Results Concerning Some Generalized Covering Properties in Bitopological Spaces

Resultados sobre algunas propiedades de cobertura generalizadas en espacios biotopológicos

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Abstract
In this paper some generalizations for compactness and paracompactness in bitopological spaces are stated.

Keywords: s-$[a,\infty)$ *compact* space, s-$[a,\infty)$ *paracompact* space, *weakly* $s-[a,\infty)$ *compact* AMS subject classification: 54 E 65, 54 B 10.

Resumen

En este artículo se establecen algunas generalizaciones para la compacidad y la paracompacidad en espacios biotopológicos.

Palabras clave: espacio s, espacio s, clasificación de materias AMS: 54 E 65, 54 B 10.

Introduction and Preliminaries:

Kelly (1963) introduced a new concept called bitopological spaces. A non-empty set with two topologies τ_1, τ_2 . After this paper many mathematician studied this topic as a generalization of a single topology, see (Datta, 1972; Fletcher et al., 1969; Fora & Hdeib, 1983; Abushaheen & Hdeib, 2016; Abushaheen & Hdeib, 2019)

In this paper, we give some generalizations of compact spaces with different nature and largest possible cardinal numbers in bitopological spaces. We obtain some characterizations of pairwise paracompact spaces. Also we provide some product theorems and several results concerning compactness and parapcompactness in bitopological spaces.

Throughout this paper, w is the smallest infinite ordinal and w_1 is the smallest uncountable ordinal. $n,\ m$ will denote regular cardinal and n^+ is the successor of n, for example , $w_1 = w^+$. For a set A , $|A|$ will denote the cardinality of A . We refer the reader to Engelking, 1989, for the concepts and terminology not defined here.

Let $X = (X, \tau_1, \tau_2)$ be bitopological space and $A \subseteq X$, $\text{int}(A^{\tau_i})$, $\overline{A}^{\tau_i}$ denote the interior and closure of A in τ_i for i=1 ,2, when $X = (X, \tau_1, \tau_2)$ has a topological property Q that means both τ_1 ,τ_2 have this property**.**

In this section some essential definitions are introduced and some basic facts which are essential in obtaining the main results are stated.

Definition: (Fletcher et al., 1969)

A cover U of a bitopological space $X = (X, \tau_1, \tau_2)$ is called $\tau_1\tau_2 - open$ if $U \subseteq \tau_1 \bigcup \tau_2$

(Its called p-open cover if its $\tau_1\tau_2 - open$ cover contains at least one non empty element of τ_1 and one non empty element of τ_2).

Definition: (Datta, 1972).

For a $\tau_1\tau_2 - open$ covers U , V in bitopological space $X = (X, \tau_1, \tau_2)$, V is called parallel refinement of U if each $v \in V \bigcap \tau_i$ is contained in some $u \in U \bigcap \tau_i$ for i = 1, 2.

Definition: (Abushaheen & Hdeib, 2016)

A space $X = (X, \tau_1, \tau_2)$ is called $s-[a,b]\ compact$ if every $\tau_1\tau_2 - open$ cover of X with cardinality $\leq b$ has a subcover with cardinality $< a$.

(If X is $s-[a,b]\ compact$ for every $a \leq b$ then its called $s-[a,\infty)\ compact$, $s-[w,\infty)\ compact$ and $s-[w_1,\infty)\ compact$ spaces are called $s-compact$ and $s-lindelof$ spaces.

Definition:

A family A of subsets of a bitopological space X is called $locally-a$ if for all $x \in X$, x meets $< a$ members of $A \bigcap \tau_i, i = 1\ or\ 2$

Definition (Abushaheen & Hdeib, 2019).

A space $X=(X,\tau_1,\tau_2)$ is called $s-[a,b]$ *paracompact* if every $\tau_1\tau_2-open$ cover of X with cardinality $\leq b$ has a $\tau_1\tau_2-locally\ open-a$, $\tau_1\tau_2-open$ parallel refinement.

