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# Generalized K-function and its properties

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**Abstract**
In this paper, we introduce a generalization of the K-function by using extended beta function. Integral representation, recurrence relation, derivative formulas, Beta, Laplace, and Mellin transforms of the given function are obtained.

**Key Words:** K-function, Extended gamma and beta function, Beta transform, Laplace transform, Mellin transform.

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## 1. INTRODUCTION

Fractional calculus is the field of mathematical analysis, which deals with the investigation and applications of integrals and derivatives of any arbitrary real or complex order, which unify and extend the notions of integrals and derivatives. It has gained significance and recognition over the last four decades, especially because of its enormous capacity of tested programs in diverse seemingly expanded fields of science, applied mathematics and engineering.

Recently, Sharma [10] introduced and studied a new special function called as K-function, which is a particular case of the Wright generalized hypergeometric function ${}_p\psi_q(.)\mathrm{and Fox's H}$-function. K-function is interesting because the Mittag-Leffler function follows as its particular cases, and these functions have recently found essential applications in solving problems in physics, biology, engineering and applied sciences. For more detail study of K-function and its special cases, we refer to cited these references [12,13,14].

The function is defined for $\alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0, a_i, b_j \in \Re(-\infty, \infty), \quad a_i, b_j \neq 0; (i = 1,2, \ldots, p; j = 1,2, \ldots, q)$ as:

$$\underset{p}{\alpha, \beta,} \gamma K_q(x) = {}_p^{\alpha,\beta,\gamma}K_q\left(a_1, \ldots, a_p; b_1, \ldots, b_q; x\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n, \ldots, \left(a_p\right)_n}{(b_1)_n, \ldots, \left(b_q\right)_n} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{x^n}{n!},$$

where $(a_j)_n$ and $(b_j)_n$ are Pochhammer symbols. If any numerator parameter $(a_j)_n$ is a negative integer or zero, then the series terminates to a polynomial in x. The series (1.1) is defined when none of parameters $(a_j)_n j = 1,2, \ldots q$ is a negative integer or zero. From the ratio test it is evident that the series is convergent for all x if $p > q + 1$. When $p = q + 1$ and $|x| = 1$, the series can converge in some cases. Let $\gamma = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j$. It can be shown that when $p = q + 1$ the series is absolutely convergent for $|x| = 1$ if $\mathfrak{k}(\gamma) < 0$, conditionally convergent for $x = -1$ if $0 \leq \Re(\gamma) < 1$ and divergent for $|x| = 1$ if $\Re(\gamma) \geq 1$.

Some important special cases of K-function are enumerated below:

(1) For $p = q = 0$, the K-function is the generalization of the Mittag-Leffler function [6] and its generalized form [8] .

$$ {}_{0}^{\alpha,\beta,\gamma}K_0(-;-;x) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{x^n}{n!} = E_{\alpha,\beta}^{\gamma}(x), $$

(2) For $\gamma = 1$ in (1.2), then K-function is the generalized Mittag-Leffler function [5] .

$$ {}_{0}^{\alpha,\beta,1}K_0(-;-;x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \beta)} = E_{\alpha,\beta}^{1}(x) = E_{\alpha,\beta}(x), $$

(3) For $\beta = 1$ in (1.3), we get Mittag-Leffler function [5] .

$$ {}_{0}^{\alpha,1,1}K_0(-;-;x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + 1)} = E_{\alpha,1}^{1}(x) = E_{\alpha,1}(x) = E_{\alpha}(x), $$

(4) For $\alpha = 1$ in (1.4), which is the exponential function [9] denoted by $e^x$.

$$ {}_{0}^{1,1,1}K_0(-;-;x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n + 1)} = E_{1,1}^{1}(x) = E_{1,1}(x) = E_1(x) $$

