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Themonophonic Vertex Covering Number Of A Graph

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Abstract

Fora connected graph Gof order n ≥ 2, a set s of vertices of G is monophonicvertexcoverofGifSisbothamonophonicsetandavertexcover of G. Theminimumcardinalityofamonophonicvertex coverofG iscalledthe monophonic vertex covering numberof G and is denoted by $m_\alpha(G)$. Any monophonic vertex cover of cardinality $m_\alpha(G)$ is a m_α-set of G. Some general propertiessatisfied by monophonic vertex cover are studied. The monophonic vertex coveringnumbers of several classes of graphs are determined. A few realization results aregivenfortheparameter$m_\alpha(G)$.

2020 Mathematical Subject Classification: AMS-05C12
Keywords: monophonic set, vertex covering set, monophonic vertex cover, monophonicvertexcoveringnumber.

1. Introduction

ByagraphG=(V,E),wemeanafiniteundirectedsimpleconnectedgraph.Theorderand sizeof G are denoted by nand mrespectively. For basic graph theoreticterminologywerefertoHarary[5].Thedistanced(u,v)betweentwoverticesuandvinaconnectedgraphGisthelengthofashortestu-vpathinG[1].ForavertexvofG,theeccentricitye(v)isthedistancebetweenvandavertexfarthest fromv.TheminimumeccentricityamongtheverticesofGistheradius,radGandthemaximumeccentricityisitsdiameter,diamG.TheneighbourhoodofavertexvofGisthesetN(v)consistingofallverticeswhichareadjacentwithv.AvertexvisasimplicalvertexoranextremevertexofGifthesubgraphinducedbyitsneighbourhoodN(v)iscomplete.Acaterpillarisatreeoforder3or more,theremovalofwhoseendverticesproducesapathcalledthespineofthecaterpillar.A diametral pathof a graph is a shortest path whose length is equal to the diameterofthegraph.Atreecontainingexactlytwonon-pendentverticesiscalleda doublestardenotedby S_{k_1,k_2} where$k_1$and$k_2$arethenumberofpendentverticesonthesetwonon-pendentvertices.AgraphGiscalledtrianglefreeifitdoesnotcontaincyclesoflength3.Asetof verticesnotwoofwhichareadjacentiscalledan independent set. By a matchingin a graph G, we mean an independent set ofedgesofG.AmaximalmatchingisamatchingMofagraphGthatisnotasubsetofanyothermatching.Theindependencenumberβ(G)ofGisthemaximum number of vertices in an independent set of vertices of G. A subset $S \subseteq V(G)$ is adominatingsetifeveryvertexinV−SisadjacenttoatleastonevertexinS.AsetS⊆V(G)iscalleda globaldominatingsetifitisadominatingsetofbothGand $\overline{G}$ (the complement of G). The minimum cardinality of a dominating set in agraphGiscalledthedominatingnumberofGanddenotedbyγ(G).

AgeodeticsetofGisasetS⊆V(G)suchthateveryvertexofGiscontained inageodesicjoiningsomepairofverticesinS.Thegeodeticnumberg(G)ofGis the minimum cardinality of its geodetic sets. The geodetic number of a graphwas introduced in [1,2] and further studied in [3,4]. A subset S ⊆V (G) is calledgeodeticglobaldominatingsetofGifSisbothgeodeticandglobaldominatingset of G. The geodetic global domination numberof a graph was introduced in [11] andfurtherstudiedin[8,9].AchordofapathPisanedgejoiningtwonon-adjacentverticesofP. A path Piscalledamonophonicpathifitisachordlesspath.AsetSofverticesofGisamonophonicset ofGifeachvertexvofGliesonanx-ymonophonic pathforsomex,y∈S.TheminimumcardinalityofamonophonicsetofGisthe monophonic numberof G and is denoted by m(G). Any monophonic set of cardinalitym(G)isaminimummonophonicsetoramonophonicbasisoram−setofG.Themonophonicnumberofagraphwasstudiedanddiscussedin[10].AsubsetS⊆V(G)issaidtobeavertexcoveringsetofGifeveryedgehasatleastoneend vertex in S. Avertex covering set of Gwith the minimum cardinalityis called aminimumvertexcoveringsetofG.ThevertexcoveringnumberofGisthecardinalityofany minimumvertexcoveringsetofG.Itisdenotedbyα(G)[12].AsetofverticesofGissaidtobemonophonicdominationsetifit isbothamonophonic setand a dominating set ofG.The minimumcardinalityof a monophonicdominationsetofGiscalledamonophonicdominationnumberofGanddenotedbyγ_m(G).Themonophonicdominationnumberwasstudiedin[7].

