

# A Hybridized Lobatto Quadrature of Precision Eleven for Numerical Integration of Analytic Functions

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## Article Info

Page Number: 8496-8506

Publication Issue:

Vol. 71 No. 4 (2022)

## Article History

Article Received: 15 September 2022

Revised: 25 October 2022

Accepted: 14 November 2022

Publication: 21 December 2022

## Abstract

A hybridizedeleven precision quadrature rule using Lobatto 6-point rule and modified form of Lobatto 4-point rule through kronrod extension is formed. This rule is capable of evaluating line integral of analytic functions. The hybridized rule has been tested both theoretically through error analysis and numerically using some test integrals. It is found that the constructed rule is more effective than that of theconstituent rules. It is alsoverified that the hybridized rulewhen appliedin adaptive environment gives significantly better results than its constituents.

**Key words:** Lobatto six point transformedrule,Hybridised rule,Kronrod extension ofLobatto four-point rule,  $SM_{L_6KEL}(f)$ .

## 1. Introduction

Several mixed quadrature rules developed in the papers [2],[4]for numerical evaluation of real definite integrals.

Some authors in their papers [8],[10] modified the mixed quadrature rules of earlier others to form transformed rules [6] for numerical evaluation of line integral of analytic functions.

The authors S.K. Mohanty, D. Das and R.B. Dash [8], S.K. Mohanty, R.B. Dash [9],[10],[11],[12] used the mixed rules as base rules to evaluate real definite integrals as well as line integrals of analytic functions in adaptive quadrature schemes, very few mixed quadrature rules of precision higher than 9 [8],[12] are available so far.

We used hybridized quadrature as a synonym of mixed quadrature in this paper. Usually, two quadraturesof identical precision are mixed are mixed suitably to get a quadrature rule of higher precision. The resulting quadrature rule is known as mixed quadrature rule. By doing this we increasing the precision of the quadrature rules in a very simplified manner unlike Richardson extrapolation and Kronrod extension.

In this paper, we designed a Hybridized rule of precision eleven out of two quadrature rules each of precision nine. The analytical error estimate of this rule and its constituent rules are studied. The theoretical predictions are verified evaluating test integrals. The highlights of the Hybridized rule have been shown in tables and figures. Using suitable adaptive scheme for

the Hybridized rule it is seen that the number of steps required to achieve some pre-assign accuracy is drastically reduced.

**2. Lobatto6-pointtransformed rule.**

The(n+1) point Gauss-Legendrerule [1],[12],[13] is given by

$$\int_{-1}^1 f(z)dz = \sum_{k=0}^n \omega_k f(z_k)(2.1)$$

Where  $\omega_k$ 's are (n + 1) weights and  $z_k$ 's are(n + 1)nodes. The (2n + 2) unknowns can be obtained by assuming the rule to be exact for all polynomials of degree (2n + 1).The Lobatto integration method [1], [13] are of Gauss types(2.1) with two end points pre-assigned as -1 and 1. For n=5, we get the weights  $\frac{1}{15}, \frac{14+\sqrt{7}}{30}, \frac{14-\sqrt{7}}{30}$  and the nodes  $\pm 1, \pm \sqrt{\frac{7-2\sqrt{7}}{21}}, \pm \sqrt{\frac{7+2\sqrt{7}}{21}}$  respectively. Using the nodes and weights, the **Lobatto 6-point transformed rule** is given by

$$L_6(f) = \int_{z_0-h}^{z_0+h} f(z)dz$$

$$= \frac{h}{15} \{f(z_0 - h) + f(z_0 + h)\} + \frac{14 + \sqrt{7}}{30} h \{f(z_0 - \alpha h) + f(z_0 + \alpha h)\}$$

$$+ \frac{14-\sqrt{7}}{30} h \{f(z_0 - \beta h) + f(z_0 + \beta h)\}(2.2)$$

where  $\alpha = \sqrt{\frac{7-2\sqrt{7}}{21}}$  and  $\beta = \sqrt{\frac{7+2\sqrt{7}}{21}}$

**Lemma1**

If  $f(z)$  is analytic in the domain  $\Omega \supset [z_0 - h, z_0 + h]$ , then the rule  $L_6(f)$  is of precision nine and the truncation error due to  $L_6(f)$  is  $EL_6(f) \cong \frac{-256 h^{11}}{6615 \cdot 11!} f^{(11)}(z_0)$  and  $O(h^{11})$ .