On 1994, Chaudhary and Zubair [3] and in 1997 Chaudhry et al. [1] extended the domain of gamma and beta function to the entire complex plane by inserting a regularization factor:

$$ \Gamma_v(x) = \int_0^{\infty} t^{x-1} e^{-t-(v/t)} dt, \Re(v) > 0. $$

$$ B_v(x,y) = \int_0^{1} t^{x-1}(1-t)^{y-1} e^{-v/(t(1-t))} dt, \Re(v) > 0. $$

The Beta [11] , Mellin and Laplace transforms [4] of a function are defined respectively as

$$ B\{f(z); x, y\} = \int_0^1 z^{x-1}(1-z)^{y-1} f(z) dz $$

$$ \mathcal{L}\{f(z); s\} = \int_0^{\infty} e^{-sz} f(z) dz;\ \Re(s) > 0 $$

$$\mathcal{M}\{f(z);s\}=\int_0^\infty z^{s-1}f(z)dz$$

We also use the following equalities in our proofs:

$$\begin{aligned}B\{z^\lambda;x,y\}=&\quad B(\lambda+x;y),\ \Re(\lambda+x)>0,\Re(y)>0\\ \mathcal{L}\{z^\lambda;s\}=&\quad s^{-(\lambda+1)}\Gamma(\lambda+1),\ \Re(\lambda)\geq 0,\Re(s)>0\\ &\quad \mathcal{M}\{B_b(x,y);s\}=\Gamma(s)B(x+s,y+s)\\ &\quad (\Re(s)>0,\Re(x+s)>0,\Re(y+s)>0)\end{aligned}$$

## 2. Generalized K-function and their Properties

Motivated by the above works, we introduce the generalized K-function by using the extended beta function. We also obtain its certain properties such as integral representation, recurrence relation, derivative formulas, Beta, Laplace and Mellin transforms.

2.1. Generalized K-function: We introduced (presumably new) the generalized K-function as

$$\begin{aligned}{}_{p}^{\alpha,\beta,\gamma}K_q(z;v)=&\ {}_{p}^{\alpha,\beta,\gamma}K_q(a_1,\dots,a_p;b_1,\dots,b_q;z;v)\\ =&\sum_{n=0}^{\infty}\frac{(a_2)_n\dots(a_p)_n}{(b_2)_n\dots(b_q)_n}\frac{B_v(a_1+n,b_1-a_1)}{B(a_1,b_1-a_1)}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!},\end{aligned}$$

where $\Re(\alpha)>0,\Re(v)>0,\Re(b_1)>\Re(a_1)>0$. The series (2.1) is defined when none of parameters $a_{jn}, j=1,2,\dots,q$ is a negative integer or zero. From the ratio test it is evident that the series is convergent for all x if $p>q+1$. When $p=q+1$ and $|x|=1$, the series can converge in some cases. Let $\gamma=\sum_{j=1}^{p}a_j-\sum_{j=1}^{q}b_j$. It can be shown that when $p=q+1$ the series is absolutely convergent for $|x|=1$ if $\Re(\gamma)<0$, conditionally convergent for $x=-1$ if $0\leq\Re(\gamma)<1$ and divergent for $|x|=1$ if $\Re(\gamma)\geq 1$.

When, we take $v=0$, then equation (2.1) takes the equation (1.1).

Also $v\neq 0$, then we obtain some special cases are as below.

- For $\alpha=\beta=\gamma=p=q=1$, then we obtain confluent hypergeometric function which is defined in Chaudhary et al [2] as:

$${}_{1}^{1}M_{1}^{1}(a;b;z;v)=\phi_v(a;b;z)=\sum_{n=0}^{\infty}\frac{B_v(a+n,b-a)}{B(a,b-a)}\frac{z^n}{n!},$$

where $\Re(b)>\Re(a)>0$. - For $p=q=2$, then we obtain generalization of Mittag-Leffler function which is denied in Ozarslan and Yilmaz [7] as:

$$\begin{aligned}&{}_{2}^{\alpha}M_{2}^{\beta}(a_1,b_1;b_1,1;z;v)\\ &=E_{\alpha,\beta}^{a_1,b_1}(z,v)=\sum_{n=0}^{\infty}\frac{B_v(a_1+n,b_1-a_1)}{B(a_1,b_1-a_1)}\frac{(b_1)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!},\end{aligned}$$

where $\Re(v) > 0$ and $\Re(b_1) > \Re(a_1) > 0$.

2.2. Properties: Further, we obtain its certain properties such as integral representation, recurrence relation, derivative formulas, Beta, Laplace and Mellin transforms.

