# Decomposition of Various Graphs in to Prime Graphs

**Dr. Rajeev Gandhi S**
Head of the department
Department of Mathematics (SF)
V H N Senthikumara Nadar College (Autonomous),Virudhunagar.
rajeevgandhi@vhnsnc.edu.in

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***Abstract***

**Abstract:** In this paper we define prime decomposition and prime decomposition number $\pi_p(G)$ of a graph. Also investigate some bounds of $\pi_p(G)$ in product graphs like Cartesian product, composition etc.

Keywords: Decomposition, Prime graph, Cardinality.

## 1. Introduction

A decomposition of $G$ is a collection $\psi_p = \{H_1, H_2, .....H_r\}$ such that $H_i$ are edge disjoint and every edges in $H_i$ belongs to $G$. If each $H_i$ is a prime graphs, then $\psi_p$ is called a prime decomposition of $G$. The minimum cardinality of a prime decomposition of $G$ is called the prime decomposition number of $G$ and it is denoted by $\pi_p(G)$.

## 2. Prime Decomposition

In this section we define graceful decomposition of a graph $G(V,E)$ some and investigate some bounds of graceful decomposition number in $G(V,E)$ .

**Definition 2.1: [20]** A prime labelling of a graph G is an injective function $f:\{1,2,3,...,|V|\}$, such that for every pair of adjacent vertices $u$ *and* $v$, $\gcd(f(u), f(v)) = 1$. The graph admits prime labeling is called a prime graph.

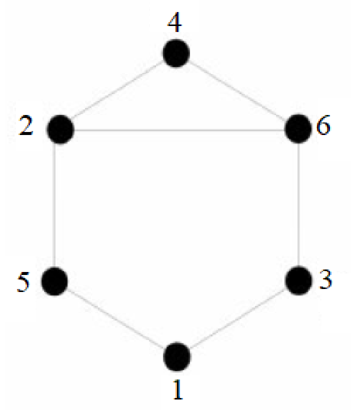



**Figure 2.1:**Prime cordial labeling

**Definition 2.3:** Let $\psi_P = \{H_1, H_2, .....H_r\}$ be a decomposition of a graph $G$. If each $H_i$ is a prime graphs, then $\psi_g$ is called a prime decomposition of $G$. The minimum cardinality of a graceful decomposition of $G$ is called the prime decomposition number of $G$ and it is denoted by $\pi_g(G)$.

**Definition 2.4: [35]** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs. The join $G_1 + G_2$ of $G_1$ *and* $G_2$ with disjoint vertex set $V_1 \& V_2$ and the edge set E of $G_1 + G_2$ is defined by the two vertices $(u_i, v_j)$ if one of the following conditions are satisfied

i) $u_i v_j \in E_1$.

ii) $u_i v_j \in E_2$.

iii) $u_i \in V_1 \; \& \; v_j \in V_2$, $u_i v_j \in E$

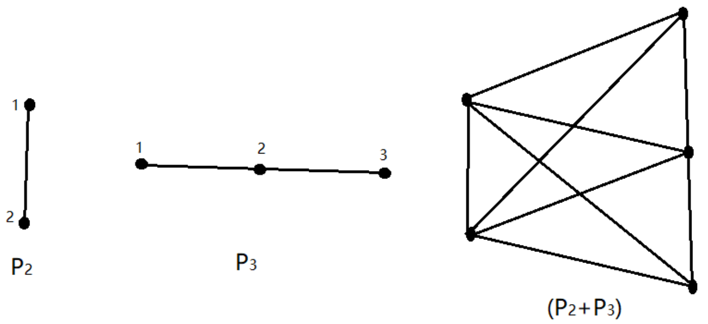
## 3. Main Results

**Theorem 3.1:** A graph $(P_m + P_n)$ is a join of two path prime graphs with $(m < n)$ .The bounds of prime decomposition number of the graph $(P_m + P_n)$ is, $3 \le \pi_p(P_m + P_n) \le (m(n+1) + (n-1))$.