(If X has a $\tau_1\tau_2-locally\ open-a$, τ_1,τ_2-open parallel refinement for all $a\leq b$ then its $s-[a,\infty)$ *paracompact*)

2 . $almost\ s-[a,\infty)\ compact$ **and** $weakly\ s-[a,\infty)\ compact$ **in bitopological spaces.**

In this section we give the definitions of $almost\ s-[a,\infty)\ compact\ weakly\ s-[a,\infty)\ compact$ in bitopological spaces and some related results.

Definition:

A bitpological space $X=(X,\tau_1,\tau_2)$ is called $weakly\ s-[a,\infty)\ compact$ space if every $locally-a$, $\tau_1\tau_2-open$ cover of X has a subcover of cardinality $<a$.

(s-$weakly\ [w_1,\infty)\ compact$ spaces are called $w_1-lidelof$ space) .

Definition:

A bitopological space $X=(X,\tau_1,\tau_2)$ is called $almost\ s-[a,\infty)\ compact$ if for every $\tau_1\tau_2-open$ cover $\underline{U}=\{U_\alpha{}^i\setminus\alpha\in\Delta\}\cup\{U_\gamma{}^j\setminus\gamma\in\Gamma\}$ has $\tau_1\tau_2$-sub collection $\underline{U}^*=\{U_{\alpha^*}{}^i\setminus\alpha^*\in\Delta^*\subseteq\Delta\}\cup\{U_{\gamma^*}{}^j\setminus\gamma^*\in\Gamma^*\subseteq\Gamma\}$ with cardinality < a ($\left|\Delta^*\cup\Gamma^*\right|<a$) such that $\overline{\bigcup\{U\setminus U\in\underline{U}^*\}}^{\tau_i}=X$, for i= 1 or 2.

Theorem (2.1):

Let $X=(X,\tau_1,\tau_2)$ be a $s-[a,\infty)$ *paracompact* space, then the following are equivalent:

(a) X has a dense $s-[a,\infty)\ compact$ space.

(b) X is an $almost\ s-[a,\infty)\ compact$ space.

(c) X is $weakly\ s-[a,\infty)\ compact$ space.

(d) X is $s-[a,\infty)\ compact$.

Proof: $(a)\Rightarrow(b),(c)\Rightarrow(d),(d)\Rightarrow(a)$ are obvious.

$(b)\Rightarrow(c)$ Suppose X is not $weakly\ s-[a,\infty)\ compact$ space then there exists a locally $a-\tau_1\tau_2-open$ cover V of X which has no subcover of cardinality $<a$

(where $V=\{W_\alpha \mid \alpha\in\Delta\}\bigcup\{U_\gamma \mid \gamma\in\Gamma\}$ where $W_\alpha\in\tau_1\ for\ all\ \alpha\in\Delta$ and $U_\gamma\in\tau_2\ for\ all\ \gamma\in\Gamma$.

Let $O_x^{\tau_i}$, i= 1 or 2 be a neighborhood of x which meets $<a$ members of V.

$V^x=\{v\in V \mid v\bigcap O_x\neq\phi\}$ where $O_x^{\tau_i}=\{W_{\alpha_x} \mid \alpha\in\Delta\}\bigcup\{U_{\gamma_x} \mid \gamma\in\Gamma\}$, i= 1 or 2.

Suppose without loss of generality that $O_x^{\tau_i}$ is τ_i -open where *i*= 1 or 2.

Since $O^{\tau_i}=\bigcup\{O_x \mid x\in X\}$ is an open cover of X and X is $almost\ s-[a,\infty)\ compact$ then O^{τ_i} has subfamily of cardinality $<a$ say $\{O_{x_\psi}{}^{\tau_i} \mid \psi\in\alpha\cup\gamma=\Lambda,\alpha\in\Delta,\gamma\in\Gamma,|\Lambda|<a\}$, such that $\overline{A}^{\tau_i}=\overline{\bigcup\{O_{x_\psi}{}^{\tau_i} \mid \psi\in\Lambda,|\Lambda|<a\}}^{\tau_i}=X$. Hence A is dense.