**2.0.1. Integral representation:**

Theorem 2.1. The following integral representation of generalized K-function are valid:

$$ {}_{p}^{\alpha,\beta,\gamma}K_q(z; v) = \frac{1}{B(a_1,\ b_1 - a_1)} \times \int_0^\infty \left[u^{a_1-1}(1+u)^{-b_1} e^{\frac{-v(u+1)^2}{u}} \underset{p-1}{\alpha,\beta,\gamma} K_{q-1}\left(a_2, \cdots, a_p; b_2, \cdots, b_q; \frac{zu}{1+u}\right)\right] du. $$

Proof: Now we take the equation (2.1) and we get

$$ {}_{p}^{\alpha,\beta,\gamma}K_q(z; v) = \sum_{n=0}^{\infty} \frac{(a_2)_n \cdots (a_p)_n}{(b_2)_n \cdots (b_q)_n} \frac{B_v(a_1 + n,\ b_1 - a_1)}{B(a_1,\ b_1 - a_1)} \frac{(\gamma)_n z^n}{\Gamma(\alpha n + \beta) n!} $$

$$ = \frac{1}{B(a_1,\ b_1 - a_1)} \sum_{n=0}^{\infty} \frac{(a_2)_n \cdots (a_p)_n}{(b_2)_n \cdots (b_q)_n} \frac{(\gamma)_n z^n}{\Gamma(\alpha n + \beta) n!} B_v(a_1 + n,\ b_1 - a_1). $$

Now, using equation (1.7) in the above equation, we get

$$ {}_{p}^{\alpha,\beta,\gamma}K_q(z; v) = \frac{1}{B(a_1,\ b_1 - a_1)} \sum_{n=0}^{\infty} \frac{(a_2)_n \cdots (a_p)_n}{(b_2)_n \cdots (b_q)_n} \frac{(\gamma)_n z^n}{\Gamma(\alpha n + \beta) n!} $$

$$ \times \int_0^1 t^{a_1+n-1}(1-t)^{b_1-a_1-1} e^{\frac{-v}{\left(\frac{1}{(1-t)}\right.}} dt = \frac{1}{B(a_1, b_1 - a_1)} $$

$$ \int_0^1 \left[t^{a_1-1}(1-t)^{b_1-a_1-1} e^{\frac{-v}{(1-t)}} p - 1K_{q-1}{}^{\alpha,\beta,\gamma}(a_2, \cdots, a_p; b_2, \cdots, b_q; zt)\right] dt. $$

Apply the transformations $t = \frac{u}{1+u}$ so that $u = \frac{t}{1-t}, dt = \frac{du}{(1+u)^2}$, $t = 0 \Rightarrow u = 0, t = 1 \Rightarrow u = \infty$, we have

$$ \times_{p-1} K_{q-1}\left(a_2, \cdots, a_p; b_2, \cdots, b_q; \frac{zu}{1+u}\right)\Big] \frac{du}{(1+u)^2} $$

$$ {}_{p}^{\alpha,\beta,\gamma}K_q(z;\ \upsilon) = \frac{1}{B(a_1,\ b_1 - a_1)} \int_0^\infty \left[\left(\frac{u}{1+u}\right)^{a_1-1} \left(1 - \frac{u}{1+u}\right)^{b_1-a_1-1} e^{\frac{-\upsilon}{\frac{u}{1+u}\left(1-\frac{u}{1+u}\right)}}\right. $$

$$ = \frac{1}{B(a_1,\ b_1 - a_1)} \int_0^\infty [u^{a_1-1}(1 \quad +u)^{-b_1} e^{\frac{-v(u+1)^2}{u}} $$

$$ {}_{p-1}K_{q-1}\left(a_2, \cdots, a_p; b_2, \cdots, b_q; \frac{zu}{1+u}\right)\Big] du. $$

which yields the result.

**2.0.2. Recurrence relation:**

Theorem 2.2. For the generalized K-function, the following recurrence relation holds true

$$\,_{p}^{\alpha,\beta,\gamma}K_q(z;v)=\frac{b_1-a_1}{b_1}\,pK_q^{\alpha,\beta,\gamma}\left(a_1,a_2,\cdots,a_p;b_1+1,b_2,\cdots,b_q;z;v\right)$$
$$+\frac{a_1}{b_1}p,K_q,\gamma,\gamma$$
$$K_1+1,a_2,\cdots,a_p;b_1+1,\ b_2,\cdots,b_q;z;v).$$

