Fixed Point Theorem and its Properties in Cone

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ABSTRACT

In this paper we prove some fixed point theorems in fuzzy cone metric spaces under some fuzzy cone contractive type conditions. The concept of metric space by replacing the real numbers with an ordered Banach space, and proved some fixed point results for nonlinear mappings satisfying some contraction conditions. After that lots of works were devoted to the problems on cone metric spaces. The notation of fuzzy cone metric space which generalized the notation of fuzzy metric space. They also presented some structural properties of fuzzy cone metric spaces and proved a fixed point theorem under a fuzzy cone contraction condition. Some fixed point theorems and common fixed point theorems concerning fuzzy cone metric spaces were obtained and some more properties for fuzzy cone metric spaces can be found. The purpose of this paper is to give further study of fixed point theory in fuzzy cone metric spaces. Some fixed point theorems are proved under some fuzzy cone contractive type conditions.

KEYWORDS: Fixed Point Theorem, fuzzy cone metric spaces, nonlinear mappings

INTRODUCTION

It is common knowledge that many problems in various subfields of mathematics may be converted into a fixed point problem of the form Tx = x for self-mapping T defined on the framework of metric space (X, d). This transformation can be done in a variety of ways. Banach presented the idea of contraction mapping in 1992 and later proved the fixed point theorem for such mapping, which is now known as the Banach contraction principle. These contributions paved the way for more research and development in the area of analysis. A number of mathematical experts made use of a variety of conditions on self-mappings in order to demonstrate a number of fixed point theorems in metric spaces and other spaces.

1969 saw the establishment of the fixed point theorem for multivalued contraction mapping by Nadler. This was accomplished by the use of the idea of Hausdorff metric, which is an extension of the traditional Banach contraction principle. After that, Kaneko generalised the findings of Jungck by extending the analogous findings of Nadler to include both single valued mapping and multivalued mapping. Following this, there are a number of findings that expand the scope of this conclusion in a variety of new ways.

On the other hand, Kada and colleagues presented the idea of w-distance in relation to a metric space. They were able to enhance Caristi's fixed point theorem, Ekland's variational principle, and Takahashi's existence theorem by using this approach. After that, Suzuki and Takahashi came up with the fixed point solution for the multivalued mapping with regard to

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the w-distance. In point of fact, this conclusion is an enhancement of the fixed point theorem proposed by Nadler. In the context of metric spaces, several mathematicians have used w distance to establish a number of fixed point theorems; for example, see. Recently, Kutbi produced a helpful lemma for w-distance, which is an improved version of the lemma provided in, and demonstrated a crucial lemma on the presence of f-orbit for extended f-contraction mappings. Both of these were accomplished by proving a key lemma on the existence of f-orbit. In addition to this, he demonstrated the existence of coincidence points as well as common fixed points for generalised f-contraction mappings that did not include the extended Hausdorff metric.

COMMON FIXED POINT THEOREM AND THEIR PROPERTIES IN CONE METRIC SPACE FOR COMMUTATIVE MAPPING AND ORBITALLY CONTINUOUS MAPPING

Definition 1. Assuming E be a genuine Banach space and a nonempty set X. Assume that the mapping fulfills

$$0 \leq d(x,y) \text{ for all } x,y \in X \qquad \text{...(1)}$$

$$d(x,y) = 0 \text{ if and only if } x = y \qquad \text{...(2)}$$

$$d(x,y) = d(y,x) \text{ for all } x,y \in X \qquad \text{..(3)}$$

$$d(x,y) \leq d(x,z) + d(y,z) \text{ for all } x,y,z \in X. \qquad \text{..(4)}$$

Then distance d is known as a cone metric on X and set X with cone metric d is called cone metric space (X, d).

An arrangement { x_n } in a cone metric space X merges to x if and provided that

$$d(x_n, x) \to 0 \text{ as } n \to \infty \qquad \text{...(5)}$$

Let a sequence { x_n} in a cone metric space X converges to x. If $\{x_n\}$

unites to- y then x = y. That is breaking point of { x_n } is extraordinary.

A sequence $\{x_n\}$ is cone metric space X is said to be a Cauchy sequence, if for any $c \in E$ with $0 << c$ there is N such that

$$d(x_n, x_m) << c \text{ for all } n, m > N. \qquad \text{...(6)}$$

A sequence $\{x_n\}$ in cone metric space X converges to x , then $\{x_n\}$ is called a Cauchy sequence.