*Proof* Let us denote truncation error of  $L_6(f)$  is by  $EL_6(f)$ .

We know that  $I(f) = L_6(f) + EL_6(f)$

$$EL_6(f) = I(f) - L_6(f)(2.3)$$

Applying Taylor's theorem [1],[7] in (2.2) and the exact value of the integral  $I(f)$  we get

$$L_6(f) = 2h \left[ f(z_0) + \frac{h^2}{3!} f^{ii}(z_0) + \frac{h^4}{5!} f^{iv}(z_0) + \frac{h^6}{7!} f^{vi}(z_0) + \frac{h^8}{9!} f^{viii}(z_0) \right] +$$

$$\frac{1226h^{11}}{6615 \times 10!} f^x(z_0) + \frac{650 h^{13}}{3969 \times 12!} f^{xii}(z_0) + \dots(2.4)$$

$$I(f) = 2h \left[ f(z_0) + \frac{h^2}{3!} f^{ii}(z_0) + \frac{h^4}{5!} f^{iv}(z_0) + \frac{h^6}{7!} f^{vi}(z_0) + \frac{h^8}{9!} f^{viii}(z_0) + \frac{h^{10}}{11!} f^x(z_0) + \frac{h^{12}}{13!} f^{xii}(z_0) + \dots \right](2.5)$$

By using (2.4) and (2.5) in (2.3), we get

$$EL_6(f) = -\frac{256}{6615} \frac{h^{11}}{11!} f^x(z_0) - \frac{512}{3969} \frac{h^{13}}{13!} f^{xii}(z_0) + \dots \quad (2.6)$$

The truncation error establishes that the degree of precision of the rule  $L_6(f)$  is nine,

$$EL_6(f) \cong -\frac{256}{6615} \frac{h^{11}}{11!} f^x(z_0) \text{ and } O(h^{11}). \square$$

### 3. Kronrod extension of Lobatto 4-point rule

The Kronrod extension of the Lobatto 4-point rule [3],[5], [11] is denoted by  $KEL_4(f)$ , is given by

$$\int_{z_0-h}^{z_0+h} f(x) dx \approx KEL_4(f)$$

where

$$KEL_4(f) = \frac{h}{1470} \left[ 77 \{ f(z_0 - h) + f(z_0 + h) \} + 432 \left\{ f \left( z_0 - \frac{\sqrt{2}}{\sqrt{3}} h \right) + f \left( z_0 + \frac{\sqrt{2}}{\sqrt{3}} h \right) \right\} + 625 \left\{ f \left( z_0 - \frac{h}{\sqrt{5}} \right) + f \left( z_0 + \frac{h}{\sqrt{5}} \right) \right\} + 672 f(z_0) \right] \quad (3.1)$$

Applying Taylor's theorem [1],[7],[12] after simplification we obtain

$$KEL_4(f) = 2h \left[ f(z_0) + \frac{h^2}{3!} f^{ii}(z_0) + \frac{h^4}{5!} f^{iv}(z_0) + \frac{h^6}{7!} f^{vi}(z_0) + \frac{h^8}{9!} f^{viii}(z_0) + \frac{4741}{4725} \frac{h^{10}}{11!} f^x(z_0) + \frac{72059}{70875} \frac{h^{12}}{13!} f^{xii}(z_0) + \dots \right] \quad (3.2)$$

#### Lemma2

Let us denote, the truncation error due to Kronrod extension of Lobatto 4-point rule by  $EKEL_4(f)$ , then  $EKEL_4(f) \cong -\frac{32}{4725} \frac{h^{11}}{11!} f^x(z_0)$  and  $O(h^{11})$ .