Proof. By using equation (2.5), we can write as below

$$\alpha,\underset{p}{\beta},\gamma K_q(z;v)=\frac{1}{B(a_1,\ b_1-a_1)}\int_0^1\left[t^{a_1-1}(1-t)^{b_1-a_1-1}(1-t+t)e^{\frac{-v}{(1(-t)}}\right.$$
$$\left.\times_{p-1}\overset{\alpha,\beta,\gamma}{K}-1_q\left(a_2,\cdots,a_p;b_2,\cdots,b_q;zt\right)\right]dt$$
$$=\frac{1}{B(a_1,b_1-a_1)}$$
$$\left.+\int_0^1\left[t^{a_1}(1-t)^{b_1-a_1-1}e^{\frac{-v}{(1-t)}}p-1\overset{\alpha,\beta,\gamma}{K}{}_{q-1}\left(a_2,\cdots,a_p;b_2,\cdots,b_q;zt\right)\right]dt\right\}$$

Using K-function defined in equation (1.1), we have

$$\,_{p}^{\alpha,\beta,\gamma}K_q(z;v)=\frac{1}{B(a_1,\ b_1-a_1)}$$
$$\left\{\int_0^1\left[t^{a_1-1}(1-t)^{b_1-a_1}e^{\frac{-v}{(1-t)}}\sum_{n=0}^{\infty}\frac{(a_2)_n\cdots(a_p)_n}{(b_2)_n\cdots(b_q)_n}\frac{(\gamma)_n(zt)^n}{\Gamma(\alpha n+\beta)n!}\right]dt\right.$$
$$\left.+\int_0^1\left[t^{a_1}(1-t)^{b_1-a_1-1}e^{\frac{-v}{(1-t)}}\sum_{n=0}^{\infty}\frac{(a_2)_n\cdots(a_p)_n}{(b_2)_n\cdots(b_q)_n}\frac{(\gamma)_n(zt)^n}{\Gamma(\alpha n+\beta)n!}\right]dt\right\}.$$

Changing the order of integration and summation, we obtain

$$\alpha,\underset{p}{\beta},\gamma K_q(z;v)=\frac{1}{B(a_1,\ b_1-a_1)}\left\{\sum_{n=0}^{\infty}\frac{(a_2)_n\cdots(a_p)_n}{(b_2)_n\cdots(b_q)_n}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)n!}\right.$$

$$\left\{\int_0^1\left[t^{a_1-1}(1-t)^{b_1-a_1}e^{\frac{-v}{t(1-t)}}{}_{p-1}\overset{\alpha,\beta,\gamma}{K}{}_{q-1}(a_2,\ \cdots,\ a_p;b_2,\ \cdots,\ b_q;zt)\right]dt\right.$$

$$\times\int_0^1\left[t^{a_1+n-1}(1-t)^{b_1-a_1}e^{\frac{-v}{r(1-t)}}\right]dt$$
$$\left.+\sum_{n=0}^{\infty}\frac{(a_2)_n\cdots(a_p)_n}{(b_2)_n\cdots(b_q)_n}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)n!}\int_0^1\left[t^{a_1+n}(1-t)^{b_1-a_1-1}e^{\frac{-v}{(1-t)}}\right]dt\right\}.$$

Further, using equation (1.7) and multiplying the terms in numerator and denominator with $B(a_1, b_1+1-a_1)$ and $B(a_1+1, b_1-a_1)$ respectively, we get

$$
{}_{p}^{\alpha,\beta,\gamma}K_q(z;v) = \frac{1}{B(a_1,\ b_1-a_1)}
\left\{\sum_{n=0}^{\infty} \frac{(a_2)_n \cdots (a_p)_n}{(b_2)_n \cdots (b_q)_n} \frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta) n!} B_v(a_1+n, b_1+1-a_1)dt
+ \sum_{n=0}^{\infty} \frac{(a_2)_n \cdots (a_p)_n}{(b_2)_n \cdots (b_q)_n} \frac{(\gamma)_n z^k}{\Gamma(\alpha n+\beta) n!} B_v(a_1+1+n, b_1-a_1)dt.\right\}
$$