Now, $V^*=\bigcup\{V^{x_\psi} \mid \psi\in\Lambda\},|\Lambda|<a$ has cardinality $<$ a and it's a subfamily of V hence V^* does not cover X. Choose $M\in V\backslash V^*$, since A is dense in X we have $M\bigcap A\neq\phi$. Thus $M\bigcap O_{x_\psi}{}^{\tau_i}\neq\phi,\ for\quad some\quad \psi$, so $M\in V^{x_\psi}$, hence $A\in V^*$ which is a contradiction.

Definition:

Let $U=\{U_\alpha{}^i \mid \alpha\in\Delta\}\cup\{U_\beta{}^j \mid \beta\in\Gamma\}$ and $V=\{V_\gamma{}^i \mid \gamma\in\Gamma\}\cup\{V_\omega{}^j \mid \omega\in W\}$ be $\tau_1\tau_2-open$ families of subsets of *X*. Then V is called $s-n-cushioned$ space of U if for $V\in V$ and $U_V\in U$ such that for every subfamily $V'=\{V_\gamma{}^i \mid \gamma\in\Gamma'\subset\Gamma\}\cup\{V_\omega{}^j \mid \omega'\in W\}$ of V with cardinality <n ($|\Gamma'\bigcup\omega'|<n$) we have $\overline{\bigcup\{V_\gamma \mid \gamma\in\Gamma'\}}^{\tau_i}\subseteq\bigcup\{U_{V_\gamma} \mid \gamma\in\Gamma\}$

Theorem:

Let $X=(X,\tau_1,\tau_2)$ be a bitopological space in which every $\tau_1\tau_2$ -open cover has s-n-cushioned parallel refinement. The following are equivalent:

(1) X is has dense $s-[n,\infty)-compact$,

(2) X is $almost\ s-[n,\infty)\ compact$,

(3) X is $s-[n,\infty)-compact$.

Proof:

$(1)\Rightarrow(2)$ let $U=\{U_\alpha \mid \alpha\in\Delta\}$ be an p-open cover of X so by (1) there exist a $\tau_1\tau_2$ -subcover of U with cardinality < n with $\bigcup\{U_\alpha \mid \alpha\in\Delta^*\subset\Delta,\left|\Delta^*\right|<n\}=X$

Clearly , $\overline{\bigcup\{U_\alpha \mid \alpha\in\Delta^*\subset\Delta,\left|\Delta^*\right|<n\}}^{\tau_i}=\overline{X}^{\tau_i}=X$,for i = 1 or 2

$(2)\Rightarrow(3)$ Let $U=\{U_\alpha \mid \alpha\in\Delta\}$ be a p-open ($\tau_1\tau_2$ -open) cover of X, let V be a p-n-cushioned space (s-n-cushioned) of U , since X is $p-almost-[n,\infty)-compact$ ($s-almost-[n,\infty)-compact$) , there exist a subcollection $X=\overline{\bigcup\{V_\gamma \mid V_\gamma\in V^*,\gamma\in\Gamma_1,\left|\Gamma_1\right|<n\}}^{\tau_i}$, i=1 or 2. Now for each $\gamma\in\Gamma$ take $\alpha_i\in\Delta$ such that $V_{\alpha_\gamma}\subseteq U_{\alpha_\gamma}$, then $X=\overline{\bigcup\{V_\gamma \mid \gamma\in\Gamma_1,\left|\Gamma_1\right|<n\}}^{\tau_i}\subseteq\bigcup\{U_{\alpha_\gamma} \mid \gamma\in\Gamma_1\}$.

Corollory:

Let $X=(X,\tau_1,\tau_2)$ be a bitopological space in which every $\tau_1\tau_2$ -open cover has an open p-n-cushioned parallel refinement . Then following are equivalent:

(1) X has dense $s-lidelof$ space .

