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A Hybridized Lobatto Quadrature of Precision Eleven for Numerical Integration of Analytic Functions

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Abstract
A hybridizedeleven precision quadrature rule using Lobatto 6-point rule and modified form of Lobatto 4-point rule through kronrod extension is formed. This rule is capable of evaluating line integral of analytic functions. The hybridized rule has been tested both theoretically through error analysis and numerically using some test integrals. It is found that the constructed rule is more effective than that of theconstituent rules. It is alsoverified that the hybridized rulewhen appliedin adaptive environment gives significantly better results than its constituents.

Key words: Lobatto six point transformedrule,Hybridised rule,Kronrod extension ofLobatto four-point rule, $\mathrm{SM}_{\mathrm{L_6KEL}}(\mathrm{f})$.

1. Introduction

Several mixed quadrature rules developed in the papers [2],[4]for numerical evaluation of real definite integrals.

Some authors in their papers [8],[10] modified the mixed quadrature rules of earlier others to form transformed rules [6] for numerical evaluation of line integral of analytic functions.

The authors S.K. Mohanty, D. Das and R.B. Dash [8], S.K. Mohanty, R.B. Dash [9],[10],[11],[12] used the mixed rules as base rules to evaluate real definite integrals as well as line integrals of analytic functions in adaptive quadrature schemes, very few mixed quadrature rules of precision higher than 9 [8],[12] are available so far.

We used hybridized quadrature as a synonym of mixed quadrature in this paper. Usually, two quadraturesof identical precision are mixed are mixed suitably to get a quadrature rule of higher precision. The resulting quadrature rule is known as mixed quadrature rule. By doing this we increasing the precision of the quadrature rules in a very simplified manner unlike Richardson extrapolation and Kronrod extension.

In this paper, we designed a Hybridized rule of precision eleven out of two quadrature rules each of precision nine. The analytical error estimate of this rule and its constituent rules are studied. The theoretical predictions are verified evaluating test integrals. The highlights of the Hybridized rule have been shown in tables and figures. Using suitable adaptive scheme for the Hybridized rule it is seen that the number of steps required to achieve some pre-assign accuracy is drastically reduced.

2. Lobatto6-pointtransformed rule.

The(n+1) point Gauss-Legendrerule [1],[12],[13] is given by

$\int_{-1}^{1} f(z)dz = \sum_{k=0}^{n} \omega_k f(z_k)$(2.1)

Where ω_k's are $(n+1)$ weights and z_k's are$(n+1)$nodes. The $(2n+2)$ unknowns can be obtained by assuming the rule to be exact for all polynomials of degree $(2n+1)$.The Lobatto integration method [1], [13] are of Gauss types(2.1) with two end points pre-assigned as -1 and 1. For n=5, we get the weights$\frac{1}{15}$, $\frac{14+\sqrt{7}}{30}$, $\frac{14-\sqrt{7}}{30}$and the nodes ±1, $\pm\sqrt{\frac{7-2\sqrt{7}}{21}}$, $\pm\sqrt{\frac{7+2\sqrt{7}}{21}}$respectively. Using the nodes and weights, the **Lobatto 6-point transformed rule** is given by

$$L_6(f) = \int_{z_0-h}^{z_0+h} f(z)dz = \frac{h}{15}\{f(z_0-h)+f(z_0+h)\} + \frac{14+\sqrt{7}}{30}h\{f(z_0-\alpha h)+f(z_0+\alpha h)\}$$

$$+\frac{14-\sqrt{7}}{30}h\{f(z_0-\beta h)+f(z_0+\beta h)\}(2.2)$$

where $\alpha = \sqrt{\frac{7-2\sqrt{7}}{21}}$ and $\beta = \sqrt{\frac{7+2\sqrt{7}}{21}}$

Lemma1

If f(z) is analytic in the domain$\boldsymbol{\Omega} \supset [z_0-h, z_0+h]$, then the rule $L_6(f)$ is of precision nine and the truncation error due to $L_6(f)$is $EL_6(f) \cong \frac{-256}{6615}\frac{h^{11}}{11!}f^{x}(z_0)andO(h^{11})$.

ProofLet us denote truncation error of $L_6(f)$ is by $EL_6(f)$.