Using the known property of gamma and beta function, we obtain

$$
\begin{aligned}
{}_{p}^{\alpha,\beta,\gamma}K_q(z;v) &= \left\{\sum_{n=0}^{\infty} \frac{(a_2)_n \cdots (a_p)_n}{(b_2)_n \cdots (b_q)_n} \frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta) n!} \frac{B_v(a_1+n, b_1+1-a_1)}{B(a_1, b_1+1-a_1)} \frac{b_1-a_1}{b_1} dt \right.\\
&\left. + \sum_{n=0}^{\infty} \frac{(a_2)_n \cdots (a_p)_n}{(b_2)_n \cdots (b_q)_n} \frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta) n!} \frac{B_v(a_1+1+n, b_1-a_1)}{B(a_1+1, b_1-a_1)} \frac{a_1}{b_1} dt \right\}\\
&= \frac{b_1-a_1}{b_1}\, {}_{p}^{\alpha,\beta,\gamma}K_q\left(a_1, a_2, \cdots, a_p; b_1+1, b_2, \cdots, b_q; z; v\right)\\
&+ \frac{a_1}{b_1} p,\beta,\gamma K_q\left(a_1+1, a_2, \cdots, a_p; b_1+1,\ b_2, \cdots, b_q; z; v\right)
\end{aligned}
$$

which yields the result.

2.2.3. Derivative formulas: Now, we investigate the derivative formulas of the generalized K-function.

Theorem 2.3. The following derivative formula holds true:

$$
\frac{d^k}{dz^k}\left\{z_p^{\beta-1}\ {}_{p}^{\alpha,\beta,\gamma}K_q(\lambda z^\alpha; v)\right\} = z^{\beta-k-1}\ {}_{p}^{\alpha,\beta-k,\gamma}K_q(\lambda z^\alpha; v).
$$

Proof. We know that the formula in term of gamma function

$$
\frac{d^k}{dz^k}\left(z^\lambda\right) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-k+1)} z^{\lambda-k}.
$$

Now, differentiate with respect to z in order to $k^{th}$ derivative of equation (2.1) yields

$$
\begin{aligned}
&\frac{d^k}{dz^k}\left\{z_p^{\beta-1}\ {}_p^{\alpha,\beta,\gamma}K_q(\lambda z^\alpha; v)\right\} \\
&= \frac{d^k}{dz^k}\left\{z^{\beta-1}\sum_{n=0}^{\infty}\frac{(a_2)_n\cdots(a_p)_n}{(b_2)_n\cdots(b_q)_n}\frac{B_v(a_1+n, b_1-a_1)}{B(a_1, b_1-a_1)}\frac{(\gamma)_n(\lambda z^\alpha)^n}{\Gamma(\alpha n+\beta)n!}\right\} \\
&= \sum_{n=0}^{\infty}\frac{(a_2)_n\cdots(a_p)_n}{(b_2)_n\cdots(b_q)_n}\frac{B_v(a_1+n, b_1-a_1)}{B(a_1, b_1-a_1)}\frac{(\gamma)_n\lambda^n}{\Gamma(\alpha n+\beta)n!}\frac{d^k}{dz^k}\left\{z^{\alpha n+\beta-1}\right\} \\
&= z^{\beta-k-1}\sum_{n=0}^{\infty}\frac{(a_2)_n\cdots(a_p)_n}{(b_2)_n\cdots(b_q)_n}\frac{B_v(a_1+n, b_1-a_1)}{B(a_1, b_1-a_1)}\frac{(\gamma)_n(\lambda z^\alpha)^n}{\Gamma(\alpha k+\beta-k)n!} \\
&= z^{\beta-k-1}\ {}_p^{\alpha,\beta-k,\gamma}K_q(\lambda z^\alpha; v).
\end{aligned}
$$

which yields the result.

### 2.0.3. Beta transform:

Theorem 2.4. The Beta transform of generalized K-function is

$$
B\left\{{}_pK_q^{\alpha,\beta,\gamma}(xz^\alpha;\ v);\ \beta,\ \zeta\right\} = \Gamma(\zeta)\ {}_pK_q^{\alpha,\beta+\zeta,\gamma}(x;\ v),
$$

where $\Re(\beta) > 0, \Re(\zeta) > 0$.