*Proof* We have  $I(f) = KEL_4(f) + EKEL_4(f)$

$$\Rightarrow EKEL_4(f) = I(f) - KEL_4(f) \quad (3.3)$$

Using (2.5) and (3.2) on (3.3), we obtain

$$EKEL_4(f) = 2h \left[ -\frac{16}{4725} \frac{h^{10}}{11!} f^x(z_0) - \frac{1184}{70875} \frac{h^{12}}{13!} f^{xii}(z_0) - \dots \right]$$

$$\text{or} \quad EKEL_4(f) = -\frac{32}{4725} \frac{h^{11}}{11!} f^x(z_0) - \frac{2368}{70875} \frac{h^{13}}{13!} f^{xii}(z_0) - \dots \quad (3.4)$$

The expression (3.4) the truncation error of the rule  $KEL_4(f)$ . From (3.4) we also concluded that the degree of precision of the Kronrod extension of Lobatto 4-point rule is 9 and of  $O(h^{11})$ .  $\square$

### 4. Formulation of the Hybridized quadrature rule of precision eleven

The construction of the proposed Hybridized quadrature rule is given in the following theorem.

**Theorem1**(Formulationof $SM_{L_6KEL}(f)$ )

If  $f(z)$  is analytic in the given domain  $\Omega \supset [z_0 - h, z_0 + h]$ , then the Hybridize rule  $SM_{L_6KEL}(f)$  and truncation error due to the Hybridize rule  $ESM_{L_6KEL}(f)$  are given by

$$SM_{L_6KEL}(f) = \frac{1}{33} [40 KEL_4(f) - 7 L_6(f)] \quad \text{and} \quad ESM_{L_6KEL}(f) = \frac{1}{33} [40 EKEL_4(f) - 7 EL_6(f)].$$

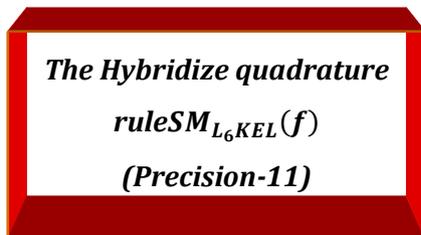
*Proof*

Recalling  $I(f) = L_6(f) + EL_6(f)$ (4.1)

$I(f) = KEL_4(f) + EKEL_4(f)$ (4.2)

Subtracting 7 times of (4.1) from 40 times of (4.2), we get

$$33 I(f) = [40 KEL_4(f) - 7 L_6(f)] + [40 EKEL_4(f) - 7 EL_6(f)]$$



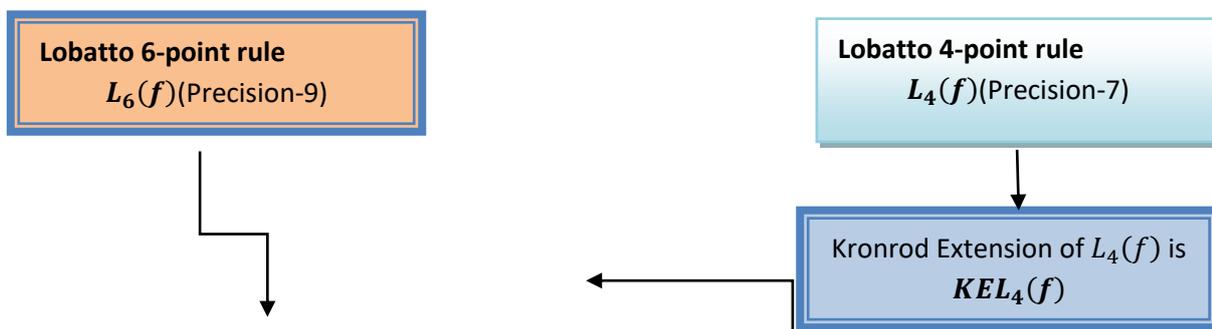
$$\Rightarrow I(f) = \frac{1}{33} [40 KEL_4(f) - 7 L_6(f)] + \frac{1}{33} [40 EKEL_4(f) - 7 EL_6(f)]$$

$$\Rightarrow I(f) = SM_{L_6KEL}(f) + ESM_{L_6KEL}(f)$$

Where  $SM_{L_6KEL}(f) = \frac{1}{33} [40 KEL_4(f) - 7 L_6(f)]$ (4.3)

and  $ESM_{L_6KEL}(f) = \frac{1}{33} [40 EKEL_4(f) - 7 EL_6(f)]$ (4.4)

The expression (4.3) is the proposed Hybridized rule and (4.4) is the truncation error associated due to the rule.  $\square$



**Figure-1:** Construction of the Hybridize rule of precision-11.

**5. Error Analysis**

An error analysis of the constructed rule has been obtained by the following Theorems.