We know that$I(f) = L_6(f) + EL_6(f)$

$EL_6(f) = I(f) - L_6(f)$(2.3)

Applying Taylor's theorem [1],[7] in (2.2) and the exact value of the integral I(f) we get

$$L_6(f) = 2h\left[f(z_0) + \frac{h^2}{3!}f^{ii}(z_0) + \frac{h^4}{5!}f^{iv}(z_0) + \frac{h^6}{7!}f^{vi}(z_0) + \frac{h^8}{9!}f^{viii}(z_0)\right] + \frac{1226h^{11}}{6615\times 10!}f^{x}(z_0) + \frac{650\,h^{13}}{3969\times 12!}f^{xii}(z_0) + \cdots(2.4)$$

$$I(f) = 2h\left[f(z_0) + \frac{h^2}{3!}f^{ii}(z_0) + \frac{h^4}{5!}f^{iv}(z_0) + \frac{h^6}{7!}f^{vi}(z_0) + \frac{h^8}{9!}f^{viii}(z_0) + \frac{h^{10}}{11!}f^{x}(z_0) + \frac{h^{12}}{13!}f^{xii}(z_0) + \dots\right] \quad (2.5)$$

By using (2.4) and (2.5) in (2.3), we get

$$EL_6(f) = -\frac{256}{6615}\frac{h^{11}}{11!}f^{x}(z_0) - \frac{512}{3969}\frac{h^{13}}{13!}f^{xii}(z_0) + \cdots \quad (2.6)$$

The truncation error establishes that the degree of precision of the rule $L_6(f)$ is nine,

$EL_6(f) \cong -\frac{256}{6615}\frac{h^{11}}{11!}f^{x}(z_0)$ and $O(h^{11})$.□

3. Kronrod extension of Lobatto 4-point rule

The Kronrod extension of the Lobatto 4-point rule [3],[5], [11] is denoted by $KEL_4(f)$, is given by

$$\int_{z_0-h}^{z_0+h} f(x)dx \approx KEL_4(f)$$

where

$$KEL_4(f) = \frac{h}{1470}\left[77\{f(z_0-h)+f(z_0+h)\} + 432\left\{f\left(z_0 - \frac{\sqrt{2}}{\sqrt{3}}h\right) + f\left(z_0 + \frac{\sqrt{2}}{\sqrt{3}}h\right)\right\} + 625\left\{f\left(z_0 - \frac{h}{\sqrt{5}}\right) + f\left(z_0 + \frac{h}{\sqrt{5}}\right)\right\} + 672f(z_0)\right] \quad (3.1)$$

Applying Taylor's theorem [1],[7],[12] after simplification we obtain

$$KEL_4(f) = 2h\left[f(z_0) + \frac{h^2}{3!}f^{ii}(z_0) + \frac{h^4}{5!}f^{iv}(z_0) + \frac{h^6}{7!}f^{vi}(z_0) + \frac{h^8}{9!}f^{viii}(z_0) + \frac{4741}{4725}\frac{h^{10}}{11!}f^{x}(z_0) + \frac{72059}{70875}\frac{h^{12}}{13!}f^{xii}(z_0) + \cdots\right] \quad (3.2)$$

Lemma2

Let us denote, the truncation error due to Kronrod extension of Lobatto 4-point rule by $EKEL_4(f)$, then $EKEL_4(f) \cong -\frac{32}{4725}\frac{h^{11}}{11!}f^{x}(z_0)$ and $O(h^{11})$.

Proof We have $I(f) = KEL_4(f) + EKEL_4(f)$

$$\Rightarrow EKEL_4(f) = I(f) - KEL_4(f) \quad (3.3)$$

Using (2.5) and (3.2) on (3.3), we obtain

$$EKEL_4(f) = 2h\left[-\frac{16}{4725}\frac{h^{10}}{11!}f^{x}(z_0) - \frac{1184}{70875}\frac{h^{12}}{13!}f^{xii}(z_0) - \cdots\right]$$

or

$$EKEL_4(f) = -\frac{32}{4725}\frac{h^{11}}{11!}f^{x}(z_0) - \frac{2368}{70875}\frac{h^{13}}{13!}f^{xii}(z_0) - \cdots \quad (3.4)$$

The expression (3.4) the truncation error of the rule $\mathrm{KEL}_4(\mathrm{f})$.From (3.4) we also concluded that the degree of precision of the Kronrod extension of Lobatto 4-point rule is 9 and of$\mathrm{O}(\mathrm{h}^{11})$. □

4.Formulation of the Hybridizedquadrature rule of precision eleven

The construction of the proposedHybridized quadrature rule is given in the following theorem.