Thefollowingtheoremswillbeusedinthesequel.

Theorem 1.1.[11] Every extreme vertex of a connected graph G belongs to everymonophonic set of G. In particular, each end vertex of G belongs to every mono-phonicsetofG.

Theorem1.2.[11]ForanytreeTwithkendvertices,m(T)=k.Infact,thesetofallendvertices ofT istheuniquemonophonicsetofT.

Theorem1.3.[9]ForthecompletegraphK_n($n\geq 2$),$\gamma_m(K_n)=n$.

Theorem1.4.[9]ForanytreeTwith$n\geq 3$vertices,$\gamma_m(T)=n-1$ifandonlyifTisastar.

2. Definitions and Main results

Definition2.1.

Let G be a connected graphofordern≥2. Aset S of vertices of G is amonophonicvertexcoverofGifSisbothamonophonicsetandavertexcoverof G. The minimum cardinality of a monophonic vertex cover of G is called themonophonic vertex covering numberof G and is denoted bym_α(G). Any monophonicvertexcoverofcardinalitym_α(G)isam$_\alpha$-setofG.

Example2.2. For the graph G given in Figure 2.1, S = {v_1, v_5} is a minimummonophonicsetofGsothatm(G)=2andS'={v_1,v_4,v_5}isaminimummonophonic vertex cover of G so that m_α(G)=3. Thus the monophonic number is differentfromthemonophonicvertexcoveringnumberofagraph.

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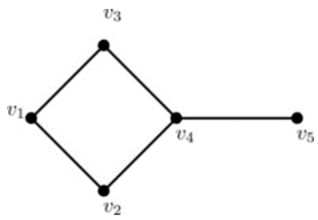

Figure2.1

Remark2.3. For the graph G given in Figure 2.2, S= $\{v_2, v_3\}$ is a minimummonophonic set of G so that m(G) = 2. S is also a minimum monophonic dominatingsetofGsothat$\gamma_m(G)=2$.$S'=\{v_1,v_2,v_3\}$isaminimummonophonicvertexcoverof Gso that$m_\alpha(G)=3$.Hence themonophonic vertexcoverof agraphis differentfromthemonophonicnumberandmonophonicdominatingnumberofagraph.

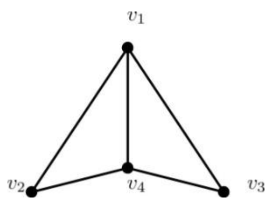

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Figure2.2

Theorem2.4. ForanyconnectedgraphG,$2\leq\max\{\alpha(G),m(G)\}\leq m_\alpha(G)\leq n$.

Proof of theorem2.4. AnymonophonicsetofGneeds at least 2 vertices. Then $2 \leq\max\{\alpha(G), m(G)\}$. From the definition of monophonic vertex cover of G, we have,$\max\{\alpha(G), m(G)\} \leq m_\alpha(G)$.ClearlyV(G)isamonophonicvertexcoverofG.Hence$m_\alpha(G)\leq n$.Thus$2\leq\max\{\alpha(G),m(G)\}\leq m_\alpha(G)\leq n$.