**Theorem2**

If  $f(z)$  is analytic in the given domain  $\Omega \supset [z_0 - h, z_0 + h]$ , then the truncation error associated due to the rule  $SM_{L_6KEL}(f)$  is given by  $ESM_{L_6KEL}(f) \cong -\frac{2048 h^{13}}{4725 13!} f^{xii}(z_0)$ .

*Proof* Using (2.6) and (3.4) on (4.4), we get

$$ESM_{L_6KEL}(f) = -\frac{2048 h^{13}}{4725 13!} f^{xii}(z_0) \dots$$

$$\Rightarrow ESM_{L_6KEL}(f)(f) \cong -\frac{2048 h^{13}}{4725 13!} f^{xii}(z_0) \text{ [Since truncation error} = O(h^{13})]$$

□

**Theorem3**

The Error bound of the constructed Hybridize quadrature rule is

$$|ESM_{L_6KEL}(f)| \leq \frac{256 M h^{11}}{31185 11!} |\xi_2 - \xi_1|, \quad \xi_1, \xi_2 \in [z_0 - h, z_0 + h], \quad \text{where } M = \max_{z_0 - h \leq z \leq z_0 + h} |f^{xi}(z)|.$$

*Proof* From (2.6), we get  $EL_6(f) \cong -\frac{256 h^{11}}{6615 11!} f^x(\xi_1), \quad \xi_1 \in [z_0 - h, z_0 + h]$ ,

and from (3.4), we get  $EKEL_4(f) \cong -\frac{32 h^{11}}{4725 11!} f^x(\xi_2), \quad \xi_2 \in [z_0 - h, z_0 + h]$ ,

using above two values on (4.4), we can write

$$\begin{aligned} ESM_{L_6KEL}(f) &= \frac{1}{33} [40 EKEL_4(f) - 7 EL_6(f)] \\ ESM_{L_6KEL}(f) &\cong \frac{1}{33} \left[ 40 \left\{ -\frac{32 h^{11}}{4725 11!} f^x(\xi_2) \right\} - 7 \left\{ -\frac{256 h^{11}}{6615 11!} f^x(\xi_1) \right\} \right] \\ &= \frac{256 h^{11}}{31185 11!} \{f^x(\xi_1) - f^x(\xi_2)\} \\ &= \frac{-256 h^{11}}{31185 11!} \{f^x(\xi_2) - f^x(\xi_1)\} \\ &= \frac{-256 h^{11}}{31185 11!} \int_{\xi_1}^{\xi_2} f^{xi}(z) dz \\ \Rightarrow |ESM_{L_6KEL}(f)| &\cong \frac{256 h^{11}}{31185 11!} \left| \int_{\xi_1}^{\xi_2} f^{xi}(z) dz \right| \\ &\leq \frac{256 h^{11}}{31185 11!} \int_{\xi_1}^{\xi_2} |f^{xi}(z)| dz \\ &\leq \frac{256 h^{11}}{31185 11!} \int_{\xi_1}^{\xi_2} M dz, \quad \text{where } M = \max_{z_0 - h \leq z \leq z_0 + h} f(z) \end{aligned}$$

$$\Rightarrow |ESM_{L_6KEL}(f)| \leq \frac{256 M h^{11}}{31185 \cdot 11!} |\xi_2 - \xi_1| \quad (5.1)$$

Since  $\xi_1$  and  $\xi_2$  are arbitrarily chosen points in the interval  $[z_0 - h, z_0 + h]$ , (5.1) shows that the absolute value of the truncation error will be less if the points  $\xi_1$  and  $\xi_2$  are close to each other.

**Corollary.**

The error bound for the truncation error is  $|ESM_{L_6KEL}(f)| \leq \frac{512 M h^{12}}{22869 \cdot 11!}$ ,  $M = \max_{z_0-h \leq z \leq z_0+h} |f^{xi}(z)|$ .