Let (X,d) be a complete cone metric space and f and g be two self-mappings on X . If

$$w = fx = gx \text{ for some } x \text{ in } X. \qquad \text{..(7)}$$

Then, at that point, x is known as a happenstance point f and g and w is known as a point of fortuitous event of f and g.

Two self-mapping f and g of a set X are supposed to be feebly viable assuming they drive at their fortuitous event point, or at least, if

$$fu = gu \text{ for some } u \in X \text{ then} \qquad \text{...(8)}$$

$$fgu = gfu \qquad \text{...(9)}$$

Let f and g be weakly compatible self-mapping of a set X . If f and g have a unique point of coincidence, that is $w = fx = gx$,then w is the unique common fixed point of f and g.

Theorem 1: Let (X,d) be a cone metric space and P be a normal cone with normal constant K . Suppose that the mappings $f, g : X \to X$ satisfy the contractive condition

$$d(fx, fy) \le \alpha[d(gx, gy) + d(fx, fy) + d(fx, gy) + d(fy, gx)$$
$$+ d(fy, gy) + d(fx, gx)] \qquad \text{.......................(10)}$$

Where $\alpha \in [0, \frac{1}{6})$ is a constant. If the range of g contains the range of f and $g(X)$ is complete subspace of X . Then f and g have a unique coincidence point in X .Moreover if f and g are weakly compatible then f and g has a unique common fixed point.

Proof. Let x_0 be an arbitrary point inX . Then since $f(X) \subset g(X)$, we choose a point x_1 in X such that $f(x_0) = g(x_1)$ continuing this process, having chosen x_n in X we obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$.

Then

$$
\begin{aligned}
& d(gx_{n+1}, gx_n) \\
&\qquad = d(fx_n, fx_{n-1}) \\
&\qquad \le \alpha[d(gx_n, gx_{n-1}) + d(fx_n, fx_{n-1}) + d(fx_n, gx_{n-1}) + d(fx_{n-1}, gx_n) \\
&\qquad +d(fx_n, gx_n) + d(fx_{n-1}, gx_{n-1})] \\
&\qquad \le \alpha[d(gx_n, gx_{n-1}) + d(gx_{n+1}, gx_n) + 2d(gx_{n+1}, gx_n) + 2d(gx_n, gx_{n-1})] \\
&\qquad \le \alpha[3d(gx_n, gx_{n-1}) + 3d(gx_{n+1}, gx_n)] \\
&\qquad \le \frac{3\alpha}{1-3\alpha} d(gx_n, gx_{n-1})
\end{aligned}
$$

$$d(gx_{n+1}, gx_n) \le hd(gx_n, gx_{n-1}) \qquad \text{..}\mathbf{(11)}$$

$$d(gx_{n+1}, gx_n) \le hd(gx_n, gx_{n-1})$$

$$h = \frac{3\alpha}{1-3\alpha}$$

Now for $n > m$ we get

$$d(gx_n, gx_m) \leq (gx_n, gx_{n-1}) + d(gx_{n-1}, gx_{n-2}) + ... + d(gx_{m+1}, gx_m)$$

$$\leq (h^{n-1} + h^{n-2} + ...h^m) d(gx_{m+1}, gx_m)$$

$$\leq \frac{h^m}{1-h} d(gx_1, gx_0)$$

.......(12)

When using the normality of cone P implies that

$$\| d(gx_n, gx_m) \| \leq \frac{h^m}{1-h} K \| d(gx_1, gx_0) \|$$

....(13)

$$d(gx_n, gx_m) \to 0 \text{ as } n, m \to \infty$$

$\{gx_n\}$ is a Cauchy sequence in X since $g(X)$ is complete subspace of X so there exists q in $g(X)$ such that

$$g(p) = q$$

$$d(gx_n, fp) \leq \alpha[d(gx_{n-1}, gp) + d(gx_n, fp) + d(gx_{n-1}, fp) + d(fp, gp)$$

$$+ d(fx_{n-1}, gp) + d(fx_{n-1}, gx_{n-1})]$$

$$\leq \alpha[d(gx_{n-1}, gp) + d(gx_n, fp) + d(gx_{n-1}, gp) + d(gx_{n-1}, gp$$

$$\leq \frac{3\alpha}{1-\alpha} [d(gx_{n-1}, gp)]$$

.......(14)

Using normality of cone

$$\| d(gx_n, fp) \| K \frac{3\alpha}{1-\alpha} \| d(gx_{n-1}, gp \| = 0$$

.....................................(14)

As $n \to \infty$.