Remark 2.5. The bounds in Theorem 2.4 are sharp. For the complete graphK_n($n \geq 2$), $m_\alpha(K_n) = n$. For the cycle C_4, $m_\alpha(C_4)=2$. For the path P_3, $m_\alpha(P_3)=3$.TheboundsarestrictinFigure2.3as$\alpha(G)=2$,$m(G)=3$,$m_\alpha(G)=4$.Here$2<3<4<5$

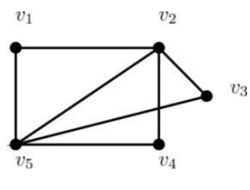

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Figure2.3

Remark2.6. Clearlyunionofavertexcoveringsetandamonophonicsetof Gis a monophonic vertex cover of G. In Figure 2.1, S = {v_1, v_4, v_5} is a monophonicvertexcoverandinFigure2.2,S={v_1,v_2,v_3,v_4}isamonophonicvertexcover.

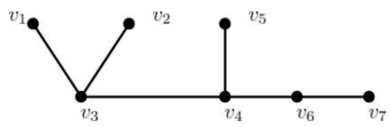

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Figure2.4

Thus$2\leq\max\{\alpha(G),m(G)\}\leq m_\alpha(G)\leq\min\{\alpha(G)+m(G),n\}$.

ForthegraphGinFigure2.3,weobserve that S_1 = {v_2, v_4} is a minimumvertexcover of G so that $\alpha(G)$ = 2, S_2 = {v_1, v_5} is a minimum monophonic setof G so that m(G)=2 and S_3 = {v_1, v_2, v_4, v_5} = $S_1 \cup S_2$ is a m_α-set of G and so$m_\alpha(G)=4=\alpha(G)+m(G)<n=6$.

Theorem 2.7. Each extreme vertex of G belongs to every monophonic vertex coverof G. In particular, each end vertex of G belongs to every monophonic vertex coverofG.

Proof of theorem 2.7. Fromthedefinitionofm_α-set,everym_α-setofGisam-setofG.HencetheresultfollowsfromTheorem1.1.

Corollary2.8. ForanygraphGwithkextremevertices,$\max\{2,k\}\leq m_\alpha(G)\leq n$.

Proof of corollary 2.8. TheresultfollowsfromTheorem2.4andTheorem2.7.

Corollary2.9. Let$K_{1,n-1}$($n\geq3$)beastar.Then$m_\alpha(K_{1,n-1})=n-1$.

Proof of corollary 2.9. LetxbethecentreandS={v_1, v_2, ..., v_{n-1}} bethe set of all extremevertices of $K_{1,n-1}$ ($n \geq 3$).Clearly S is a minimum monophonic vertex cover of$K_{1,n-1}$($n\geq3$)byTheorem2.7.Hence$m_\alpha(K_{1,n-1})=n-1$.

Corollary2.10. ForthecompletegraphK$_n$($n\geq2$),$m_\alpha(K_n)=n$.

Proof of corollary 2.10. WehaveeveryvertexofthecompletegraphK_n($n\geq2$)isanextremevertex.Then byTheorem2.7,thevertexsetistheuniquemonophonicvertexcover of K_n.Then$m_\alpha(K_n)=n$.

Theorem2.11. IfGisaconnectedgraphofordern ≥2,then

(i) $m_\alpha(G)=2$ifandonlyifGiseitherK_2or$K_{2,n-2}$($n\geq3$).

(ii) $m_\alpha(G)=n$ifandonlyifG=K_n($n\geq2$).

Proof of theorem 2.11.

(i) Let$m_\alpha(G)=2$.LetS={u,v}beaminimummonophonicvertexcoverofG.We claim that G = K_2or $K_{2,n-2}$ ($n \geq 3$). Suppose that G = K_2. Thenthereisnothingto prove. If not, then $n \geq 3$ and since S = {u, v} is a m_α-set of G, u and v cannot be adjacent in G.

