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Characterization of Steiner Domination with Chromatic Number in Fuzzy Graphs

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Abstract
If G is a connected graph and S is a subset of $V(G)$, then the Steiner distance is defined as the minimum size among all connected minimal sub graphs whose node sets contain S. These sub graphs are called Steiner trees of S. The Steiner interval, $IG(S)$or $I(S)$, of a set S is defined by IG(S) = {w ∈ V (G) / w lies on a Steiner tree for S in G}. If I(S)=V(G) then S is called a Steiner set. This article characterizes fuzzy graphs with crisp nodes using steiner dominating number and chromatic number.

Keywords and phrases: Characterization, steinerdomination , chromatic number.

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1. Introduction

The concept of fuzziness is one of embellishment. It is very essential to consider the fuzzy situations which occur frequently and reflects more precisely the real life problems. Fuzzy graph theory introduced by Rosenfeld has many real life applications. Several fuzzy graph theoretic concepts has been studied from [7] and [8]. In [1] and [2] the authors described about domination in fuzzy graphs. The notion of chromatic number in fuzzy graphs was initiated by Munoz et al [3]. Steiner domination in crisp graphs was studied from [4], [5] and [6]. In this article we characterize the fuzzy graphs with crisp nodes using steiner dominating number along with the chromatic number.

2. Characterization of Graphs using steinerdominating number and chromaticnumber

Definition :

Let G (V,σ,μ) be a connected fuzzy graph with 'n' nodes. To color the fuzzy graph we treat the fuzzy graphs with crisp nodes and edges with membership values in the interval [0,1]. Let $L = \{0, \alpha_1, \alpha_2, \ldots\ldots, \alpha_k\}$ be the level set of nodes of G. If $\alpha \in L$, then $E_\alpha = \{ij/1 \leq i < j \leq n, \mu(i,j) \geq \alpha\}$. Let $G_\alpha = (V, E_\alpha)$ and $\chi_\alpha = \chi(G_\alpha)$ which is the chromatic number of G_α. The fuzzy chromatic number of G is defined as $\chi^f(G) = \max\{\chi_\alpha/\alpha \in L\}$. A r-coloring of G partitions the node set of G into r color classes. It has been determined that
$\chi^f(G) = \chi(\tilde{G})$where$\tilde{G}$ is the underlying crisp graph of G. To characterize the graphs using steiner dominating number and chromatic number the fuzzy graphs with crisp nodes are considered here.

i) **Theorem :**For any connected graph G with 'n' nodes, $\gamma^{fs}(G) + \chi(G) \leq 2n$ and equality holds iff $G = K_n$, the complete graph with 'n' nodes.

Proof :

It is clear that $\gamma^{fs}(G) \leq n$ and it is known that if $\Delta(G)$ is the maximum degree of a node in G, then $\chi(G) \leq \Delta(G) + 1$. Also $\Delta(G) \leq n - 1$. Therefore
$\gamma^{fs}(G) + \chi(G) \leq n + \Delta(G) + 1 \leq n + (n-1) + 1 \leq 2n$. Now if equality holds,
$\gamma^{fs}(G) + \chi(G) = 2n \Leftrightarrow \gamma^{fs}(G) = n \;\&\chi(G) = n \Leftrightarrow G = K_n^f$.

ii) **Theorem**

$\gamma^{fs}(G) + \chi(G) = 2n - 1$does not holds for any connected graph G with $n > 1$ nodes.

Proof :

Suppose $\gamma^{fs}(G) + \chi(G) = 2n - 1$. Here there are two cases either i) $\gamma^{fs}(G) = n$, $\chi(G) = n - 1$ or ii)$\gamma^{fs}(G) = n - 1$, $\chi(G) = n$

Case (i) $\gamma^{fs}(G) = n$, $\chi(G) = n - 1$

Since $\chi(G) = n - 1$, G must have a set of n-1 nodes whose induced subgraph is complete or is a (n-1)-clique. Let $U = \{u_1, u_2, \ldots\ldots, u_{n-1}\}$ be the nodes of (n-1)-clique. Now if u_n is the remaining node it must be adjacent to some 'r' nodes in U. The remaining $n - r - 1$ nodes of U are extreme nodes and hence must belong to any steiner set. Along with the node 'u_n' these extreme nodes forms a minimumsteiner dominating set and contains $n - r - 1 + 1 = n - r$ nodes. But $\gamma^{fs}(G) = n$. Hence $r = 0$. Therefore the graph G is disconnected which is a contradiction. Thus this case is not possible.

Case (i) $\gamma^{fs}(G) = n - 1$, $\chi(G) = n$

If $\chi(G) = n$,then G must be a complete graph with 'n' nodes. But then $\gamma^{fs}(G) = n$. Hence this case is also impossible. Thus the above theorem does not holds good for connected graphs.

iii) Theorem

Let G be a connected graph with 'n' nodes where $n > 2$,
then$\gamma^{fs}(G) + \chi(G) = 2n - 2$ iff $G \cong K_{1,2}$ or G is composed of K_2&K_{n-1}.

