The Fixed Points of Mobius Transformation

Pallavi Pawar, Dr. Vishwajeet S. Goswami

Department of Mathematics, Chandigarh University, Mohali, Punjab

Pallavipawar258@gmail.com,vishwajeet.e9858@cumail.in

Article Info
Page Number: 409-412
Publication Issue:
Vol. 72 No. 1 (2023)

Article History
Article Received: 12 October 2022
Revised: 24 November 2022
Accepted: 18 December 2022

Abstract - In Complex Examination, a Bilinear Change of complex plane is a target limit of one complex variable z. Numerically, a Bilinear Change can be gotten by performing from plane to unit two-circle, turning and moving the circle to another space besides, bearing in space and performing stereographic projection to the plane.

Keywords – Mobius Transformation, Inverse Transformation, translation, Magnification and rotation, Inversion, Fixed points, Normal form, Cross Ratio.

1.Introduction:

In this research paper, a quick presentation has been given on Mobius Transformation, cross-ratio, fixed points and a few properties related to this type of transformation. We know that Mobius Transformation has a large number of applications that deal with the complex issues in Engineering, Physics and Mathematics.

1.1 ***Definition***: A Mobius Change of the complex plane is a rational function of the form

$w = f(z) = \frac{xz+y}{cz+d}$ where x, y, c, d are complex numbers, 0 ≠xd-yc

Which relates a one of a kind mark of w-plane to any place of z-plane aside from z=-d/c, c≠0. If xd - yc =1 , then, at that point this change is called standardized change.

1.2 ***Inverse of Mobius Transformation***: Let f(z) = $\frac{xz+y}{cz+d}$ = w, xd -yc≠0 be Mobius Transformation

W (cz + d) = xz + y

⇒z (wc – x) = y -wd

⇒ f^{-1}(w)= z= $\frac{wd-y}{-cw+x}$ is called Backwards Change which partners special place of z-plane to any mark of w-plane aside from w= x/c, c≠0.

1.3 ***Theorem***: Each Mobius Change comprises of four composition functions.

Proof:- Let w=f(z)= $\frac{xz+y}{cz+d}$, xd-yc≠0 be Mobius Transformation

a.) f_1(z) = z + $\frac{d}{c}$, i.e. translation by d/c b.) $f_2(z) = \frac{1}{f_1(z)} = \frac{1}{z+\frac{d}{c}}$, i.e. reflection and inversion with respect to real axis.

c.) $f_3(z)= \frac{-xd+yc}{c^2} f_2(z)$ i.e. magnification and rotation

d.) $f_4(z) = f_3(z) + \frac{x}{c}$ i.e. translation by x/c

$$f_4(z) = \frac{x}{c} + f_3(z) = \frac{x}{c} + \frac{-xd+yc}{c^2} f_2(2) \qquad = \frac{x}{c} + \frac{xd+yc}{c^2}\frac{1}{z+\frac{d}{c}}$$

$$= \frac{x}{c} + \frac{\frac{x}{c}\left[\frac{y}{x} - \frac{d}{c}\right]}{z+\frac{d}{c}} = \frac{xz+y}{cz+d} = f(z)$$

i.e., Mobius Change is comprises of elementary transformations.

2. Fixed Points

2.1 ***Definition:*** The focuses which correspond with their change i.e., f(z)= z or w= z.
Thus z = $\frac{xz+y}{cz+d} \Rightarrow cz^2$ – (x-d) z – y = 0 and for roots applying quadratic formula:
$z = \frac{(x-d)\pm\sqrt{(x-d)^2+4yc}}{2c}$. Let $Z_1 = \frac{(x-d)+\sqrt{(x-d)^2+4yc}}{2c}$, $Z_2 = \frac{(x-d)-\sqrt{(x-d)^2+4yc}}{2c}$.
Then Z_1, Z_2 are particular focuses and are fixed focuses for the change.

2.2 ***Theorem*** In case there are two unmistakable fixed focuses and of Mobius change w = f(z)= $\frac{xz+y}{cz+d}$ then, at that point the change take the structure $\frac{w-\alpha}{w-\beta} = k\left(\frac{z-\alpha}{z-\beta}\right)$.