*Proof* From the theorem-4

$$|ESM_{L_6KEL}(f)| \leq \frac{256 M h^{11}}{31185 \cdot 11!} |\xi_2 - \xi_1|, \quad \xi_1, \xi_2 \in [z_0 - h, z_0 + h], \quad \text{where} \quad M = \max_{z_0-h \leq z \leq z_0+h} |f^{xi}(z)|$$

Again  $|\xi_2 - \xi_1| \leq 2h$ , ref [15].

Using on the above inequation, we have

$$|ESM_{L_6KEL}(f)| \leq \frac{512 M h^{12}}{22869 \cdot 11!}. \quad \square$$

**Theorem 4**

The error committed due to the Hybridize rule  $SM_{L_6KEL}(f)$  is less than its constituent rules.

*Proof* Using (2.6) and Theorem2  $|ESM_{L_6KEL}(f)| \leq |EL_6(f)|$

Using (3.4) and Theorem2  $|ESM_{L_6KEL}(f)| \leq |EKEL_4(f)| \quad \square$

**6. Numerical verification**

**Table-1:** Values of different test integrals using Constructed Hybridize rule and its constituent rules.

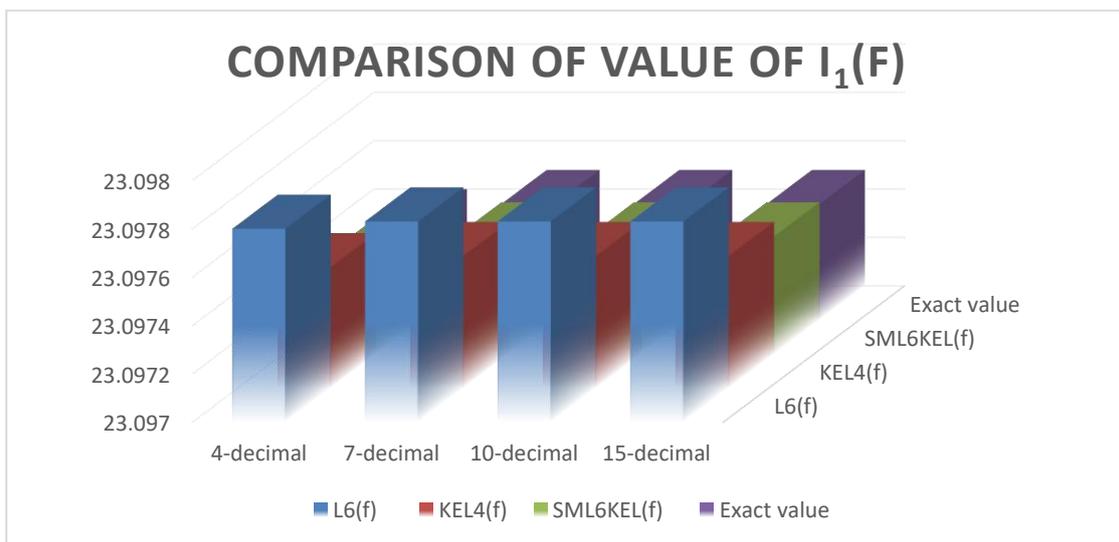
Integrals	Values obtained by different quadrature rules		
	$L_6(f)$	$KEL_4(f)$	$SM_{L_6KEL}(f)$
$I_1 = \int_{-\pi i}^{\pi i} \cos z \, dz$	23.0978303270584i	23.097546272400 4683 i	23.097486018382119151 515151515152
$I_2 = \int_{-\sqrt{3}i}^{\sqrt{3}i} z^{10} \, dz$	- 78.005912696795898 5i	- 76.784286657824 8i	- 76.525153861679415469 69696969697i
$I_3 = \int_0^{2i} \sinh z \, dz$	-1.41614683574858	- 1.4161468364088 3306	- 1.4161468365488867393 939393939394