(2) X is $s-$ $almost\ lidelof$ space.

(3) X is $s-lidelof$ space.

Corollory:

For a $s-[w_1,\infty)\ paracompact\ X=(X,\tau_1,\tau_2)$. The following are equivalent:

(1) X has a dense $s-[w_1,\infty)\ compact$ subspace.

(2) X is $almost\ s-[w_1,\infty)\ compact$ space.

(3) X is $weakly\ s-[w_1,\infty)\ compact$ space .

(4) X is $s-[w_1,\infty)$ *compact* space.

Corollory:

A $s-[w_1,\infty)$ *paracompact* $p-T_3-space$ with a dense $s-[w_1,\infty)$ *compact* space is $s-[w_1,\infty)$ *compact* space.

Proposition:

A $T_1-space$ is $s-[w_0,\infty)$ *paracompact* if and only if every s-open cover has $s-w_0-cushioned$ parallel refinement.

<u>Theorem</u>:

Let $X=(X,\tau_1,\tau_2)$ be an $s-[n,\infty)$ *paracompact*, $p-T_1-space$, $T_1-space$, then the following are equivalent:

(1) X is $s-[w_0,\infty)$ *paracompact* space.

(2) Every $\tau_1\tau_2-open$ cover of X has an s-n-cushioned parallel refinement.

Proof:

$(1)\Rightarrow(2)$

let $U=\{U_\alpha \mid \alpha\in\Delta\}$ be a $\tau_1\tau_2-open$ cover of X, then by the assumption there exists s-locally-n, s- open parallel refinement $V=\{V_\sigma \mid \sigma\in\Gamma,|\Gamma|<n\}$ such that fir each $V\in V$ and $U(V)\in U$ we have :

$$\bigcup\{V \mid V\in V\}\subseteq\bigcup\{U(V)\mid U\in U\}=X$$

and

$$\overline{\bigcup\{V \mid V\in V\}}^{\tau_i}\subseteq\bigcup\{U(V)\mid U\in U\},\ i=1,2$$

$(2)\Rightarrow(1)$

Let $U=\{U_\rho \mid \rho\in\Gamma\}$ be any $\tau_1\tau_2-open$ cover of X, then U has a s-n-cushioned parallel refinement $M=\{M_\alpha \setminus \alpha\in\Delta\}$, since X is $s-[n,\infty)$ *paracompact* then M has a $\tau_1\tau_2-open$ cover, locally-n, parallel refinement say $V=\{V_\alpha\setminus\alpha\in\Delta\}$, *such that* $V_\alpha\subseteq M_\alpha$ for each $\alpha\in\Delta$.

Claim:

$\mathcal{V}$ is $s-w_0$ *cushioned* parallel refinement of $\mathcal{U}$.

Proof:

For each $\alpha \in \Delta$ *assign* $B_\alpha \in \Gamma$ *such that* $V_\alpha \in U_{B_\alpha} \in \tau_i,\ i=1,2$. *Let* $\Delta_1 \in \Delta$ *and* $x \in \overline{\bigcup\{V_\alpha \ |\alpha \in \Delta_1\}}$, *then there exist an* τ_i*-open set* U *of* x *such that the set* $\Delta_2 = \{\alpha \in \Delta_1 | U \ \cap V_\alpha \ \neq \phi\}$ *with* $|\Delta_2| < n$, *hence* $x \notin \overline{\bigcup\{V_\alpha \ |\alpha \in \Delta_1 - \Delta_2\}}$. *Now since* $x \in \overline{\bigcup\{V_\alpha \ |\alpha \in \Delta_1\}}^{\tau_i} = \overline{\{V_\alpha \ |\alpha \in \Delta_1 - \Delta_2\}}^{\tau_i} \bigcup \overline{\{V_\alpha \ |\alpha \in \Delta_2\}}^{\tau_i}$.*Therefore:* $x \in \overline{\bigcup\{V_\alpha \ |\alpha \in \Delta_2\}}^{\tau_i}$.