Theorem1(Formulationof$\mathrm{SM}_{\mathrm{L}_6\mathrm{KEL}}(\mathrm{f})$)

If f(z) is analytic in the given domain$\mathbf{\Omega} \supset [\mathrm{z}_0 - \mathrm{h}, \mathrm{z}_0 + \mathrm{h}]$, then the Hybridize rule $\mathrm{SM}_{\mathrm{L}_6\mathrm{KEL}}(\mathrm{f})$ and truncation error due to the Hybridize rule $\mathrm{ESM}_{\mathrm{L}_6\mathrm{KEL}}(\mathrm{f})$ are given by

$\mathrm{SM}_{\mathrm{L}_6\mathrm{KEL}}(\mathrm{f}) = \frac{1}{33}[40\ \mathrm{KEL}_4(\mathrm{f}) - 7\ \mathrm{L}_6(\mathrm{f})]$ and $\mathrm{ESM}_{\mathrm{L}_6\mathrm{KEL}}(\mathrm{f}) = \frac{1}{33}[40\ \mathrm{EKEL}_4(\mathrm{f}) - 7\ \mathrm{EL}_6(\mathrm{f})]$.

Proof

Recalling$\mathrm{I}(\mathrm{f}) = \mathrm{L}_6(\mathrm{f}) + \mathrm{EL}_6(\mathrm{f})$(4.1)

$\mathrm{I}(\mathrm{f}) = \mathrm{KEL}_4(\mathrm{f}) + \mathrm{EKEL}_4(\mathrm{f})$(4.2)

Subtracting 7 times of (4.1) from40 times of (4.2), we get

$$33\ \mathrm{I}(\mathrm{f}) = [40\ \mathrm{KEL}_4(\mathrm{f}) - 7\ \mathrm{L}_6(\mathrm{f})] + [40\ \mathrm{EKEL}_4(\mathrm{f}) - 7\ \mathrm{EL}_6(\mathrm{f})]$$

$$\Rightarrow \mathrm{I}(\mathrm{f}) = \frac{1}{33}[40\ \mathrm{KEL}_4(\mathrm{f}) - 7\ \mathrm{L}_6(\mathrm{f})] + \frac{1}{33}[40\ \mathrm{EKEL}_4(\mathrm{f}) - 7\ \mathrm{EL}_6(\mathrm{f})]$$

$$\Rightarrow \mathrm{I}(\mathrm{f}) = \mathrm{SM}_{\mathrm{L}_6\mathrm{KEL}}(\mathrm{f}) + \mathrm{ESM}_{\mathrm{L}_6\mathrm{KEL}}(\mathrm{f})$$

Where $\mathrm{SM}_{\mathrm{L}_6\mathrm{KEL}}(\mathrm{f}) = \frac{1}{33}[40\ \mathrm{KEL}_4(\mathrm{f}) - 7\ \mathrm{L}_6(\mathrm{f})]$(4.3)

and $\mathrm{ESM}_{\mathrm{L}_6\mathrm{KEL}}(\mathrm{f}) = \frac{1}{33}[40\ \mathrm{EKEL}_4(\mathrm{f}) - 7\ \mathrm{EL}_6(\mathrm{f})]$(4.4)

The expression (4.3) is the proposed Hybridized rule and (4.4) is the truncation error associated due to the rule. □