Proof: As above, α, β are distinct fixed points and are given

by

$\alpha = \frac{(x-d)+\sqrt{(x-d)^2+4yc}}{2c}$, $\beta = \frac{(x-d)-\sqrt{(x-d)^2+4yc}}{2c}$

Since α, β are the roots of quadratic equation cz^2 – (x - d) z -y=0.
⇒ $c\alpha^2$ – (x-d) α – y= 0 & $c\beta^2$ – (x-d) β – y= 0
Thus, $C\alpha^2$ – xα = y – dα , $c\beta^2$ – xβ = y – dβ
Consider, $\frac{w-\alpha}{w-\beta} = \frac{\frac{xz+y}{cz+d}-\alpha}{\frac{xz+y}{cz+d}-\beta} = \frac{xz+y-\alpha cz-\alpha d}{xz+y-\beta cz-\beta d} = \frac{(x-\alpha c)z+(y-d\alpha)}{(x-\beta c)z+(y-\beta d)}$

$$= \frac{(x-\alpha c)z + (c\alpha^2 - a\alpha)}{(x-\beta c)z + (c\beta^2 - a\beta)}$$

$= \frac{(z-\alpha)(x-\alpha c)}{(z-\beta)(x-\beta c)} = k\frac{(z-\alpha)}{(z-\beta)}$

Where k = $\frac{(x-\alpha c)}{(y-\beta c)}$

If there should be an occurrence of two unmistakable fixed focuses, than the change is Exaggerated if k>0 and ELLIPTIC if k= $e^{i\alpha}$, α≠0 and LOXODROMIC if k = $xe^{i\alpha}$ where x ≠ 1, α≠ 0 and x, α are both genuine numbers, x>0.

2.3 ***Theorem***: In the event that there is just a single limited fixed place of Mobius Change, the change is in the structure

$\frac{1}{w-\alpha} = \frac{1}{z-\alpha} + k$

Proof: Since for fixed point, we put w = z $\Rightarrow \frac{xz+y}{cz+d} = z$

$$\Rightarrow cz^2 - (x-d)z - y = 0$$

On solving, $z = \frac{(x-d)\pm\sqrt{(x-d)^2+4yc}}{2c}$ are two fixed points, according to given condition, there is just a single limited fixed point $\Rightarrow (x-d)^2 + 4yc = 0$

$\Rightarrow \alpha = \frac{x-d}{2c} \Rightarrow 2\alpha c = x-d \Rightarrow d = x - 2\alpha c$

Again, $cz^2 - (x-d)z - y = 0 \Rightarrow c\alpha^2 - (x-d)\alpha - y = 0 \Rightarrow c\alpha^2 - x\alpha = y - d\alpha$

Now, $\frac{1}{w-\alpha} = \frac{1}{\frac{xz+y}{cz+d}-\alpha} = \frac{cz+d}{xz+y-\alpha cz-\alpha d} = \frac{cz+d}{(x-\alpha c)z+(y-d\alpha)} = \frac{cz+x-2\alpha c}{(x-\alpha c)z+c\alpha^2-x\alpha} = \frac{cz+x-\alpha c-\alpha c}{(z-\alpha)(x-\alpha c)} = \frac{(cz-\alpha c)+(x-\alpha c)}{(z-\alpha)(x-\alpha c)} = \frac{c}{(x-\alpha c)} + \frac{1}{(z-\alpha)}$

Taking $k = \frac{c}{(x-\alpha c)} \Rightarrow \frac{1}{w-\alpha} = k + \frac{1}{(z-\alpha)}$, which is normal form of change. In the event of one limited fixed point, the change is called illustrative.

3. CROSS RATIO

3.1 ***Definition:*** In math, the cross proportion is a number related with a rundown of four collinear focuses, especially focuses on a projective line. If x_1, x_2, x_3, x_4 are four distinct complex numbers then the ratio $\frac{(x_1-x_2)(x_3-x_4)}{(x_2-x_3)(x_4-x_1)}$ is called cross ratio of x_1, x_2, x_3, x_4.