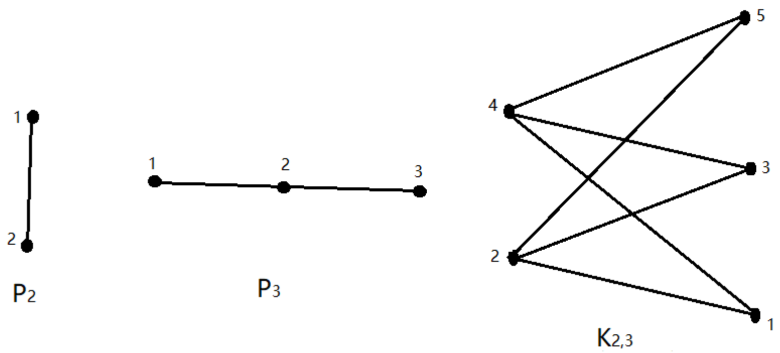
**Proof:** Let $P_m$ and $P_n$ be two path prime graphs of order m and n $(m > n)$ respectively and $(P_m + P_n)$ is a join of $P_m$ and $P_n$ with edge set E. The graph $(P_m + P_n)$ contains $(m+n)$ vertices and the edge set is $E = E_1 \cup E_2 \cup S(K_{m,n})$, Here $S(K_{m,n})$ is a size of a complete bipartite graph $K_{m,n}$. In the graph $(P_m + P_n)$ there are graphs $P_m$ , $P_n$ and the complete bipartite graphs $K_{m,n}$ .Note that $P_m$ and $P_n$ be two prime graphs and complete bipartite graphs $K_{m,n}$ also prime graph. This implies $\psi_p \supseteq \{\cup P_m \cup P_n \cup K_{mn}\}$ and $|\psi_p| \geq |\{\cup P_m \cup P_n \cup K_{mn}\}|$. Note that the graphs $P_m$ , $P_n$ and $K_{m,n}$ are prime graphs. Hence $\pi_p(P_m + P_n) \geq (3)$.

In $(P_m + P_n)$ there are $(m-1)+(n-1)+mn \Rightarrow m(n+1)+(n-1)$ edges. Note that every edge is a prime graph. This implies $\pi_p(P_m + P_n) \leq (m(n+1)+(n-1))$. Hence we get $3 \leq \pi_p(P_m + P_n) \leq (m(n+1)+(n-1))$.

**Illustration 3.2** The Join of two prime graphs $P_2$ & $P_3$ is given in figure.2.1




Decomposition of the graph $(P_2 + P_3)$ in to minimum copies of prime graph



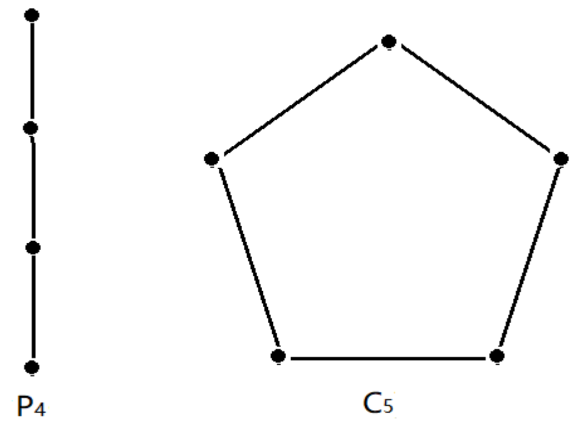
There are $((3-1)+(2-1)+6)=9$ edges in $(P_2+P_3)$, this implies the bound of $\pi_p(P_2+P_3)$ is $3 \le \pi_p(P_2+P_3) \le 9$.

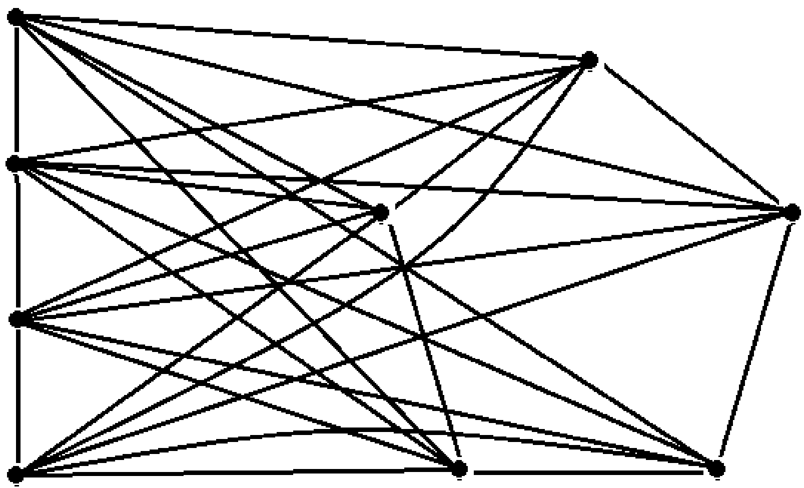
**Theorem 3.3.** A graph $(P_m+C_n)$ is a join of path prime graphs with $(m<n)$ and cycle $C_n$. The bounds of prime decomposition number of the graph $(P_m+C_n)$ is, $3 \le \pi_p(P_m+C_n) \le (m(n+1)+(n-1))$..