Let W= V − S. We claim thatevery vertex of W is adjacent to both u and v and no two vertices of W areadjacent. Suppose there is a vertex w ∈W such that w is adjacent to at mostonevertexinS.Thenwliesonau-vmonophonicpathoflengthatleast3.Let P: u $=v_0v_1v_2..., v_i= w, v_{i+1}, ..., v_m= v$ be a u-v monophonic.Then theedges in E(P)-$\{v_0v_1, v_{m-1}v_m\}$ are not covered by any of the vertices u and v,whichisacontradictiontoSisam$_\alpha$-set.HenceeveryvertexofWisadjacenttobothuandv.Supposethereexistverticesw$_i$,w$_j$∈Wsuchthatw$_i$andw$_j$are adjacent.SinceeveryvertexofWisadjacenttobothuandvandS={u,v}isam$_\alpha$-setofG,w$_i$andw$_j$lieontheu-vmonophonicpathsuw$_i$vanduw$_j$ vrespectively.Thentheedgew$_i$w$_j$isnotcoveredbyanyofverticesofS,whichis acontradictiontoSisam$_\alpha$-setofG.HencenotwoverticesofWareadjacentinG.ThusGisthecompletebipartitegraph$K_{2,n-2}$(n≥3)withthepartitesetsSandW.

Conversely assume that G = K_2or $K_{2,n-2}$(n ≥ 3). If G=K_2, then byCorollary 2.10, $m_\alpha(K_2) = 2$. If not, let G = $K_{2,n-2}$(n ≥ 3). Let U = $\{u_1, u_2\}$andW=$\{w_1,w_2,...,w_{n-2}\}$ bethebipartitionofG.Clearlyeveryvertexw_i(1≤i≤n−2)liesonthemonophonicpath$u_1w_iu_2$andthe vertices$u_1$and$u_2$cover

alltheedgesofG.HenceUisamonophonicvertexcoverofGandso$m_\alpha(G)=2$.

(2)AssumethatG=K_n(n≥2).ThenbyCorollary2.10,$m_\alpha(G)=n$.

Converselyassumethat$m_\alpha(G)=n$.WeclaimthatG=K_n(n≥2).Forn=2, the result holds from 1. Let n ≥ 3. Suppose there exist two non-adjacentverticesuandvinG.Letavertexxbeadjacenttoulyingonau-vmonophonic.

ThenV(G)−{x}isamonophonicvertexcoverofG,whichisacontradictionto$m_\alpha(G)=n$.ThusG =K_n.

Theorem. 2.12. ForaconnectedgraphGwithm(G)≥n−1,$m_\alpha(G)=m(G)$.

Proof of theorem 2.12. Let G be a connected graph with m(G) ≥ n − 1. Then by Theorem 2.4,m(G)≤$m_\alpha(G)$≤n.Now,ifm(G)=n,then$m_\alpha(G)=n$.Hence$m_\alpha(G)=m(G)$.If m(G) = n− 1, then let S = $\{x_1, x_2, ..., x_{n-1}\}$ be a minimum monophonic set of G.Letx ∉ SbeavertexofG.Thenanyedgexx_i(1≤i≤n−1)liesonamonophonicpathjoiningpairofverticesof SandeveryedgeofGhasatleastoneendpointin S. HenceSisaminimummonophonicvertexcoverofGandso$m_\alpha(G)$ = m(G).

Remark.2.13. TheconverseofTheorem2.12neednotbetrue.

ForthegraphinFigure2.5,S=$\{v_1,v_2\}$isbotham-setofGandam$_\alpha$-setofG.Hence$m_\alpha(G)$=m(G)=2butm(G)<n−1.

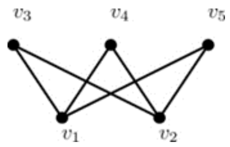

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Figure2.5

Theorem.2.14. For a connected graph G of order n ≥ 2, $m_\alpha(G) = m(G)$ if andonly if there exists a minimum monophonic set of G such that V (G) –Sis eitheremptyoranindependentset.