Proof :

Suppose G is the composition of any of the above then it can be easily checked that the result is true. Conversely suppose $\gamma^{fs}(G) + \chi(G) = 2n - 2$, then there are three cases here.
Case (i)$\gamma^{fs}(G) = n$ &$\chi(G) = n - 2$, *Case (ii)*$\gamma^{fs}(G) = n - 2$ &$\chi(G) = n$ and
Case (iii)$\gamma^{fs}(G) = n - 1$ &$\chi(G) = n - 1$. The first two cases are not possible because $\gamma^{fs}(G) = n \Leftrightarrow G = K_n^f$ and $\chi(G) = n \Leftrightarrow G = K_n^f$
Now considering the third case.
Case (iii)$\gamma^{fs}(G) = n - 1$ & $\chi(G) = n - 1$

Assume that G has a (n-1)-clique with node set U, let the remaining node be 'x'. Each node in U has degree n-2 in $\langle U \rangle$. But since $\gamma^{fs}(G) = n - 1$, G has a cut node of degree n-1. 'x' cannot be such cut node. Also if 'x' is adjacent to more than one node in U, then no such cut node exists. Hence exactly one node from U must be adjacent to 'x' so that it must be the cut node of degree n-1. Thus G has components $\boldsymbol{K_2}$**&**$\boldsymbol{K_{n-1}}$**.** On the other hand suppose G has no (n-1)-clique, then let 's' be the cut node of degree n-1 and let S be the neighborhood of 's'. If S is independent then G has 2-coloring.
So $n - 1 = 2 \Longrightarrow n = 3$ and G is isomorphic to $\boldsymbol{K_{1,2}}$**.** If not (i.e) if S is not independent, then S has 'r' coloringwhere $2 \leq r \leq n - 2$. If $r = n - 1$, then there is a n-1 clique. Hence G has r+1 coloring and $r + 1 = n - 1 \Longrightarrow r = n - 2$ and hence the induced subgraph of S has a n-2 clique. Along with the node 's' a n-1 clique is formed. Hence this S must be independent. Hence the theorem.

iv) Characterization of Graphs with $\boldsymbol{\gamma^{fs}(G) + \chi(G) = 2n - 3}$

Let G be a connected graph with 'n' nodes where $n > 3$,
then$\gamma^{fs}(G) + \chi(G) = 2n - 3$ iff G can be decomposed into any one of the following components. a) $K_{1,3}$andK_{n-2} b) P_3 and K_{n-2} c)C_4 d)P_4 e)$K_{2,3}$ and K_{n-2} f)P_3 and K_{n-1} g)K_{n-2} and C_3.

Proof :

If G is the composition of any of the above then the result is obvious. Conversely suppose $\gamma^{fs}(G) + \chi(G) = 2n - 3$, then there are four cases here.

Case (i)$\gamma^{fs}(G) = n$ &$\chi(G) = n - 3$ and *Case (ii)*$\gamma^{fs}(G) = n - 3$ &$\chi(G) = n$ are not possible because $\gamma^{fs}(G) = n \Leftrightarrow G = K_n^f$ and $\chi(G) = n \Leftrightarrow G = K_n^f$

We shall prove the result for the remaining 2 cases.

Case (iii)$\gamma^{fs}(G) = n - 1$ & $\chi(G) = n - 2$

Suppose G has a (n-2)-clique $U = \{u_1, u_2, \dots, u_{n-2}\}$, let the remaining two nodes be $S = \{u, v\}$. There are two subcases

Subcase (i)$\langle S \rangle = K_2$

Suppose either u or v is a pendant node then the other node has ‘r’ neighbors in U(say). Then the end node and the nodes in U which are non-adjacent to the nodes of S are extreme nodes. It is known that all the extreme nodes must belong to any Steiner set. Hence these nodes forms a minimum Steiner dominating set which has

$n - r + 1 = n - (r - 1)$nodes. But $\gamma^{fs}(G) = n - 1$. Thus $r - 1 = 1$. So $r = 2$. Thus G consists of a $\boldsymbol{K_{1,3}}$ component and a $\boldsymbol{K_{n-2}}$ clique as in fig 1.

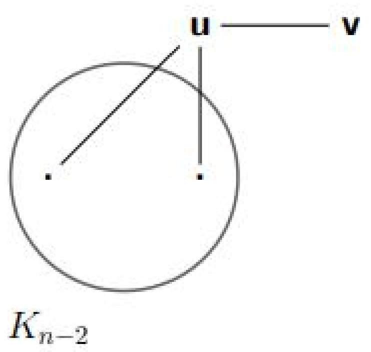

Fig 1

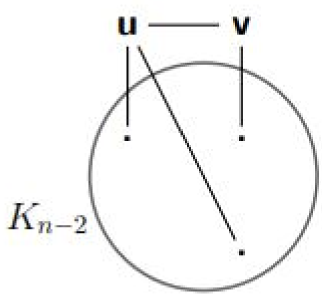

Fig 2

If both the nodes u and v are adjacent to atleast one node in U, then let they be adjacent to some ‘r’ nodes jointly(counting each node exactly once).

