Even Sum Domination Number of Join Two Paths

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Article Info

Page Number: 8894 - 8904

Publication Issue:

Vol 71 No. 4 (2022)

Article History

***Article Received**: 25 March 2022*

***Revised**: 30 April 2022*

***Accepted**: 15 June 2022*

***Publication**: 19 August 2022*

Abstract

Throughout this paper, some properties of the sum even domination are introduced. Moreover, the operation join of two paths is defined. Finally, for all possibly cases of join of two paths, the even sum domination number is determined.

Keywords: Even sum dominating set, even sum domination number, and the join of two paths.

Introduction

A graph theory $G(V, E)$depends on two sets, the first set (vertex set) contains the vertices of this graph and denoted by $V(G)$ or V and the second set (edge set) contains the edges of this graph and denoted by $E(G)$ or E.Nowadays, graph theory has become a new language that most other sciences deal with to find alternative or new solutions to their problems, such as medicine, engineering, statistics, physics, chemistry and others.The most important notions in graph theory that deal with many application in most sciences is the concept of dominating set. The first used this concept is Berge in 1962 [7].After thisthis concept appear in various sciences especially, in mathematics where it is deal with most subjects as topological graph [2] and [15] , labeled graph [3], number theory graph [1], topological indices [5-6], and others. Moreover, many new definition presented to solve variant life problems [4], [11-14], and [16-22]. Omran and Ihsan[19],introduced a new parameter of domination number which is called even sum domination. They determined many bound of this number and get important results, also, they calculated this number for most the certain graphs such as path, cycle, complete, and others. In this paper, the resulting graph of the join operationis discussed. All possible cases resulting from this binary operation have been discussed and thus the even sum domination number for each case has been calculated.

Definition 2.1. [19]Let G be a graph and $D \subseteq G$ be a dominating set and for all $\forall u \in V - D$, there is adjacent vertex in the set D say v such that $deg(v) + deg(u)$ is even, so the set D is called even sum dominating set (ESDS).

Definition 2.2. [19] Let D be a minimal ESDS, so the minimum value of all these set is called even sum domination number and written by $\gamma_{es}\ (G)$.

The set D has property that mentioned above is called γ_{es}-set.
Observation 1.4. [19] Consider G be a graph and $D \subseteq G$ is γ_{es} −set, so
1) Every ESDS contains each an isolated vertex.
2) For all a vertex in $V - D$, there is a vertex in the set D such that the two vertices are odd or even together.
3) If the degree of a vertex is odd and all the neighborhood of this vertex are even 9or vice versa), then this vertex belong to every ESDS.
4) $\gamma_{es}(G) \leq \gamma(G)$.

Corollary 1.6.[19]
1) $\gamma_{es}(K_n) = \gamma(K_n) = 1$

$$2)\ \gamma_{es}(C_n) = \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$$

3) $\gamma_{es}(N_n) = \gamma(N_n) = n$

2 Main results.

Proposition 2.1. Let G be a graph isomorphic to the graph $P_n + P_m; n = 1,2$, then

$$\gamma_{sc}(G) = \begin{cases} 1, & if\ n = 1\ and\ m = 1 \\ 3, & if\ n = 1\ and\ m\ is\ odd; m > 1\ or\ n = 2\ and\ m\ is\ odd; m > 1 \\ 1 + \left\lceil \frac{m-2}{3} \right\rceil, & if\ n = 1\ and\ m\ is\ even; m \geq 2\ or\ \ n = 2\ and\ m\ is\ even; m \geq 2 \end{cases}$$

Proof. Let $\{u_1\}$ and $\{v_1, v_2, \dots, v_m\}$ be the vertex set of the two paths P_1 and P_m respectively, sotwo cases depend on n are classification:
Case 1. If $n = 1$, then there are two cases are classification:.
I) If $m = 1$, then $P_n + P_m \equiv P_2$ this yields $\gamma_{es}(G) = 1$.