Now $x \in \overline{\bigcup\{V_\alpha \ |\alpha \in \Delta_2\}}^{\tau_i} \subseteq \bigcup\{U_{B_\alpha} \ |\alpha \in \Delta_2\} \subseteq \bigcup\{U_{B_\alpha} \ |\alpha \in \Delta_1\}$.

Hence $x \in \overline{\bigcup\{V_\alpha \ |\alpha \in \Delta_1\}}^{\tau_i} \subseteq \bigcup\{U_{B_\alpha} \ |\alpha \in \Delta_1\}$, *so* $\mathcal{V}$ *is s-n-cushioned parallel refinement of* $\mathcal{U}$, *then* X *is* $s-[w_0,\infty)$ *paracompact space.*

Corollary:

Let X be an $s-[w_1,\infty)$ *paracompact* if and only if every $\tau_1\tau_2 - open$ cover of X has a $\tau_1\tau_2 - open, s - w_1 - cushioned$ parallel refinement.

Definition:

In a bitopological space $X = (X, \tau_1, \tau_2)$:

(1) A point $x \in X$ is called $\tau_i - n -$ limit point of $A \subseteq X$, if every $\tau_i - open$ set of x, we have $|U \bigcap A| = n$

(2) A space X is called n-compact if every subset of cardinality n^+ has an n^+-limit point.

(3) A subset $M \subseteq X$ is said to be $\tau_1\tau_2 -$ distinguished with respect to $\tau_1\tau_2 -$ cover $\mathcal{U}$ of X, if for each $x, y \in M$ with $x \neq y$ we have $x \in U$ implies $y \notin U$ for all $U \in \mathcal{U}$.

We say that M is $\tau_1\tau_2 -$ maximal distinguished set with respect to $\tau_1\tau_2 -$ cover of $\mathcal{U}$ of X if it is not a proper subset of any $\tau_1\tau_2 -$ distinguished subset of X with respect to $\mathcal{U}$.

Lemma:

Let $X=(X,\tau_1,\tau_2)$ *be a* $p-T_1-space$ *and* $\underset{\sim}{U}$ *is* $\tau_1\tau_2-$*open, then there exists a maximal distinguished subset* M_{τ_i} *, such that* M_{τ_i} *discrete and closed and* $U\bigcap M_{\tau_i}\neq\phi$.

Theorem:

Let $X=(X,\tau_1,\tau_2)$ be a $p-T_1-space$ then the following are equivalent:

a) X is an $s-[n^+,\infty)-compact$ space.

b) X is n-compact and $s-[n^+,\infty)-metacompact$ space.

Proof:

$(a)\Rightarrow(b)$ obviously, $X=(X,\tau_1,\tau_2)$ is $s-[n^+,\infty)-metacompact$. To show that X is n-compact.

Let B a subset of X with $|B|=n^+$. Suppose that B has n^+-limit point , hence for every $x\in X$ there is a τ_i-open set M_x such that $|M_x\bigcap B|\leq n$ Now $\underset{\sim}{M}=\{M_x\mid x\in X\}$ is a $\tau_1\tau_2-$ open cover of X, so X has a $\tau_1\tau_2-$open subcover $\underset{\sim}{M}^*=\{M_x\mid x\in\Delta,|\Delta|\leq n\}$. Hence $|B|\leq|\bigcup\{M_x\bigcap B,x\in\Delta\}|\leq n$ which is a contradiction with $|B|=n^+$, hence X is n-compact.

$(b)\Rightarrow(c)$ It is suffices to show that every $\tau_1\tau_2-$ point –n , $\tau_1\tau_2-$ open cover of X has a $\tau_1\tau_2-$ subcover of cardinality < n.