$$d(gx_n, fp) \to 0 \text{ as } n \to \infty$$

The uniqueness of a cutoff in a cone metric space infers that $f(p) = g(p)$

Again we show f and g have a one of a kind point of occurrence, assuming conceivable accept that there exists one more point t in X to such an extent that

$$f(t) = g(t)$$

$$\begin{aligned} d(gt, gp) &= d(ft, fp) \\ &\le \alpha[d(gt, gp) + d(ft, fp) + d(ft, gp) + d(fp, gt) + d(ft, gt) + d(fp, gp)] \\ &\le \alpha[d(gt, gp) + d(ft, fp) + d(gp, gt) + d(gp, gt)] \end{aligned}$$

$$d(gt, gp) \le \frac{\alpha}{1-2\alpha}[d(gp, fp)] \to 0$$

$$f(t) = g(t)$$

$$\begin{aligned} d(gt, gp) &= d(ft, fp) \\ &\le \alpha[d(gt, gp) + d(ft, fp) + d(ft, gp) + d(fp, gt) + d(ft, gt) + d(fp, gp)] \\ &\le \alpha[d(gt, gp) + d(ft, fp) + d(gp, gt) + d(gp, gt)] \end{aligned}$$

$$d(gt, gp) \le \frac{\alpha}{1-2\alpha}[d(gp, fp)] \to 0$$

This show f and g both have same fixed point.

Theorem 2 : Let (X, d) be a complete cone metric space and f ,g are two orbitally continuous mapping from X into itself. Also let there exists a finite number of function $\{\phi_j\} : X \to [0, \infty)$ such that

$$d(fx,gy) \le \alpha d(x,y) + \sum_{j=1}^{N}[\phi_j(x) - \phi_j(fx) + \phi_j(y) - \phi_j(gy)], \forall x, y \in X.$$

................(15)

$$d(fx,gy) \le \alpha d(x,y) + \sum_{j=1}^{N}[\phi_j(x) - \phi_j(fx) + \phi_j(y) - \phi_j(gy)], \forall x, y \in X. \quad (1.12)$$

Where $o \le \alpha < 1.$ Then f ,g has a unique common fixed point.

Proof. Let x_0, y_0 are arbitrary point in X . Let us Consider the sequences $\{x_n\}$ and $\{y_n\}$ in X such that, $x_n = f^n x_0$ and $y_0 = g^n y_0.$

Putting $x = x_{i-1}, y = y_{i-1}$ in the above disparity we acquire,

$$d(x_i, y_i) = d(fx_{i-1}, gy_{i-1})$$

$$\le \alpha d(x_{i-1}, y_{i-1}) + \sum_{j=1}^{N}[\phi_j(x_{i-1}) - \phi_j(fx_{i-1}) + \phi_j(y_i) - \phi_j(gy_{i-1})]$$

$$= \alpha d(x_{i-1}, y_{i-1}) + \sum_{j=1}^{N}[\phi_j(x_{i-1}) - \phi_j(x_i) + \phi_j(y_{i-1}) - \phi_j(y_i)]$$

Putting i= 1,2,3,........n and summing up we get,

$$\sum_{i=1}^{n} d(x_i, y_i) \le \alpha \sum_{i=1}^{n} d(x_{i-1}, y_{i-1}) + \sum_{i=1}^{n}[\phi_j(x_{i-1}) - \phi_j(x_i) + \phi_j(y_{i-1}) - \phi_j(y_i)]$$

$$\Rightarrow (1-\alpha)\sum_{i=1}^{n} d(x_i, y_i) \le \alpha \sum_{i=1}^{n} d(x_0, y_0) + \sum_{i=1}^{n}[\phi_j(x_0) - \phi_j(x_n) + \phi_j(y_0) - \phi_j(y_n)]$$

$$\Rightarrow \sum_{r=1}^{n} d(x_i, y_i) \le \frac{\alpha}{1-\alpha}(x_0, y_0)$$

$$+\frac{1}{1-\alpha}\sum_{i=1}^{n}[\phi_j(x_0)-\phi_j(x_n)+\phi_j(y_0)-\phi_j(y_n)]$$
....................................(16)

Again for , $x = x_i, y = y_i$, we have

$$d(x_{i+1},y_i) = d(fx_i, gy_{i-1})$$

$$\leq \alpha d(x_i,y_{i-1}) + \sum_{i=1}^{n}\left[\phi_j(x_i)-\phi_j(fx_i)+\phi_j(y_{i-1})-\phi_j(gy_{i-1})\right]$$

$$= \alpha d(x_i,y_{i-1}) + \sum_{i=1}^{n}[\phi_j(x_i)-\phi_j(x_{i+1})+\phi_j(y_{i-1})-\phi_j(y_i)]$$