II) If $m \geq 2$, then two subcases are classification:

Subcase 1.If $m\ is\ odd; m > 1$, then $deg(u_1)$ is odd, $deg(v_1)\ and\ deg(v_n)$ are even, and $deg(v_i); i = 2, \dots, n-1$ are odd (as an example, see Figure 2.1).

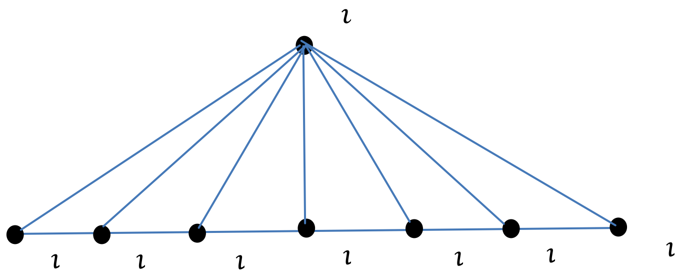


Fig. 2.1. The graph $P_1 \times P_7$

The vertex $u_1$even sum dominates to all vertices $\{v_i, i = 2, \dots, n-1\}$, since u_1 is adjacent to all vertices $\{v_i, i = 2, \dots, n-1\}$ and $deg(u_1) + deg(v_i)$is even $\forall\ i;\ i = 2, \dots, n-1$. The two vertices $v_1\ and\ v_n$ belong to every even sum dominating set according to the Observation 1.3(3). Thus, the set $D_1 = \{u_1, v_1, v_n\}$ is the minimum ESD, and $\gamma_{es}(G) = 3$.

Subcase2. If $m\ is\ even; m > 2$, then $deg(u_1)$ is even, $deg(v_1)\ and\ deg(v_n)$ are even, and $deg(v_i); i = 2, \dots, n-1$ are odd (as an example, see Figure 2.2).

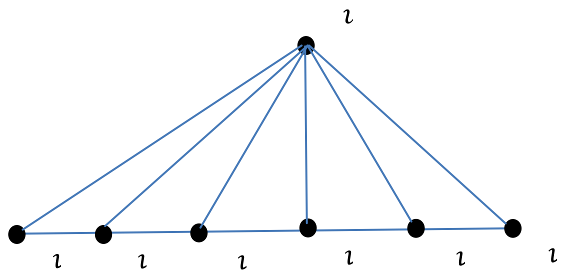


Fig. 2.2. The graph $P_1 \times P_6$

So, the vertex u_1 even sum dominates to the two vertices $v_1\ and\ v_n$. The remain vertices not dominate by the vertex is the set of vertices $S = \{v_i, i = 2, \dots, n-1\}$ and the induced subgraph of the set S is isomorphic to the P_{m-2}, but the special case here is all vertices of the induced subgraph have odd degree. Thus, $\gamma_{sc}(P_{m-2}) = \left\lceil \frac{m-2}{3} \right\rceil$, so $\gamma_{sc}(G) = 1 + \left\lceil \frac{m-2}{3} \right\rceil$.

Case 2. If $n = 2$, then three subcases are discussed as the following.

Let $\{u_1, u_2\}$ and $\{v_1, v_2, \dots, v_m\}$ be the vertex sets of the paths P_2 and P_m respectively. then three subcases are discussed as the following.

Subcase 1. If $n = m = 2$, then $P_2 + P_2 \equiv K_4$, then $\gamma_{es}(G) = 1$, according to the Corollary 1.6.

Subcase 2. If$n = 2\ and\ m\ is\ odd; m > 2,$ then $deg(u_1), deg(u_1),\ and\ deg(v_i); i = 2, \dots, n-1$ are even and $deg(v_1), and\ deg(v_n)$are odd, and (as an example, see Figure 2.3).

