]} \quad \mathbf{(2.2)}$$

where α is a constant.

Proof. From definition of Wronskian, we obtain

$W[\varphi_1+\alpha, \varphi_2+\alpha] = (\varphi_1+\alpha)\varphi_1{}' - \varphi_2{}' \quad (\varphi_2+\alpha)$

$= W[\varphi_1, \varphi_2] + \alpha \ d/dx\ [\varphi_2 - \varphi_1]$

Hence the lemma is proved

Example: Let Φ_1 = Sin3x, Φ_2 = Cos5x, and let α = 7,

$$W[\text{Sin}3x+7, \text{Cos}5x+7 = \begin{vmatrix} \text{Sin}3x+7 & \text{Cos}5x+7 \\ 3\,Cos3x & -5Sin5x \end{vmatrix}$$

$= (\text{Sin}3x+7)(-5Sin5x) - (3\,Cos3x)(\text{Sin}3x) - (3\,Cos3x)(\text{Cos}5x$

$= W[\Phi_1, \Phi_2] + 7\frac{d}{dx}[\Phi_2 - \Phi_1]$.

Example: A boundary value problem is given by,

y'' + 8y = 0, subject to the condition, y (0) = 3 and y $(\frac{\pi}{2})$ = 0.

Auxiliary equation is, $m^2 + 8 = 0$

$\Rightarrow m = 0 + i2\sqrt{2}$

Hence general solution, $y = k_1 \text{Cos}(2\sqrt{2})x + k_2 \text{Sin}(2\sqrt{2})x$

using the boundary condition, y(0) = 3 and y $(\frac{\pi}{2})$ = 0.

$3 = k_1 . 1 + k_2 . 0$

$k_1 = 3$

Given condition is y $(\frac{\pi}{2})$ = 0

$0 = 3.\text{Cos}(2\sqrt{2}) + k_2 \text{Sin}(2\sqrt{2})$

$k_2 = -\frac{3.\text{Cos}(\sqrt{2}\pi)}{\text{Sin}(\sqrt{2}\pi)}$

$= -3\text{Cot}(\sqrt{2}\pi)$

Hence, Particular solution is given by

$y = 3\text{Cos}(2\sqrt{2}\,t) - 3\,\text{Cot}(\sqrt{2}\,\pi)\ \text{Sin}(2\sqrt{2}\,t)$.

Wronskian:

Jozef Hoene-Wronski coined the term "Wronkian" in the field of mathematics to describe a determinant (1776). It is necessary in the study of differential equations since it aids in determining the linear independence of the solutions [1] and [2] set. In [3] and [4], the characteristics and solution of the Wronskian diff erential equation were explored.

Let a linear homogeneous equation of the form, y'' + p(t) y' + q(t) y = 0,

Let two solution of this equation are u and v, So Wronskian of this equation can be written as W[u, v] = uv' - vu'. 1.1

(a) If u is a constant multiple of v then W [u, v] is identically zero. Then u and v are linearly dependent.

(b) If u and v agree at some point t_o and their derivative also exist at t_o, then W[u, v] vanishes at t_o, thais if u and v are two solution of same initial value problem then their Wronskian vanishes t_o

If there are non-zero constants, two differential functions, f(t) and g(t), are linearly dependent. and c_2 with, C_1 f(t) + C_2 g(t) =0, for all t, otherwise they are called linearly independent.

Example: The functions, f(t) =10$t^2 + t^3$ and g(t) = $t^4 - t^4$ are linearly independent.

There would be a nonzero constant and such that if the function f(t) and g(t) are directly dependant.

$C_1 f(t) + C_2 g(t) = 0$

$$C_1(10t^2 + t^3) + C_2(-t^4) = 0$$

When t= - 1, then,
$9c_1 - c_2 = 0$ (1)
When t= - 2, then,
$2c_1 - c_2 = 0$

The linear solution system is represented by equations (1) and (2). The relevant coefficient matrix's current determinant is
$\begin{vmatrix} 9 & -1 \\ 2 & -1 \end{vmatrix} = -9 + 2 = -7 \neq 0.$
As a result of the determinant being nonnegative, only the trivial solution can be found.
That is, $c_1 = c_2$.
The two functions supplied are therefore linearly independent.

Theorem:1

$W[\ \emptyset_1\ \emptyset_2] = -\ W[\ \emptyset_2\ \emptyset_1]$

Example: Φ_1= tanx and Φ_2 = Cosecx, then

$$W[\ \emptyset_1\ \emptyset_2] = \begin{vmatrix} tanx & cosecx \\ sec^2x & -cosecx\ Cotx \end{vmatrix}$$
$$= -\text{tanx. Cosecx.Cotx} - \text{Cosecx.}Sec^2x$$
$$= -\text{Cosecx}\ (1 + Sec^2x),$$

$$W[\ \emptyset_2\ \emptyset_1] = \begin{vmatrix} cosecx & tanx \\ -cosecx\ Cotx & sec^2x \end{vmatrix}$$

$$= \text{Cosecx.}Sec^2x + \text{tanx. Cosecx.Cotx}$$
$$= \text{Cosecx}(1 + Sec^2x)$$

$$= -\ W[\ \emptyset_1\ \emptyset_2]$$

Hence, $W[\ \emptyset_1\ \emptyset_2] = -\ W[\ \emptyset_2\ \emptyset_1]$

Theorem :2

$W[\alpha\emptyset_1, \beta\emptyset_2] = \alpha\beta\ W[\emptyset_1, \emptyset_2]$, where α and β are constant,

Example: $\emptyset_1 = a^x$ and $\emptyset_2 = e^x$, and α = 5 and β = 7, then
$W[\alpha\emptyset_1, \beta\emptyset_2] = W[5a^x, 7e^x]$
$$= \begin{vmatrix} 5a^x & 7e^x \\ 5a^x logx & 7e^x \end{vmatrix}$$
$$= 5\text{x}\ 7 \begin{vmatrix} a^x & e^x \\ a^x logx & e^x \end{vmatrix}$$
$$= 5 \times 7\ W[\emptyset_1, \emptyset_2].$$

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