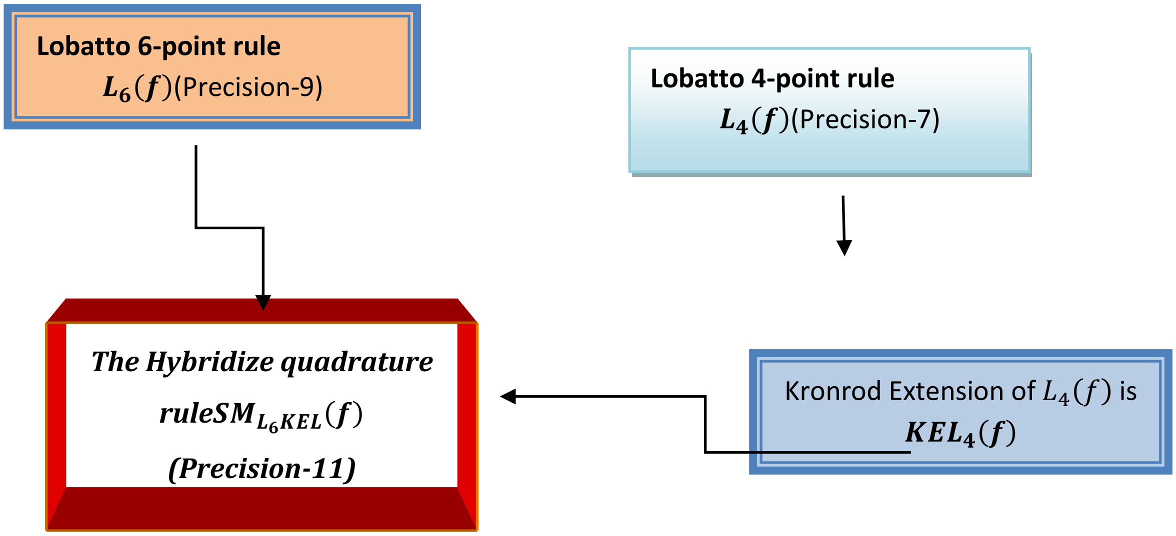

Figure-1: Construction of the Hybridize rule of precision-11.

5. Error Analysis

An error analysis of the constructed rule has been obtained by the following Theorems.

Theorem2

If f(z) is analytic in the given domain $\mathbf{\Omega} \supset [z_0 - h, z_0 + h]$, then the truncation error associated due to the rule $SM_{L_6KEL}(f)$is given by$ESM_{L_6KEL}(f) \cong -\frac{2048}{4725}\frac{h^{13}}{13!}f^{xii}(z_0)$.

Proof Using (2.6) and (3.4) on (4.4), we get

$$ESM_{L_6KEL}(f) = -\frac{2048}{4725}\frac{h^{13}}{13!}f^{xii}(z_0)\text{-..}$$

$$\Rightarrow ESM_{L_6KEL}(f)(f) \cong -\frac{2048}{4725}\frac{h^{13}}{13!}f^{xii}(z_0) \text{ [Since truncation error= } O(h^{13})]$$

□

Theorem3

The Error bound of the constructed Hybridizequadrature rule is

$$|ESM_{L_6KEL}(f)| \leq \frac{256\,M}{31185}\frac{h^{11}}{11!}|\xi_2 - \xi_1|, \quad \xi_1, \xi_2 \epsilon [z_0 - h, z_0 + h], \text{ where } M = \max_{z_0 - h \leq z \leq z_0 - h} |f^{xi}(z)|.$$

ProofFrom (2.6), we get$EL_6(f) \cong -\frac{256}{6615}\frac{h^{11}}{11!}\frac{h^{11}}{11!}f^{x}(\xi_1), \quad \xi_1 \epsilon [z_0 - h, z_0 + h]$,

and from (3.4), we get $EKEL_4(f) \cong -\frac{32}{4725}\frac{h^{11}}{11!}f^{x}(\xi_2), \quad \xi_2 \epsilon [z_0 - h, z_0 + h]$,

using above two values on (4.4), we can write

$$\mathrm{ESM}_{\mathrm{L_6KEL}}(\mathrm{f}) = \frac{1}{33}[40\ \mathrm{EKEL}_4(\mathrm{f}) - 7\ \mathrm{EL}_6(\mathrm{f})]$$

$$\mathrm{ESM}_{\mathrm{L_6KEL}}(\mathrm{f}) \cong \frac{1}{33}\left[40\left\{-\frac{32}{4725}\frac{h^{11}}{11!}f^{x}(\xi_2)\right\} - 7\left\{-\frac{256}{6615}\frac{h^{11}}{11!}f^{x}(\xi_1)\right\}\right]$$

$$= \frac{256}{31185}\frac{h^{11}}{11!}\{f^{x}(\xi_1) - f^{x}(\xi_2)\}$$

$$= \frac{-256}{31185}\frac{h^{11}}{11!}\{f^{x}(\xi_2) - f^{x}(\xi_1)\}$$

$$= \frac{-256}{31185}\frac{h^{11}}{11!}\int_{\xi_1}^{\xi_2} f^{xi}(z)dz$$

$$\Rightarrow |\mathrm{ESM}_{\mathrm{L_6KEL}}(\mathrm{f})| \cong \frac{256}{31185}\frac{h^{11}}{11!}\left|\int_{\xi_1}^{\xi_2} f^{xi}(z)dz\right|$$

$$\leq \frac{256}{31185}\frac{h^{11}}{11!}\int_{\xi_1}^{\xi_2} \left|f^{xi}(z)\right|dz$$