$I_4 = \int_{1-\frac{i}{4}}^{1+\frac{i}{4}} \ln z dz$	0.0051134817804912 8i	0.0051134817196 792386i	0.0051134817067797146 6666666666667i
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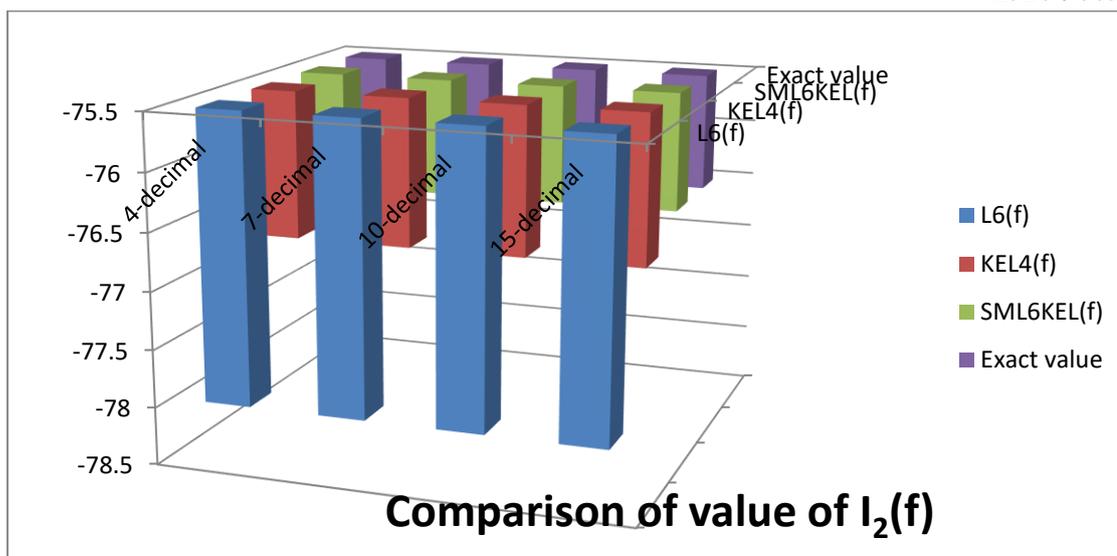
**Table-2:** Absolute value of Truncation error due to Hybridize rule and its constituent rules.

Integrals	Exact value	Error  obtained by different quadrature rules		
		$EL_6(f)$	$EKEL_4(f)$	$ESM_{GLKEL}(f)$
$I_1$	23.097478714515 496i	0.0003516125 42904	0.0000675578849 723	0.0000073038666 23151515151515
$I_2$	- 76.525153861679 48769584i	1.4807588351 1641080416	0.2591327961453 1230416	0.0000000000000 72226143030303
$I_3$	- 1.4161468365471 4238	0.0000000007 9856238	0.0000000001383 0932	0.0000000000017 44359393939393 93
$I_4$	0.0051134817078 3701898765i	0.0000000000 726542610123	0.0000000000118 4221961235	0.0000000000010 57304320983333

**Graphical Representation of data obtain from table-1**



**Figure-2** For the integral  $I_1(f)$



**Figure-3** For the integral  $I_2(f)$

**Analysis from the figures and tables**

- (i) **In Figure-2** the graph of values of  $SM_{L_6KEL}(f)$  coincides with the exact value of  $I_1(f)$  up to seven decimal places. However, the constituent rules  $L_6(f)$  and  $KEL_4(f)$  coincide with the exact value up to three and four decimal places respectively.
- (ii) **In Figure-3** the graph of values of  $SM_{L_6KEL}(f)$  coincides with the exact value of  $I_2(f)$  up to thirteen decimal places. However, the constituent rules  $L_6(f)$  and  $KEL_4(f)$  do not coincide with the exact value to a single decimal place.
- (iii) **From Table-1 & table-2**, we observed that the value of  $SM_{L_6KEL}(f)$  coincides with the exact value of  $I_3(f)$  up to eleven decimal places. However, the constituent rules  $L_6(f)$  and  $KEL_4(f)$  coincide with the exact value up to nine decimal places.
- (iv) **From Table-1 & table-2**, we observed that the value of  $SM_{L_6KEL}(f)$  coincides with the exact value of  $I_4(f)$  up to eleven decimal places. However, the constituent rules  $L_6(f)$  and  $KEL_4(f)$  coincide with the exact value up to ten decimal places.