**Proof:** Let $P_m$ and $C_n$ be two path prime graphs of order m and n $(m>n)$ respectively and $(P_m+C_n)$ is a join of $P_m$ and $C_n$ with edge set E. The graph $(P_m+C_n)$ contains $(m+n)$ vertices and the edge set is $E=E_1 \cup E_2 \cup S(K_{m,n})$, Here $S(K_{m,n})$ is a size of a complete bipartite graph $K_{m,n}$. In the graph $(P_m+C_n)$ there are graphs $P_m$ , $P_n$ and the complete bipartite graphs $K_{m,n}$ .Note that $P_m$ and $C_n$ be two prime graphs and complete bipartite graphs $K_{m,n}$ also prime graph. This implies $\psi_p \supseteq \{\cup P_m \cup C_n \cup K_{mn}\}$ and $|\psi_p| \ge |\{\cup P_m \cup C_n \cup K_{mn}\}|$. Note that the graphs $P_m$ , $C_n$ and $K_{m,n}$ are prime graphs. Hence $\pi_p(P_m+C_n) \ge (3)$.

In $(P_m+C_n)$ there are $(m-1)+(n)+mn \Rightarrow n(m+1)+(m-1)$ edges. Note that every edge is a prime graph. This implies $\pi_p(P_m+P_n) \le (n(m+1)+(m-1))$. Hence we get $3 \le \pi_p(P_m+C_n) \le (n(m+1)+(m-1))$.

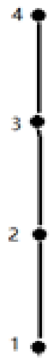
**Illustration 3.4:** The Join of two prime graphs $P_2$ & $P_3$ is given in figure.4.2.2



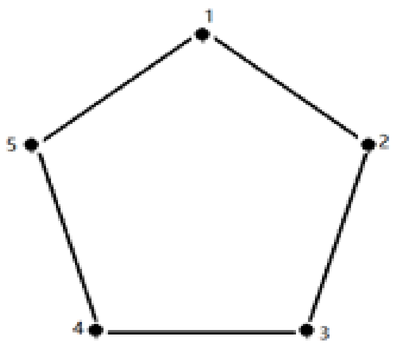


$(P_4+C_5)$

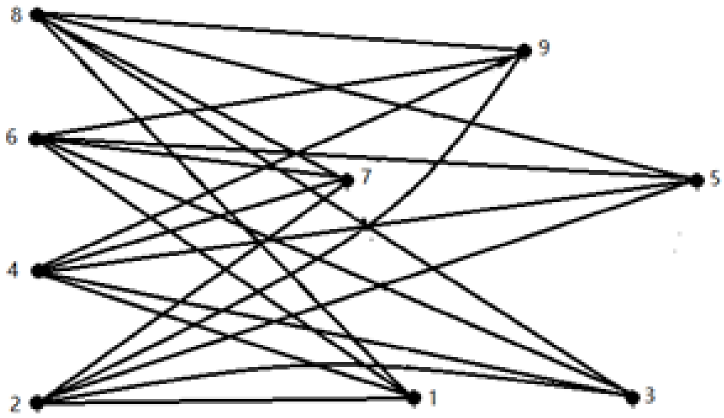
Decomposition of the graph $(P_4 + C_5)$ in to minimum copies of prime graph



$P_4$



$C_5$



$K_{4,5}$

There are $((4-1)+5+20)=28$ edges in $(P_4+C_5)$, this implies the bound of $\pi_p(P_4+C_5)$ is $3 \le \pi_p(P_4+C_5) \le 28.$

**Theorem 3.5:** A graph $(P_m \times P_n)$ is a Cartesian product of two prime graphs $(P_m \times P_n)$ with order m and n. Then bounds of prime decomposition number of the graph $(P_m \times P_n)$ is, $m+n \le \pi_p(P_m \times P_n) \le 2(mn)-(m+n)$.