$\leq \frac{256}{31185}\frac{h^{11}}{11!}\int_{\xi_1}^{\xi_2} M dz$, where $M = \max_{z_0 - h \leq z \leq z_0 - h} f(z)$

$\Rightarrow |\mathrm{ESM}_{\mathrm{L_6KEL}}(\mathrm{f})| \leq \frac{256\ M}{31185}\frac{h^{11}}{11!}|\xi_2 - \xi_1|$(5.1)

Since ξ_1 and ξ_2 are arbitrarily chosen points in the interval $[z_0 - h, z_0 + h]$, (5.1) shows that the absolute value of the truncation error will be less if the points ξ_1 and ξ_2 are close to each other.

Corollary.

The error bound for the truncation error is $|\mathrm{ESM}_{\mathrm{L_6KEL}}(\mathrm{f})| \leq \frac{512\ M}{22869}\frac{h^{12}}{11!}$, $M = \max_{z_0 - h \leq z \leq z_0 - h} |f^{xi}(z)|$.

ProofFrom the theorem-4

$|\mathrm{ESM}_{\mathrm{L_6KEL}}(\mathrm{f})| \leq \frac{256\ M}{31185}\frac{h^{11}}{11!}|\xi_2 - \xi_1|$, $\xi_1, \xi_2 \epsilon [z_0 - h, z_0 + h]$, where $M = \max_{z_0 - h \leq z \leq z_0 - h} |f^{xi}(z)|$

Again $|\xi_2 - \xi_1| \leq 2h$, ref [15].

Using on the above inequation, we have

$|\mathrm{ESM}_{\mathrm{L_6KEL}}(\mathrm{f})| \leq \frac{512\ M}{22869}\frac{h^{12}}{11!}$. □

Theorem 4

The error committed due to the Hybridize rule $SM_{L_6KEL}(f)$ is less than its constituent rules.

ProofUsing (2.6) and Theorem2 $|ESM_{L_6KEL}(f)| \leq |EL_6(f)|$
Using (3.4) and Theorem2$|ESM_{L_6KEL}(f)| \leq |EKEL_4(f)|$ □

6. Numerical verification

Table-1: Values of different test integrals using ConstructedHybridize rule and its constituent rules.

Integrals	Values obtained by different quadrature rules		
	$L_6(f)$	$KEL_4(f)$	$SM_{L_6KEL}(f)$
$I_1 = \int_{-\pi i}^{\pi i} \cos z\, dz$	23.0978303270584i	23.0975462724004683 i	23.09748601838211915 1515151515152
$I_2 = \int_{-\sqrt{3}i}^{\sqrt{3}i} z^{10} dz$	-78.0059126967958985i	-76.7842866578248i	-76.52515386167941546 969696969697i
$I_3 = \int_0^{2i} \sinh z\, dz$	-1.41614683574858	-1.41614683640883306	-1.416146836548886739 3939393939394
$I_4 = \int_{1-\frac{i}{4}}^{1+\frac{i}{4}} \ln z\, dz$	0.00511348178049128i	0.0051134817196792386i	0.005113481706779714 66666666666667i

Table-2: Absolute value of Truncation error due to Hybridize rule and its constituent rules.