Assume that $\underset{\sim}{U}$ is a $\tau_1\tau_2-$ point –n , $\tau_1\tau_2-$ open cover of X which has no $\tau_1\tau_2-$ subcover of cardinality < n. Let K be the $\tau_1\tau_2-$ maximal distinguished subset of X with respect to $\underset{\sim}{U}$, then by lemma K is $\tau_i-closed$ and discrete and $\{U\in\underset{\sim}{U}\mid U\bigcap K\neq\phi\}$ is $\tau_1\tau_2-$ covers X .

If $|K|<n$, then $\underset{\sim}{U}$ will have a $\tau_1\tau_2-$subcover of cardinality <n which is contradiction with X is $n^+-compact$ m hence the result.

Theorem:

Every s-locally-n family of subsets of the $[n,n]-compact$ space X has cardinality <n.

Theorem:

Let $X=(X,\tau_1,\tau_2)$ be a $p-T_1-space$ then the following are equivalent:

a) X is an $s-[n,n]-compact$ space.

b) Every $\tau_i-closed$ subspace of X is $weakly\ s-[n,\infty)\ compact$ space , i= 1 or 2.

Proof: $(a)\Rightarrow(b)$

Let A be a $\tau_i-closed$ subspace of X and $\underset{\sim}{U}$ be as-locally-n, $\tau_1\tau_2-$open cover of A, hence By previous lemma the result is clear.

$(b)\Rightarrow(a)$

Let $\underset{\sim}{U}=\{W_\alpha\setminus\alpha\in\Delta\}\cup\{V_\gamma\setminus\gamma\in\Gamma\}$, where $W_\alpha\in\tau_1$ for all $\alpha\in\Delta$ and $V_\alpha\in\tau_i$ for all $\gamma\in\Gamma$ be a $\tau_1\tau_2-$open cover of X with $|\Delta\bigcup\Gamma|\leq n$ with no $\tau_1\tau_2-$subcover with cardinality < n, well order the sets $\Delta=\{1,2,....,\alpha,...\}$, $\Gamma=\{1,2,....,\alpha,...\}$. Now construct $\{X_\alpha,\alpha\in\Delta\bigcup\Gamma\}$ as follows: $x_1\in W_1$ or $x_1\in V_1$, $x_2\in W_2-W_1$ or $x_2\in V_2-V_1$.

Suppose we select x_β for all $\beta<\alpha$ such that $x_\alpha\in W_\alpha-\bigcup_{\beta<\alpha}W_\beta$ or $x_\alpha\in V_\alpha-\bigcup_{\beta<\alpha}V_\beta$, then $\{X_\alpha,\alpha\in\Delta\bigcup\Gamma\}$ is a $\tau_i-closed$ and $\tau_i-discrete$ subspace of X for i = 1 or 2, hence $\{X_\alpha,\alpha\in\Delta\bigcup\Gamma\}$ is $\tau_i-closed$ subspace of X and not $weakly\ s-[n,\infty)\ compact$ which is a contradiction, hence the result.

References

1. Abushaheen, F. A., & Hdeib, H. Z. (2016). ON $[a, b] $-COMPACTNESS IN BITOPOLOGICAL SPACES. International Journal of Pure and Applied Mathematics, 110(3), 519-535.
2. Abushaheen, F. A., & Hdeib, H. Z. (2019). p−[a, b]-paracompactness in bitopological spaces. Italian. J. Pure. App. Math, 42, 242-250.
3. Datta, M. C. (1972). Projective bitopological spaces. Journal of the Australian Mathematical Society, 13(3), 327-334.
4. Engelking, R. (1989). General topology (Revised and complete edition), Sigma Ser. Pure Math, 6.
5. Fletcher, P., Hoyle III, H. B., & Patty, C. W. (1969). The comparison of topologies. Duke Mathematical Journal, 36(2), 325-331.
6. Fora, A. A., & Hdeib, H. Z. (1983). On pairwise Lindelöf spaces. Revista Colombiana de Matemáticas, 17(1-2), 37-57.
7. Kelly, J. (1963). Bitopological spaces. Proceedings of the London Mathematical Society, 3(1), 71-89.