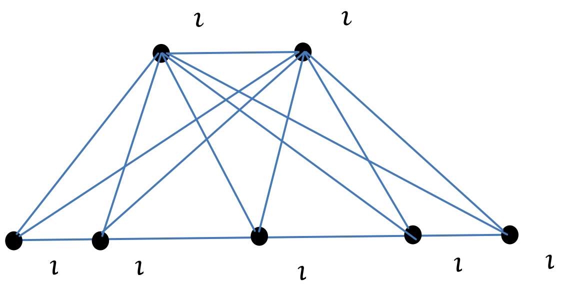


Fig. 2.3. The graph $P_2 \times P_5$

Now, according to Observation 1.2 (3), the two vertices v_1 and v_n belong to every ESDS. The vertex u_1 is dominates all vertices u_2 and all vertices of the set $S = \{v_i, i = 2, \dots, n-1\}$ and the summation of degree of the vertex u_1 with any vertex of the set S is even, so the set $D = \{u_1, v_1, v_n\}$ is the minimum ESDS. Thus, $\gamma_{sc}(G) = 3$.

Subcase 3. If $n = 2\ and\ m\ is\ even; m > 2,$ then $deg(u_1), deg(u_2), deg(v_1), and\ deg(v_n)$ are odd, and $deg(v_i); i = 2, \dots, n-1$ are even (as an example, see Figure 2.4).

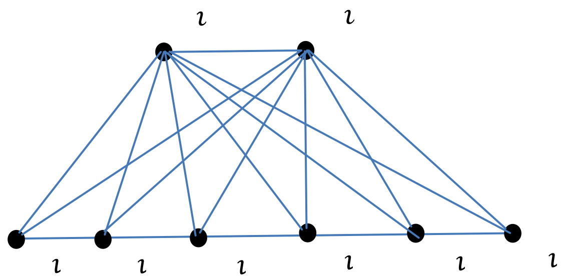


Fig. 2.4. The graph $P_2 \times P_6$

So, the vertex u_1 even sum dominates to the three vertices u_2, v_1 and v_n. The remain vertices not dominate by the vertex is the set of vertices $S = \{v_i, i = 2, \dots, n-1\}$ and the induced subgraph of the set S is isomorphic to the P_{m-2}, but the special case here is all vertices of the induced subgraph have even degree. Thus, $\gamma_{sc}(P_{m-2}) = \left\lceil \frac{m-2}{3} \right\rceil$, so $\gamma_{es}(G) = 1 + \left\lceil \frac{m-2}{3} \right\rceil$.

Depending of all cases above, the required is obtained. □

Proposition 2.2. Consider$G$ be a graph isomorphic to the graph

$P_n + P_m; n, m \geq 3$, then

$$\gamma_{sc}(G) = \begin{cases} 2, & if \text{ n is even and m is odd and } m \leq 5 \ and \ n = 4 \\ 3, & if \ n \ and \ m \ are \ odd \ and \ n \leq 5 \\ & or \ if \text{ n is even and m is odd and } m \leq 5 \ and \ n > 4 \\ & or \ if \text{ n is even and m is odd and } m > 5 \ and \ n = 4 \\ & or \ if \text{ n is even and m is even and } n \ or \ m \ equal \ to \ 4 \\ 4, & if \text{ n is even and m is odd } and \ n > 5 \ and \ m > 4 \\ & or \ if \text{ n is even and m is even and } n, m > 4 \\ & orif \ n \ and \ m \ are \ odd \ and \ n > 5 \end{cases}$$

Proof. Let $\{u_1, u_2, \dots, u_n\}$ and $\{v_1, v_2, \dots, v_m\}$ be the vertex sets of the two paths P_n and P_m respectively. Three cases are discussed as the following.

Case 1. If n is even and m is even, then $deg(u_1), deg(u_1), deg(v_1), and \ deg(v_n)$ are odd, and $deg(u_i); i = 2, \dots, n-1$ and $deg(v_i); i = 2, \dots, m-1$are even , so there are discussed as the following.