**Proof:** Let $P_m$ and $P_n$ be two path prime graphs of order m and n respectively and $(P_m \times P_n)$ is a Cartesian product of $P_n$ & $P_m$ with edge set E. An edge $((x_1x_2)(y_1y_2)) \in E$ satisfies one of the following conditions

i) $x_1 = y_1$ and $x_2, y_2$ are adjacent vertices in $G_2 = (V_2, E_2)$.

ii) $x_2 = y_2$ and $x_1, y_1$ are adjacent vertices in $G_1 = (V_1, E_1)$.

**Case (i):** If $x_1 = y_1$ and $x_2, y_2$ are adjacent vertices in $G_2 = (V_2, E_2)$

If $x_1 = y_1$ and $x_2, y_2$ are adjacent vertices in $P_n$ . Let the sub graph $H_i$ is isomorphic to the graph $P_n$ . In $(P_m \times P_n)$ there are 'm' copies of graph $P_n$ and it is prime graph .This implies $H_i$ is also aprime graph. This implies $H_i \subset \psi$

**Case (ii):** If $x_2 = y_2$ and $x_1, y_1$ are adjacent vertices in $P_m$

If $x_2 = y_2$ and $x_1, y_1$ are adjacent vertices in $P_m$ . Note that the sub graph $H_j$ is isomorphic to the graph $P_m$ . In $(P_m \times P_n)$ there are 'n' copies of graph $P_m$ and it is prime graph This implies $H_j$ is also aprime graph. This implies $H_j \subset \psi$ .

From case (i) and (ii), we get $\psi = \left\{ \left( \bigcup_{i=1}^{m} H_i \right) \cup \left( \bigcup_{j=1}^{n} H_j \right) \right\}$ this implies

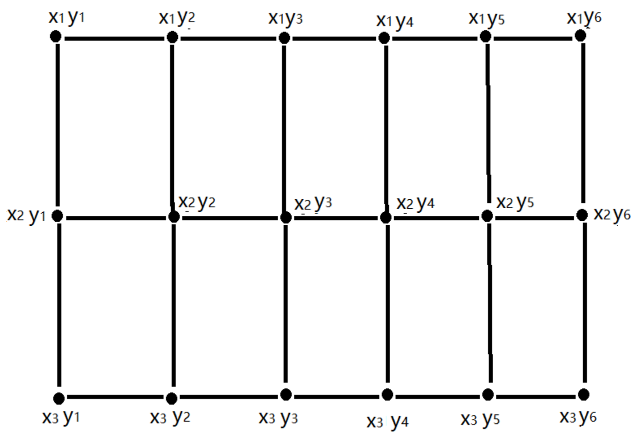
$$|\psi| = \sum_{i=1}^{m} H_i + \sum_{j=1}^{n} H_j = m+n.$$

In $(P_m \times P_n)$ there are $n(m-1)+m(n-1) \Rightarrow 2(nm)-(m+n)$ edges. Note that every edge is a prime graph. This implies $\pi_p(P_m + P_n) \le 2(nm-1)$. Hence we get $m+n \le \pi_p(P_m + P_n) \le 2(nm)-(m+n)$.