Integrals	Exact value	|Error| obtained by different quadrature rules		
		$EL_6(f)$	$EKEL_4(f)$	$ESM_{GLKEL}(f)$
I_1	23.097478714515496i	0.000351612542904	0.0000675578849723	0.000007303866 623151515151515 15
I_2	-76.52515386167948769584i	1.48075883511641080416	0.25913279614531230416	0.000000000000 0722261430303 03
I_3	-1.41614683654714238	0.00000000079856238	0.00000000013830932	0.000000000001 7443593939393 9393
I_4	0.00511348170783701898765i	0.000000000072654261012	0.00000000001184221961235	0.000000000001 0573043209833

		3		33

Graphical Representation of data obtain from table-1

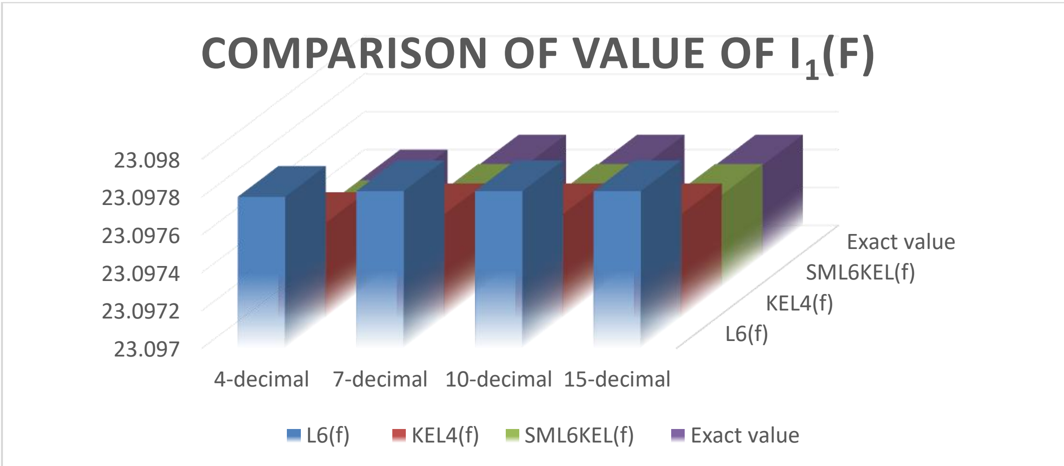

Figure-2 For the integral $I_1(f)$

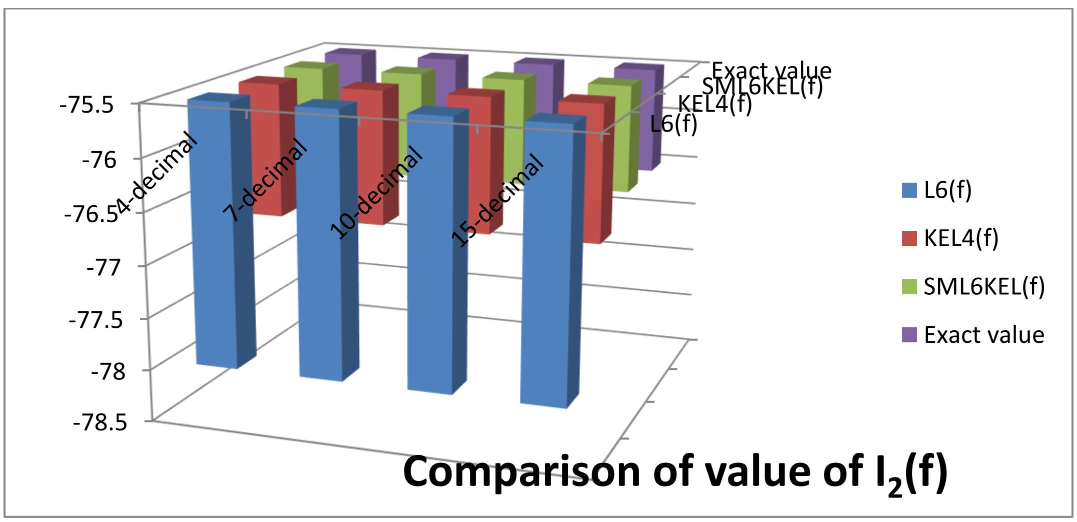

Figure-3 For the integral $I_2(f)$

Analysis from the figures and tables

(i) **In Figure-2** the graph of values of $SM_{L_6KEL}(f)$coincidewith the exact value of $I_1(f)$upto seven decimal places. However, the constituent rules $L_6(f)$and $KEL_4(f)$ coincide with theexact value up to three and four decimal places respectively.

(ii) **In Figure-3** the graph of values of $SM_{L_6KEL}(f)$ coincide with the exact value of $I_2(f)$uptothirteen decimal places. However, the constituent rules $L_6(f)$and $KEL_4(f)$ do not coincide with theexact value to a single decimal place.