Subcase 1. If$n \ or \ m \ equal \ to$ 4, then without loss of generality, suppose that $n = 4$. The vertex u_1 dominates the two vertices v_1 and v_n . the remain vertex has odd degree not dominated by the vertex u_1 is u_n so the two vertices u_1 and $u_n$are dominate the four vertices $u_1, u_2, v_1, and \ v_n$. The remain vertices have even degree which are the two vertices u_2 and u_3 and the set $S = \{v_i, i = 2, \dots, n-1\}$. The vertex u_2 dominates the vertex u_3 and all vertices in the set S, moreover, the summation of the degree of the vertex u_2 and the degree of each vertex in the S is even. Thus, the set $D = \{u_1, u_2, u_n\}$ is a minimum ESDS (as an example, see Figure 2.5) and $\gamma_{sc}(G) = 3$.

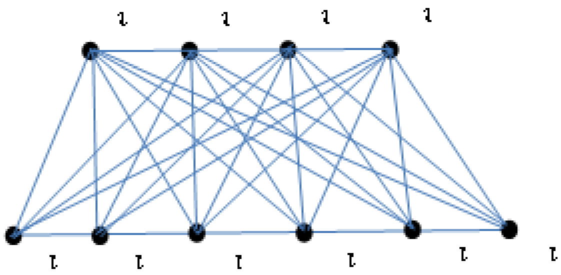


Fig. 2.5. The graph $P_4 + P_6$

Subcase 2. If $n\,, m\; > 4$, then in the same manner in previous subcase, the two vertices $u_1\ and\ u_n$ are dominate the four vertices $u_1, u_2, v_1, and\ v_n$. Also, The remain vertices have even degree which are the two sets $S_1 = \{u_i, i = 2, \ldots, n-1\}$ and $S_2 = \{v_i, i = 2, \ldots, m-1\}$. There is no vertex in the above sets dominates all other vertices in these sets, since if the vertex taken from any sets say S_1, then this vertex not dominates the other vertices in the set S_1 because $n > 5$. Let $D = \{u_1, u_2, v_2, u_n\}$, so as mentioned above two vertices $u_1\ and\ u_n$ are dominate the four vertices $u_1, u_2, v_1, and\ v_n$. Also, the vertex u_2 dominates all vertices in the set S_2, and the vertex v_2 dominates all vertices in the set S_1. Furthermore, the degree of each vertex $u_1 or\ u_n$ with each vertex $v_1\ or\ v_n$ is even, also the degree of the degree of the vertex u_2 with any vertex in the set S_2 is even and the degree of the degree of the vertex v_2 with any vertex in the set S_1 is even. Thus, the set D is a minimum ESDS, and (as an example, see Figure 2.6) and $\gamma_{sc}(G) = 4$.

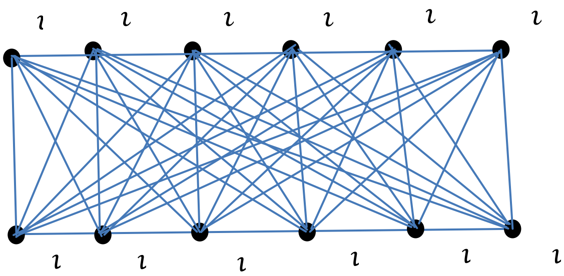


Fig. 2.6. The graph $P_6 + P_6$

Case 2. If n and m is odd, then $deg(u_1), deg(u_n), deg(v_1), and\ deg(v_n)$ are even, and $deg(u_i); i = 2, \ldots, n-1$ and $deg(v_i); i = 2, \ldots, m-1$ are odd, so two cases are discussed as the following.

Subcase 1. If $n\ or\ m \leq 5$, then without loss of generality, suppose that $n\ \leq 5$. In the same manner in previous subcase the two vertices $u_1\ and\ u_n$ are dominate the four vertices $u_1, u_2, v_1, and\ v_n$.

Also, the remain vertices haveodddegree which is one vertex if $n = 3$ or two vertices if $n = 5$ and the set $S = \{v_i, i = 2, \ldots, n-1\}$. If $n = 3$, then the vertex u_2 even sum dominates all vertices in the set S while the vertex u_3 even sum dominates the two vertices u_2 and

u_4 and all vertices in the set S. Thus, the set $D = \{u_1, u_2, u_n\}$ is a minimum ESDS where $n = 3$ or $D = \{u_1, u_3, u_n\}$ is a minimum ESDS where $n = 5$ (as an example, see Figure 2.7) and for each cases, $\gamma_{sc}(G) = 3$.