**Illustration 3.6:** The Cartesian product of two graceful graphs $P_3$ & $P_6$ is given in Figure 3.2.3

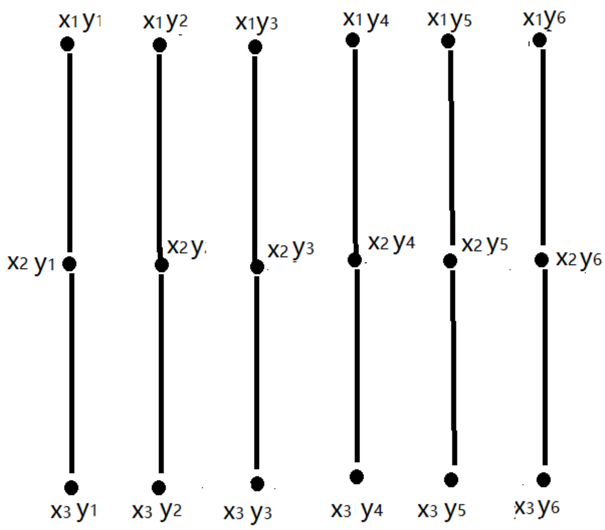

P3


P6

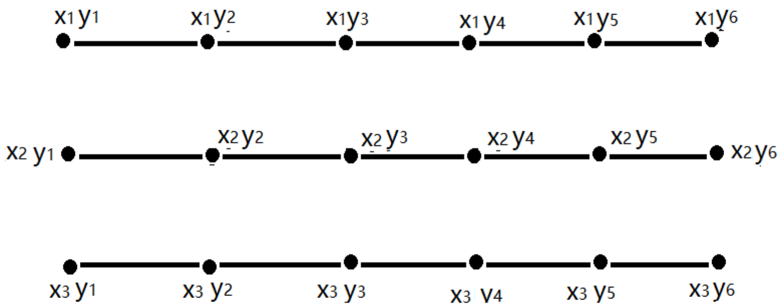



($P_3 \times P_6$)

**6 copies of $P_3$**




**3 copies of $P_6$**




There are $(2(18)-(9)=27$ edges in $(P_4+C_5)$, this implies the bound of $\pi_p(P_3+P_6)$ is $9 \leq \pi_p(P_3+P_6) \leq 27$.

**Theorem 3.7:** A graph $(P_m \times C_n)$ is a Cartesian product of two prime graphs $(P_m \times C_n)$ with order m and n. Then bounds of prime decomposition number of the graph $(P_m \times C_n)$ is, $m+n \leq \pi_p(P_m \times C_n) \leq 2(nm-1)$.

**Proof:** Let $P_m$ and $C_n$ be two path prime graphs of order m and n respectively and $(P_m \times C_n)$ is a Cartesian product of $P_m$ & $C_n$ with edge set E. An edge $((x_1x_2)(y_1y_2)) \in E$ satisfies one of the following conditions

i) $x_1 = y_1$ and $x_2, y_2$ are adjacent vertices in $G_2 = (V_2, E_2)$.

ii) $x_2 = y_2$ and $x_1, y_1$ are adjacent vertices in $G_1 = (V_1, E_1)$.

**Case (i):** If $x_1 = y_1$ and $x_2, y_2$ are adjacent vertices in $C_n$.

If $x_1 = y_1$ and $x_2, y_2$ are adjacent vertices in $C_n$ . Let the sub graph $H_i$ is isomorphic to the graph $C_n$. In $(P_m \times C_n)$ there are 'm' copies of graph $C_n$ and it is prime graph .This implies $H_i$ is also aprime graph. This implies $H_i \subset \psi$

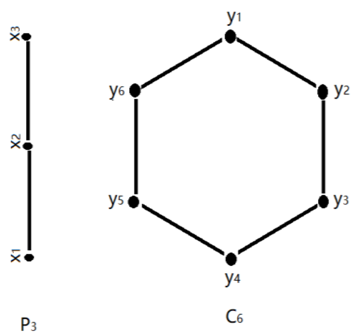
**Case (ii):** If $x_2 = y_2$ and $x_1, y_1$ are adjacent vertices in $P_m$

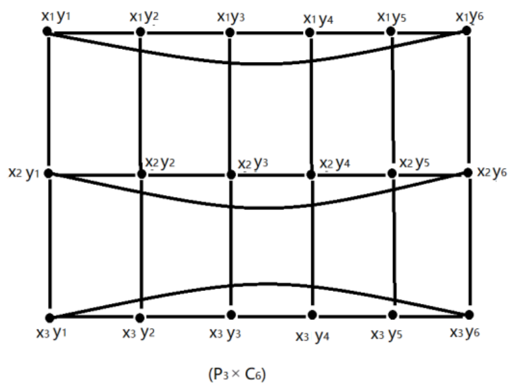
If $x_2 = y_2$ and $x_1, y_1$ are adjacent vertices in $P_m$ . Note that the sub graph $H_j$ is isomorphic to the graph $P_m$ . In $(P_m \times C_n)$ there are 'n' copies of graph $P_m$ and it is prime graph This implies $H_j$ is also aprime graph. This implies $H_j \subset \psi$ .

From case (i) and (ii), we get $\psi = \left\{ \left( \bigcup_{i=1}^{m} H_i \right) \cup \left( \bigcup_{j=1}^{n} H_j \right) \right\}$ this implies $|\psi| = \sum_{i=1}^{m} H_i + \sum_{j=1}^{n} H_j = m+n$.