Applying same procedure of above we get,

$$(1-\alpha)\sum_{i=1}^{n} d(x_{i+1},y_i)$$
$$\leq \frac{\alpha}{1-\alpha}d(x_1,y_0)+\frac{1}{1-\alpha}\sum_{i=1}^{n}[\phi_j(x_i)-\phi_j(x_{i+1})+\phi_j(y_{i-1})-\phi_j(y_i)]$$
$$= \frac{\alpha}{1-\alpha}d(x_1,y_0)+\frac{1}{1-\alpha}\sum_{i=1}^{n}[\phi_j(x_i)-\phi_j(x_{n+1})+\phi_j(y_0)-\phi_j(y_n)]$$

$$\Rightarrow \sum_{i=1}^{n} d(x_{i+1},y_i) \leq \frac{\alpha}{1-\alpha}d(x_1,y_0)$$

$$+\frac{1}{1-\alpha}\sum_{i=1}^{n}[\phi_j(x_i)-\phi_j(x_{n+1})+\phi_j(y_0)-\phi_j(y_n)]$$
.............(17)

COMMON FIXED POINT THEOREM IN CONE RECTANGULAR METRIC SPACE

Definition 1. allow X to be a nonempty set and E be a genuine Banach space. Assume the mapping

$d : X \times X \to E$ satisfies

$$0 \le d(x,y), \forall x,y \in X. \quad \text{...(18)}$$

$$d(x,y)=0 \text{ if and only if } x=y. \quad \text{...(19)}$$

$$d(x,y)=d(y,x)\ \forall x,y \in X. \quad \text{...(20)}$$

$$d(x,y) \le d(x,w)+d(w,z)+d(z,y), \forall x,y \in X \quad \text{................................(21)}$$

and for all distinct point $w,z \in X-\{x,y\}$ (rectangular property).

Then d is known as a cone rectangular metric on X and (X,d) is called cone rectangular metric space.

Lemma 1. A sequence $\{x_n\}$ in cone rectangular metric space X is said to be convergent if for every $c \in E$ with $0 << c$ there is $n_0 \in N$ such that

$$d(x_n,x) << c \text{ for all } n > n_0. \quad \text{..................................(22)}$$

Lemma 2. A sequence $\{x_n\}$ in cone rectangular metric space X is said to be Cauchy if for every

$c \in E$ with $0 << c$ there is $n_0 \in N$ such that

$$d(x_n,x_m) << c \text{ for all } n,m > n_0. \quad \text{................................(23)}$$

On the off chance that each Cauchy arrangement in cone rectangular metric space X is merged, X is supposed to be finished cone rectangular metric space. For there is such that for all is called Cauchy sequence.

A cone rectangular metric space is supposed to be finished cone rectangular metric space assuming that each Cauchy arrangement in X is united.

Theorem 1. Let (X, d) is a finished cone rectangular metric space and P is a typical cone with ordinary consistent K . Let f is self-mapping from X into itself fulfilling.

$$d(fx, fy) \leq \alpha \left\{ \frac{d(x,y) + d(x,fx) + d(fx,y)}{2} + d(y,fy) \right\} \quad \text{......................(24)}$$

$$\forall x, y \in X, \alpha \in [0,1) \text{ and } 0 < \frac{\alpha}{1-\alpha} < 1$$

then f has an unique fixed point in X .

CONCLUSION

Fuzzy logic became the most powerful instrument in a variety of technological domains, including artificial intelligence, computer science, control engineering, medical science, and robotics, among others. Fuzzy set theory is a mathematical breakthrough that allows us to solve a variety of uncertain and real-world situations. Our "fuzzy fixed point" findings also have some applications in the realm of "dynamic programming." There are eight chapters in this thesis, followed by references and a list of publications. In the first chapter, we provide an overview of our study issue, as well as its significance and applications. We discuss some of the domains in which "fuzzy fixed point theorems" are used. We provide fresh discoveries in several domains such as FMS, IFMS, MIFMS, and -FMS in this thesis. We also look at how our discoveries on "fixed point theorems" might be used to "dynamic programming." Furthermore, by changing the concept of -FMS, we propose the innovative notion of MI -FMS. Following that, we look at some "fixed point" findings in the newly specified MI -FMS.

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