**7. Application in Adaptive quadrature routines**

Considering the effective adaptive strategy [8],[12],[14].

**Table-3:** Approximation of the test integrals Hybridized rule  $SM_{L_6KEL}(f)$  and the constituent rules using the adaptive quadrature routines.

Prescribed tolerance  $\epsilon = 1.0 \times 10^{-10}$ .

Constituent rules	$KEL_4(f)$			$L_6(f)$		
	Approximat	No of	Error =	Approxim	No of	Error =

Integrals	e value(P)	steps required	P-I	ate value(P)	steps required	P-I
$AI_1 = \int_{-i}^i \cos z \, dz$	2.350402387 28760309i	03	$1.854 \times 10^{-16}$	2.3504023 87287603 99i	03	$10.8268 \times 10^{-16}$
$AI_2 = \int_{-\sqrt{2}i}^{\sqrt{2}i} e^z \, dz$	- 8.228151635 62530605 i	15	$2.6055 \times 10^{-14}$	- 8.2281516 35625424 91i	15	$14.492 \times 10^{-14}$
$AI_3 = \int_0^{2i} \sinh z \, dz$	- 1.416146836 54714226	03	$1.1753 \times 10^{-16}$	- 1.4161468 36547141 72	03	$6.575 \times 10^{-16}$
$AI_4 = \int_0^i e^{-z^2} \, dz$	1.462651745 90721566 i	03	$3.366 \times 10^{-14}$	1.4626517 45907196 48	05	$1.4478 \times 10^{-14}$

Integral s	Exact value	For the Hybridize rule $SM_{L_6KEL}(f)$		
		Approximate value(P)	No of steps required	Error = P-I
$AI_1$	2.35040238728760 2913i	2.35040238728760348i	01	$5.7 \times 10^{-16}$
$AI_2$	- 8.22815163562528 0283937i	- 8.22815163562528028i	01	$2.8449 \times 10^{-16}$
$AI_3$	- 1.41614683654714 238	-1.41614683654714277	01	$3.9237 \times 10^{-16}$
$AI_4$	1.46265174590718 2 i	1.4626517459071818 i	03	$1.9732 \times 10^{-16}$

**Observation from the table-3**

Using prescribed tolerance  $\epsilon = 1.0 \times 10^{-10}$ , we draw following conclusions.

- (i) For the integral  $AI_1$ , the mixed rule  $SM_{L_6KEL}(f)$  takes only one step, whereas  $KEL_4(f)$  and  $L_6(f)$  take three steps each to satisfy the prescribed tolerance.
- (ii) For the integral  $AI_2$ , the mixed rule  $SM_{L_6KEL}(f)$  takes only one step whereas  $KEL_4(f)$  and  $L_6(f)$  take fifteen steps each to satisfy the prescribed tolerance.
- (iii) For the integral  $AI_3$ , the mixed rule  $SM_{L_6KEL}(f)$  takes only one step whereas  $KEL_4(f)$  and  $L_6(f)$  take three steps each to satisfy prescribed tolerance.

- (iv) For the integral  $AI_4$ , all the rules  $SM_{L_6KEL}(f)$ ,  $KEL_4(f)$  and  $L_6(f)$  take three steps each to satisfy prescribed tolerance, whereas in the final step  $SM_{L_6KEL}(f)$  gives very less error in comparison to the rules  $KEL_4(f)$  and  $L_6(f)$ .

We finally conclude that the Hybridize rule  $SM_{L_6KEL}(f)$  gives significantly better results in adaptive environment.

## 8. Conclusions

*From the tables and figures it is evident that the new Hybridize quadrature rule when applied, each of the four integrals gives better result than that of constituent rules (Lobatto 6-point rule  $L_6(f)$  and Kronrod extension of Lobatto 4-point rule  $KEL_4(f)$ ). This Hybridize quadrature rule  $SM_{L_6KEL}(f)$  also gives better result in comparison to its constituent rules which was verified by evaluating test integrals in adaptive mode.*

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