Proof. Using beta transform equation (1.8) and (2.1), we have

$$
\begin{aligned}
&B\left\{{}_p^{\alpha,\beta,\gamma}K_q(xz^\alpha; v); \beta, \zeta\right\} = \int_0^1 (z^\alpha)^{\beta-1}(1-z^\alpha)^{\zeta-1}\alpha, \underset{p}{\beta}, \gamma K_q(xz^\alpha; v)dz \\
&= \int_0^1 (z^\alpha)^{\beta-1}(1-z^\alpha)^{\zeta-1}\sum_{n=0}^{\infty}\frac{(a_2)_n\cdots(a_p)_n}{(b_2)_n\cdots(b_q)_n}\frac{B_v(a_1+n, b_1-a_1)}{B(a_1, b_1-a_1)}\frac{(\gamma)_n(xz^\alpha)^n}{\Gamma(\alpha n+\beta)n!}dz
\end{aligned}
$$

After changing the order of the integration with summation and using the equality (1.11), we get

$$
\begin{aligned}
&B\left\{{}_p^{\alpha,\beta,\gamma}K_q(xz^\alpha; v); \beta, \zeta\right\} = \sum_{n=0}^{\infty}\frac{(a_2)_n\cdots(a_p)_n}{(b_2)_n\cdots(b_q)_n}\frac{B_v(a_1+n, b_1-a_1)}{B(a_1, b_1-a_1)}\frac{(\gamma)_n x^n}{\Gamma(\alpha n+\beta)n!} \\
&\times\int_0^1 (z^\alpha)^{\beta-1}(1-z^\alpha)^{\zeta-1}z^{\alpha n}dz \\
&= \sum_{n=0}^{\infty}\frac{(a_2)_n\cdots(a_p)_n}{(b_2)_n\cdots(b_q)_n}\frac{B_v(a_1+n, b_1-a_1)}{B(a_1, b_1-a_1)}\frac{(\gamma)_n x^n}{\Gamma(\alpha n+\beta)n!}B\{z^{\alpha n}; \beta; \zeta\}
\end{aligned}
$$

$$= \Gamma(\zeta) \sum_{n=0}^{\infty} \frac{(a_2)_n \cdots (a_p)_n}{(b_2)_n \cdots (b_q)_n} \frac{B_v(a_1 + n, b_1 - a_1)}{B(a_1, b_1 - a_1)} \frac{(\gamma)_n x^n}{\Gamma(\alpha n + \beta + \zeta) n!}$$

$$= \Gamma(\zeta) \; {}_p^{\alpha,\beta+\zeta,\gamma}K_q(x; v).$$

### 2.0.4. Laplace transform:

Theorem 2.5. The Laplace transform of generalized K-function is

$$\mathcal{L}\left\{ {}_p^{\alpha,\beta,\gamma}K_q(xz; v); \; s \right\} = s^{-1} \, {}_{p+1}^{\alpha,\beta,\gamma}K_q\left(a_1, \cdots, a_p, 1; b_1, \cdots, b_q; \frac{x}{s}; v\right)$$

where $\Re(s) > 0$.

Proof. Using Laplace transform equation (1.9) and generalized K-function (2.1), we have

$$= \int_0^{\infty} e^{-sz} \sum_{n=0}^{\infty} \frac{(a_2)_n \cdots (a_p)_n}{(b_2)_n \cdots (b_q)_n} \frac{B_v(a_1 + n, b_1 - a_1)}{B(a_1, b_1 - a_1)} \frac{(\gamma)_n (xz)^n}{\Gamma(\alpha n + \beta) n!} dz.$$

After changing the order of the integration with summation and using the equality (1.12), we obtain

$$\mathcal{L}\left\{ {}_p^{\alpha,\beta,\gamma}K_q(xz; v); s \right\} = \sum_{n=0}^{\infty} \frac{(a_2)_n \cdots (a_p)_n}{(b_2)_n \cdots (b_q)_n} \frac{B_v(a_1 + n, b_1 - a_1)}{B(a_1, b_1 - a_1)} \frac{(\gamma)_n x^n}{\Gamma(\alpha n + \beta) n!}$$

$$\times \int_0^{\infty} e^{-sz} z^n dz$$

$$= \sum_{n=0}^{\infty} \frac{(a_2)_n \cdots (a_p)_n}{(b_2)_n \cdots (b_q)_n} \frac{B_v(a_1 + n, b_1 - a_1)}{B(a_1, b_1 - a_1)} \frac{(\gamma)_n x^n}{\Gamma(\alpha n + \beta) n!} s^{-(n+1)} \Gamma(n+1)$$