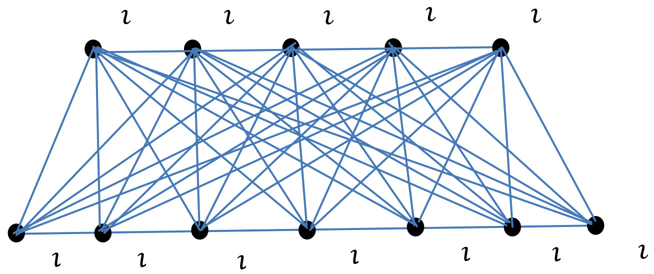


Fig. 2.7. The graph $P_5 + P_7$

Subcase 2. If $n, m > 5$, then as the same technique in the case 1 (Subcase 2) above, one can be concluded that $\gamma_{sc}(G) = 4$, (as an example, see Figure 2.8).

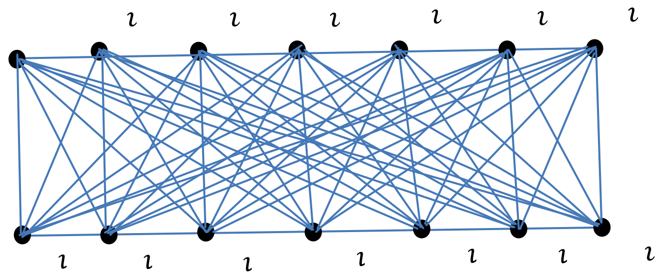


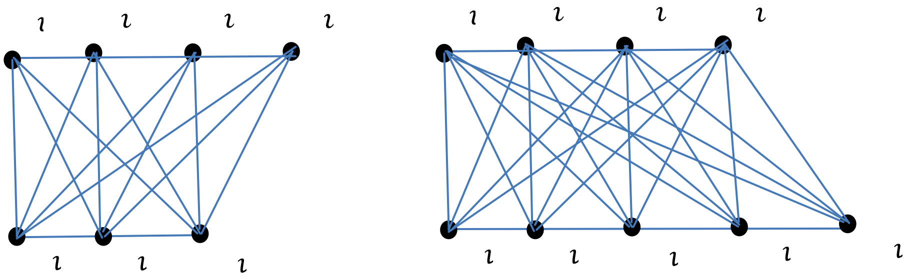
Fig. 2.8. The graph $P_7 + P_7$

Case 3. If n is even and m is odd or vice versa, without loos of generality suppose that n is even and m is odd, then $deg(u_1), deg(u_n)$ and $deg(v_i); i = 2, \ldots, m-1$ are even degree. Also, $deg(v_1), deg(v_n)$ and $deg(u_i); i = 2, \ldots, n-1$ are odd degree, so four casesare discussed as the following.

Subcase 1. If $n = 4 \; and \; m \leq 5$, then The vertex u_2 dominates the three vertices $u_3, v_1, and \; v_n$ and the summation of the degree of the vertex u_2 with each one of the vertices $u_3, v_1, and \; v_n$ is even.

Moreover, the vertex v_2 dominates the vertices u_1 and u_n if $m = 3$ while the vertex v_3 dominates the vertices u_1, u_n, v_2 and v_4 if $m = 5$, (as an example, see Figure 2.9). Thus, for each cases above, the set $D = \{u_2, v_2\}$ where $m = 3$ or $D = \{u_2, v_3\}$ where $m = 5$ is a minimum ESDS, and