3.2 ***Theorem*** A Bilinear Change jam cross proportion.

Proof: Assume $w = \frac{xz+y}{cz+d}$ where $xd-yc \neq 0$ be Bilinear Change.

Assume this change changes z_1, z_2, z_3, z_4 to corresponding points w_1, w_2, w_3, w_4 in w-plane.

$W_a = \frac{xz_a+y}{cz_a+d}$, a = 1,2,3,4

$w_a - w_b = \frac{xz_a+y}{cz_a+d} - \frac{xz_b+y}{cz_b+d}$

$= \frac{(xz_a+y)(cz_b+d)-(xz_a+y)(cz_a+d)}{(cz_a+d)(cz_b+d)}$

$= \frac{(x\,d-yc)(z_a-z_b)}{(cz_a+d)(cz_b+d)}$

Examine $(w_1, w_2, w_3, w_4) = \frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)}$

$$= \frac{\frac{(x\,d-yc)(z_1-z_2)(x\,d-yc)(z_3-z_4)}{(cz_1+d)(cz_2+d)(cz_3+d)(cz_4+d)}}{\frac{(x\,d-yc)(z_2-z_3)(x\,d-yc)(z_4-z_1)}{(cz_2+d)(cz_3+d)(cz_4+d)(cz_1+d)}}$$

$$= \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)} = (z_1, z_2, z_3, z_4)$$

3.3 ***Theorem***: Assume $f(z) = \frac{xz+y}{cz+d}$ where $xd-yc\neq0$ be any Bilinear change other than f(z) = z. This proves that f(z)is equal to inverse of f(z) if and only if d = - x

Proof: Assume $f(z) = f^{-1}(z)$, $f(z) = \frac{xz+y}{cz+d}$

$f^{-1}(w) = \frac{wd-y}{-cw+x} \Rightarrow f^{-1}(z) = \frac{zd-y}{-cz+x}$

At present, $f(z) = f^{-1}(z)$

$\frac{xz+y}{cz+d} = \frac{zd-y}{-cz+x} \Rightarrow (xz+y)(-cz+x) = (zd-y)(cz+d)$

$\Rightarrow -xcz^2 + x^2z - ycz + xy = -ycz - yd + cdz^2 + zd^2$

$\Rightarrow -xcz^2 + (x^2 - yc)z + xy = cdz^2 + (d^2 - yc)z - yd$

contrasting the factors of z^2, z, z^0,

we obtain $-xc = cd$; $x^2 - yc = d^2 - yc$; $xy = -yd$

$\Rightarrow x = -d$

On the contrary, if $d = -x$

$\Rightarrow f(z) = \frac{xz+y}{cz+d} = \frac{zd-y}{-cz+x} = \frac{-(zd-y)}{cz-x} = \frac{zd-d}{-cz+x} = f^{-1}(z)$

Thus, $f(z) = f^{-1}(z)$

Conclusion:

Mobius Transformation shows the transformation of the components over a complex plane underlying certain conditions. In Mobius Transformation of the complex plane which relates a novel mark of w-plane to any place of z-plane with the exception of z=-d/c, c≠0. There exists an converse of mobius change. Each mobius change is the composition of elementary transformation and fixed point of transformation are the points which coincide with their transformation. We discuss some theorem related to fixed points which tells that the transformation is hyperbolic, elliptic, loxodromic, and parabolic. It preserves cross proportion which is a number related with a rundown of four collinear focuses, especially focuses on a projection on a projective line.

Reference:

1. Nehari, Z. Conformal mapping, MC Graw – Hill Book, New York
2. https://planetmath.org/proofofmobiustransformationcrossratiopreservationtheorem.
3. https://www.uio.no/studier/emner/matnat/math/MAT2410/h15/extra-material/mobius_transformation_in_tex.pdf
4. Theory of Functions of Complex Variable [Shanti Narayan, Dr. P.K. Mittal]