$$= \frac{1}{s} \, {}_{p+1}^{\alpha,\beta,\gamma}K_q\left(\frac{x}{s}; v\right)$$

$$= s^{-1} \, {}_{p+1}^{\alpha,\beta,\gamma}K_q\left(a_1, \cdots, a_p, 1; b_1, \cdots, b_q; \frac{x}{s}; v\right).$$

$$\mathcal{L}\left\{ {}_p^{\alpha,\beta,\gamma}K_q(xz; \; v); \; s \right\} = \int_0^{\infty} e^{-sz} \, {}_p^{\alpha,\beta,\gamma}K_q(xz; \; v) \, dz$$

### 2.0.5. Mellin transform:

Theorem 2.6. The Mellin transform of the generalized K-function is

$$\mathcal{M}\left\{ {}_p^{\alpha,\beta,\gamma}K_q(z; v); s \right\} = \Gamma(s) \frac{B(a_1 + s, \; b_1 - a_1 + s)}{B(a_1, \; b_1 - a_1)}$$

$$\times {}_p K_q(a_1 + s, a_2, \cdots, a_p; b_1 + 2s, b_2, \cdots, b_q; z).$$

where $\Re(s) > 0$.

Proof. Using Mellin transform equation (1.10) and (2.1), we have

$$\mathcal{M}\left\{ {}_{p}^{\alpha,\beta,\gamma}K_q(z;v);s\right\} = \int_0^\infty z^{s-1} \, {}_{p}^{\alpha,\beta,\gamma}K_q(z;v)dz$$
$$= \int_0^\infty z^{s-1} \sum_{n=0}^{\infty} \frac{(a_2)_n \cdots (a_p)_n}{(b_2)_n \cdots (b_q)_n} \frac{B_v(a_1+n, b_1-a_1)}{B(a_1, b_1-a_1)} \frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta) n!} dz$$

After changing the order of the integration with summation and using the equality (1.13), we have

$$\mathcal{M}\left\{ {}_{p}^{\alpha,\beta,\gamma}K_q(z;v);s\right\} = \sum_{n=0}^{\infty} \frac{(a_2)_n \cdots (a_p)_n}{(b_2)_n \cdots (b_q)_n} \frac{1}{B(a_1, b_1-a_1)} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta) n!}$$
$$\times \int_0^\infty z^{s-1} B_v(a_1+n, b_1-a_1) z^n dz$$
$$= \sum_{n=0}^{\infty} \frac{(a_2)_n \cdots (a_p)_n}{(b_2)_n \cdots (b_q)_n} \frac{1}{B(a_1, b_1-a_1)} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta) n!} \mathcal{M}\{z^n B_v(a_1+n, b_1-a_1);s\}$$
$$= \frac{\Gamma(s)}{B(a_1, b_1-a_1)} \sum_{n=0}^{\infty} \frac{(a_2)_n \cdots (a_p)_n}{(b_2)_n \cdots (b_q)_n} \frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta) n!} \mathcal{B}(a_1+s+n, b_1-a_1+s)$$

After multiplying with $\frac{B(a_1+s,\, b_1-a_1+s)}{B(a_1+s,\, b_1-a_1+s)}$, we obtain

$$= \frac{\Gamma(s) B(a_1+s,\ b_1-a_1+s)}{B(a_1, b_1-a_1)} \, {}_{p}^{\alpha,\beta,\gamma}K_q\left(a_1+s, a_2, \cdots, a_p; b_1+2s, b_2, \cdots, b_q; z\right).$$

## 3. Conclusion

In this article, we established a new definition of the function which is called generalized K-function. Further, we represented the function in the integral and recurrence form and evaluated nth order derivative. Also, evaluated the Beta, Laplace and Mellin transforms of the newly defined function. Also, some deductions from results of this paper are connected with already published results if we use $\gamma = 1$.

## 4. Competing interests

The authors declare that they have no competing interests.

## 5. Authors' contributions

AA made significant contributions to the creation of the work. AA contributed to the design of the work and handled the analysis. GV conceptualized and doublechecked the Analysis part. GV was involved in the manuscript's drafting or critical revision for important intellectual content. All authors read and approved the final version of manuscript.

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