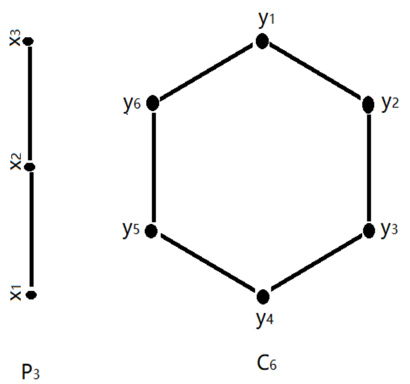
In $(P_m \times C_n)$ there are $n(m-1)+m(n) \Rightarrow 2(nm)-1$ edges. Note that every edge is a prime graph. This implies $\pi_p(P_m \times C_n) \le 2(nm)-1$. Hence we get $m+n \le \pi_p(P_m \times C_n) \le 2(nm)-1$.

**Illustration 3.8:** The Cartesian product of two Prime graphs $P_3 \,\&\, C_6$ is given in Figure.2.3

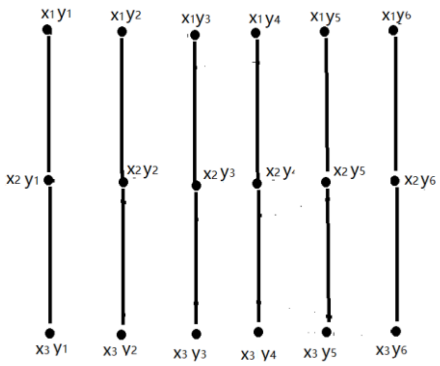


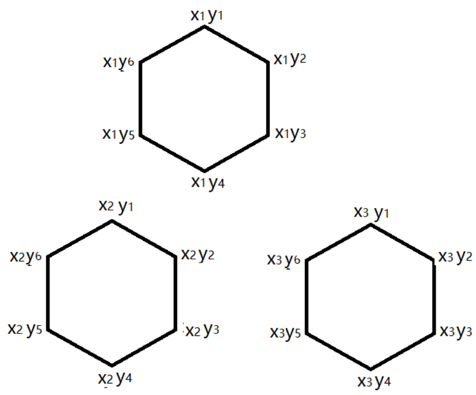

Prime decomposition $P_3 \,\&\, C_6$



**6 copies of** $P_3$




**3 copies of** $C_6$

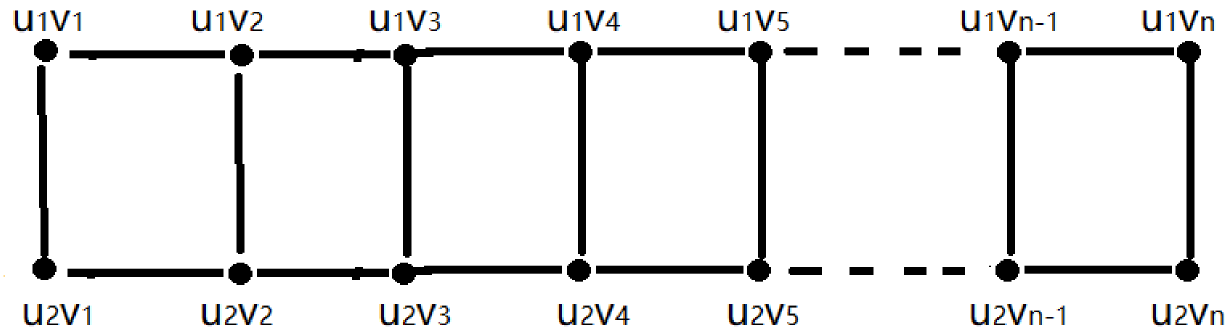



There are $2(18)-1=35$ edges in $(P_4 \times C_5)$, this implies the bound of $\pi_p(P_3 \times C_6)$ is $9 \leq \pi_p(P_3 \times C_6) \leq 35.$

**Theorem 3.9:** In a ladder graph $L_n$, the bounds of prime decomposition number is, $(n+2) \leq \pi_p(L_n) \leq (3n-2)$.

**Proof:** The ladder graph $L_n$ constructed by graphs $P_2$ and $P_n$ . The composition of the graphs $P_2$ and $P_n$ . From theorem 2.3, the bounds of the graph $(P_m \times P_n)$ is, $m+n \leq \pi_p(P_m \times P_n) \leq 2(nm-1)$. In ladder graph $m=2, n=n$ this implies we get $(n+2) \leq \pi_p(L_n) \leq n(2-1)+2(n-1)=3n-2$.