(iii) **From Table-1 & table-2**, we observed thatthevalue of $SM_{L_6KEL}(f)$coincides with the exact value of $I_3(f)$uptoeleven decimal places. However, the constituent rules $L_6(f)$and $KEL_4(f)$ coincide with the exact value up to nine decimal places.

(iv) **From Table-1 & table-2**, we observed that the value of $SM_{L_6KEL}(f)$coincides with the exact value of $I_4(f)$uptoeleven decimal places. However, the constituent rules $L_6(f)$and $KEL_4(f)$ coincide with the exact value up to ten decimal places.

7. Application in Adaptive quadrature routines

Considering the effective adaptive strategy [8],[12],[14].

Table-3: Approximation of the test integrals Hybridized rule $SM_{L6KEL}(f)$ and the constituent rules usingthe adaptive quadrature routines.

Prescribed tolerance $\in = 1.0 \times 10^{-10}$.

Constituent rules / Integrals	$KEL_4(f)$			$L_6(f)$		
	Approximate value(P)	No of steps required	|Error|= |P-I|	Approximate value(P)	No of steps required	|Error|= |P-I|
$AI_1 = \int_{-i}^{i} \cos z\, dz$	2.35040238728760309i	03	1.854×10^{-16}	2.350402387287603 99i	03	10.8268×10^{-16}
$AI_2 = \int_{-\sqrt{2}i}^{\sqrt{2}i} e^z dz$	-8.22815163562530605 i	15	2.6055×10^{-14}	-8.228151635625424 91i	15	14.492×10^{-14}
$AI_3 = \int_{0}^{2i} \sinh z\, dz$	-1.41614683654714226	03	1.1753×10^{-16}	-1.416146836547141 72	03	6.575×10^{-16}
$AI_4 = \int_{0}^{i} e^{-z^2} dz$	1.46265174590721566 i	03	3.366×10^{-14}	1.462651745907196 48	05	1.4478×10^{-14}

Integrals	Exact value	For the Hybridize rule $SM_{L6KEL}(f)$		
		Approximate value(P)	No of steps required	|Error|=|P-I|
AI_1	2.350402387287602913i	2.35040238728760348i	01	5.7×10^{-16}
AI_2	-8.228151635625280283937i	-8.22815163562528028i	01	2.8449×10^{-16}
AI_3	-1.41614683654714238	-1.41614683654714277	01	3.9237×10^{-16}
AI_4	1.462651745907182 i	1.4626517459071818 i	03	1.9732×10^{-16}

Observation from the table-3

Using prescribed tolerance $\in= 1.0 \times 10^{-10}$, we draw following conclusions.

(i) For the integral AI_1, the mixed rule $SM_{L6KEL}(f)$ takes only one step, whereas $KEL_4(f)$ and $L_6(f)$ take three steps eachto satisfythe prescribedtolerance.
(ii) For the integral AI_2, the mixed rule $SM_{L6KEL}(f)$ takes only one stepwhereas $KEL_4(f)$ and $L_6(f)$ take fifteen steps eachto satisfy the prescribed tolerance.
(iii) For the integral AI_3, the mixed rule $SM_{L6KEL}(f)$ takes only one stepwhereas $KEL_4(f)$ and $L_6(f)$ take three stepseachto satisfy prescribed tolerance.
(iv) For the integral AI_4, all the rules $SM_{L6KEL}(f)$, $KEL_4(f)$ and $L_6(f)$ takethreesteps each to satisfy prescribed tolerance, whereas in the final step $SM_{L6KEL}(f)$gives very less error in comparison to the rules $KEL_4(f)$ and $L_6(f)$.

We finally conclude that the Hybridize rule $SM_{L6KEL}(f)$ gives significantly better results in adaptive environment.

8. Conclusions

From the tablesand figures it is evident that thenew Hybridize quadrature rule when applied, each of the four integrals gives better result than that of constituent rules (Lobatto 6-point rule$L_6(f)$ and Kronrod extension of Lobatto 4- point rule $KEL_4(f)$). This Hybridize quadrature rule $SM_{L_6KEL}(f)$ also gives better result in comparison to its constituentrules which was verified by evaluating test integrals in adaptive mode.

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