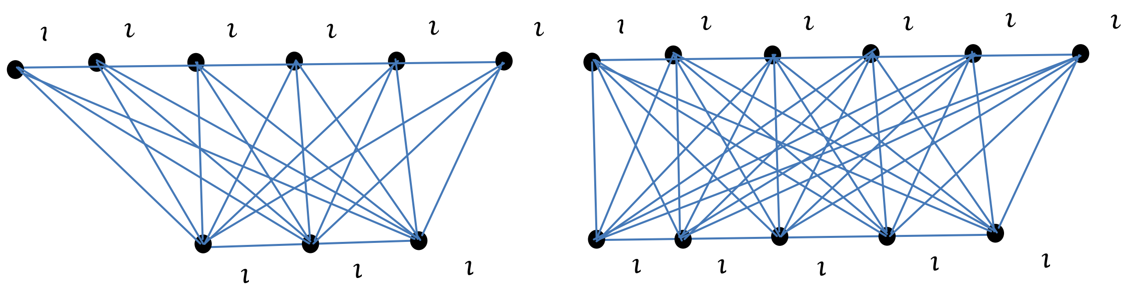
$\gamma_{sc}(G) = 2$.



A B

Fig. 2.9. The graphs (A) $P_4 + P_3$ and (B) $P_4 + P_5$.

Subcase 2. If $n > 4$ and $m \leq 5$, again the vertex v_2 dominates the vertices u_1 and u_n if $m = 3$ while the vertex v_3 dominates the vertices u_1, u_n, v_2 and v_4 if $m = 5$. There is no a vertex dominates all vertices which have odd degree, since $n > 4$, then the two vertices v_1 and v_n dominate the all vertices have odd degree. Thus, the set $D = \{v_1, v_2, v_n\}$ is a minimum ESDS (as an example, see Figure 2.10), and $\gamma_{sc}(G) = 2$.



A B

Fig. 2.10. The graphs (A) $P_6 + P_3$ and (B) $P_6 + P_5$.

Subcase 3. If $n = 4$ and $m > 5$, then The vertex u_2 dominates the three vertices $u_3, v_1, and\ v_n$ and the summation of the degree of the vertex u_2 with each one of the vertices $u_3, v_1, and\ v_n$ is even.The other vertices in the graph G have even degree and there is no vertex dominates all these vertices, since $m > 5$ (as an example, see Figure 2.11). Let $D = \{u_2, v_1, v_n\}$ it is clear that D is ESDS and has minimum cardinality, so $\gamma_{sc}(G) = 3$.

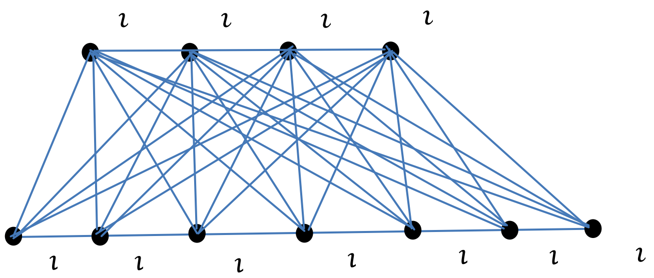


Fig. 2.11. The graph $P_4 + P_7$

Subcase 4. If $n > 4$ and $m > 5$, there is no one vertex dominates the vertices which haven odd degree, since $n > 4$. Also, there is no one vertex dominates the vertices which haven even degree, since $m > 5$ (as an example, see Figure 2.12). Let $D = \{u_1, u_n, v_1, v_n\}$ it is clear that D is ESDS and has minimum cardinality, so $\gamma_{sc}(G) = 4$.

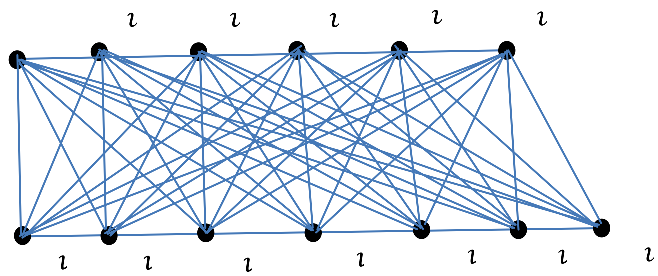


Fig. 2.12. The graph $P_6 + P_7$

Depending of all cases above, the required is obtained□

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