**Illustration 4.2.5:** The ladder graph $L_n$ is given in Figure




Ladder Graph Ln (or) (P2 × Pn)

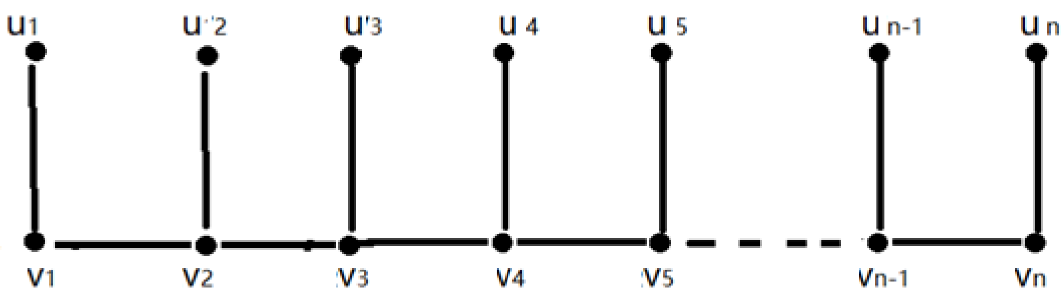
There are $3(n)-2$ edges inThe ladder graph $L_n$, this implies the bound of $\pi_p(L_n)$ is $(n+2) \le \pi_p(L_n) \le (3n-2)$.

**Theorem 3.10:** In a Brush graph $B_n$, the bounds of prime decomposition number is, $(n+1) \le \pi_p(B_n) \le (2n-1)$.

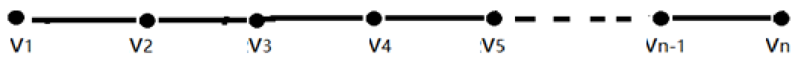
**Proof:** The Brush graph $B_n$ constructed by graphs $P_n$ and $K_{1,1}$. The Brush graph $B_n$ contains $(2n)$ vertices and the edge set E is $E = E_1 \cup (K_{1,1}, K_{1,1}, K_{1,1} ... n\, times)$, $E_1$ is set of edges in the path $P_n$. Note that $P_m$ and complete bipartite graphs $K_{1,1}$ are also prime graph. This implies $\psi_p \supseteq \{P_n \cup K_{1.1} \cup K_{1.1} \cup ... n\, times\}$ and $\psi_p \ge |P_n \cup K_{1.1} \cup K_{1.1} \cup ... n\, times|$. Hence $(n+1) \le \pi_p(B_n)$.

In Brush graph $B_n$ there are $n+(n-1) \Rightarrow 2n-1$ edges. Note that every edge is a prime graph. This implies $\pi_p(B_n) \le 2(n)-1$. Hence we get $(n+1) \le \pi_p(B_n) \le (2n-1)$.

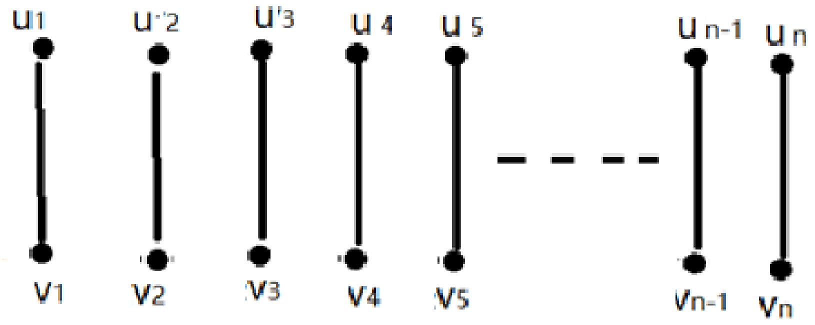
**Illustration 3.11:** The brush graph $B_n$ is given in Figure.




Brush Graph Bn




$P_n$




N copies of $K_{1,1}$

There are $2(n)-1$ edges in $(B_n)$, this implies the bound of $\pi_p(B_n)$ is $(n+1) \leq \pi_p(B_n) \leq (2n-1)$.

## 4. Conclusion

In this chapter, we define prime decomposition and prime decomposition number $\pi_p(G)$ of graphs. Also investigate some bounds of $\pi_p(G)$ in product graphs like Cartesian product, composition etc. In future we will investigate the decomposition number various labeling in graphs.

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