Proof of theorem 2.14. Assumethat$m_\alpha(G)=m(G)$.LetS={v_1,v_2, ...,v_k}beaminimummonophonic vertexcoverofG.ThenSisalsoaminimummonophonicsetofG.

Ifn=k,then V(G)– Sis empty. Let n >k. If not, there exist twoverticesu, v∈V (G) − Ssuchthatuv∈E(G).Thentheedgeuvhasnoneofitsendverticesin S,whichisacontradiction.Hence thereexistsaminimummonophonicsetofGsuchthatV(G) – Siseitheremptyoranindependentset.

Conversely assume that there exists a minimum monophonic set of G such thatV(G)– Siseitheremptyoranindependentset. Let S = {v_1, v_2, ..., v_k} so thatm(G)=|S|.SupposeV(G)– Sisempty.Thenn=kandS=V(G).HenceSisaminimum monophonic vertex cover of G so that $m_\alpha(G) = m(G)$. If not, let V(G)– Sbe independent. Then every edge of G has at least one end in V(G)−(V(G)−S) = SandsoSisavertexcoverofG.ThusSisaminimummonophonicvertexcoverof G. Thus $m_\alpha(G) = m(G)$.

Theorem.2.15. ForthecycleC$_n$(n≥4),$m_\alpha(C_n) = \left\lceil \frac{n}{2} \right\rceil$

Proof of theorem 2.15. LetC_n:$v_1v_2....v_nv_1$beacycleofordern.HereS={v_1,v_3,v_5,…,$v_{2\left\lceil \frac{n}{2}\right\rceil-1}$ }isaminimummonophonicvertexcoverofC_n.Hence$m_\alpha(C_n) = \left\lceil \frac{n}{2} \right\rceil$

Theorem.2.16. LetTbeatreeofordern≥2.Thenthefollowingstatementsareequivalent.

(1)$m_\alpha(T)=m(T)$.

(2) T is a star.

(3)α(T)=1.

(4) ThesetofallendverticesofTisavertexcoverofT.

Proof of theorem 2.16. LetSbethesetofallendverticesofT.SinceTisatree,fromtheTheorem1.2,wehave,Sis theuniquem-setofT.

(1)⇒(2) Assume that $m_\alpha(T) = m(T)$.We claim that T is a star. If not, thendiamT≥3.ThenThasatleastoneedgeotherthantheendedges.LetS′bethesetofalledgesofT whicharenotendedges.ThenclearlynoedgesofS′haveitsendverticesinS.HenceSisnotaverte xcoverofT.ByTheorem2.7,anymonophonicvertexcoverofTcontainsS.Hence$m_\alpha(T)>|S| =m(T)$,whichisacontradiction to$m_\alpha(T)=m(T)$.

(2)⇒(3) Assume that T is a star. If n=2, then an end vertex of T will cover the edge of T. If n ≥ 3, then the cut vertex of T will cover all the edges in T. Henceα(T)=1.

(3)⇒(4)Assumethatα(T)=1.ThenthereexistsavertexsayxinTsuchthatxisanendvertexofallt heedgesinT.HencealltheedgesinTaretheendedgesinTandsoSformsavertexcoverofT.

(4)⇒(1)AssumethatSisavertexcoverofT.ThenbyTheorem1.2,Sisam-setofT andbyTheorem2.7,Sisam_α-setofT.Hence$m_\alpha(T)=m(T)$.

Remark.2.17. The results in Theorem 2.16 are not equivalent for any connectedgraphGofordern≥2.

HereS={v_1,v_2,v_3}isbothm-setandm_α-setofG.So$m_\alpha(G)=m(G)=3$.

Also,Sisaminimumvertexcoveringsetandsoα(G)=3.AndhereGisnota

star.

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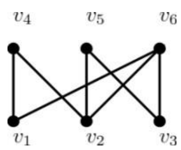

Figure2.6

3. Conclusion

In this paper we analyzed the monophonic vertex covering number of a graph. It is more interesting to continue my research in this area and it is very useful for further research.

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