These two nodes along with the extreme nodes in U form a minimum steiner dominating set with $n - r + 2 = n - (r - 2)$ nodes. Hence $r - 2 = 1 \Rightarrow r = 3$. If ‘u’ is adjacent to exactly one node then ‘v’ must be adjacent to two nodes in U. Here G has two $\boldsymbol{P_3}$ components and $\boldsymbol{K_{n-2}}$as in fig 2. We shall consider the other subcase.

Subcase (ii)$\langle S \rangle \neq K_2$

In this case since G is connected, both u and v must have neighbors in U. Suppose both u and v have s e ‘r’ common neighbors in S, then S along with the extreme nodes in U forms a minimum steiner dominating set and has $n - r + 2 = n - (r - 2)$ nodes. So $r = 3$. If ‘u’ is adjacent to 2 nodes and ‘v’ is adjacent to 1 node then G has components $\boldsymbol{P_3}$,$\boldsymbol{P_2}$and$\boldsymbol{K_{n-2}}$in fig 3. If both ‘u’ and ‘v’ are adjacent to ‘3’ common nodes in U, then the components are $\boldsymbol{K_{2,3}}$and$\boldsymbol{K_{n-2}}$ in fig 4.

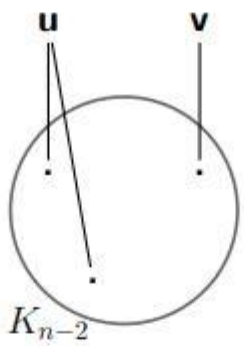

Fig 3

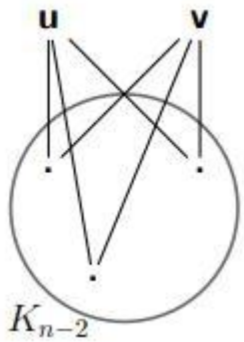

Fig 4

Let us assume that G has no (n-2)-clique. We have the result $\chi(G) \leq \Delta(G) + 1$. Hence $\Delta(G) \geq \chi(G) - 1 = n - 3$. So $\Delta(G) \geq n - 3$. Let ‘s’ be the node with maximum degree and hence adjacent to atleast $n - 3$ nodes.

Suppose ‘s’ is adjacent to exactly $n - 3$ nodes, then let S be the neighborhood of ‘s’ and x, y be the nodes which are not in S. If S is independent, then the nodes of S can be assigned ‘1’color. Also ‘s’ is non-adjacent to x,y and hence are assigned the color of ‘s’. Hence G can be assigned ‘2’ colors which is the minimum 2-coloring.

Hence $\chi(G) = n - 2 = 2$. So $n = 4$. ‘s’ is adjacent to ‘2’ nodes. Also ‘x’ must have atleast one neighbor in S. If ‘x’ is adjacent to one node in S then G is a **4-path**. If it is adjacent to two nodes in S, then G is a **4-cycle**. If S is not independent then it can be assigned ‘r’ colorswhere $2 \leq r \leq n - 3$. If $r = n - 3$, then $\langle S \rangle$ contains a (n-3)-clique, then including the node ‘s’ we acquire a (n-2) –clique which is contradiction.

Thus $r < n - 3$. Now x and y can be assigned the color of ‘s’.

Hence $\chi(G) = r + 1 = n - 2$. So $r = n - 3$ which is impossible.

Now suppose ‘s’ is adjacent to exactly $n - 2$ nodes then ‘s’ is non-adjacent to a node ‘x’ and let S be the neighborhood of ‘s’. If S is independent then as in the previous argument, G can be assigned 2 colors. Hence $\chi(G) = n - 2 = 2$. Thus $n = 4$. If not then if ‘r’ minimum colors are assigned to the nodes in S, then a total of $r + 1$ colors can be assigned to G which is again not possible. Therefore G must be isomorphic to either **4-path** or **4-cycle**.

Suppose 's' is adjacent to exactly $n-1$ nodes, then the neighborhood of 's', S has $n-1$ nodes. If S is independent, then G can be assigned a minimum of 2-coloring.
So $n=4$. On the otherhand if S is not independent, then since $\gamma^{fs}(G)=n-1$ G must have a cutnode of degree $n-1$. Let 'u' be the cutnode and U' its open neighborhood. If U' is independent, then G has 2-coloring. So $n-2=2 \Longrightarrow n=4$. Hence G is a **claw**or $\boldsymbol{K_{1,3}}$. If U' is not independent, assuming that U' has r-coloring.
Therefore $\chi(G)=r+1=n-2 \Longrightarrow r=n-3$. Now $\langle U' \rangle$ has a $n-3$ clique which leads to $n-2$ clique including the node 'u' which is a contradiction.
Case (iv) $\gamma^{fs}(G)=n-2$ & $\chi(G)=n-1$
Suppose G has a (n-1)-clique say $U=\{u_1,u_2,\dots,u_{n-1}\}$. Let 'u' be the remaining node. Since G is connected 'u' must be adjacent to atleast one node in U. Let 'u' be adjacent to some 'r' nodes in U. If $r=n-1$, then the graph is a complete graph with 'n' nodes. Hence by theorem (i), $\gamma^{fs}(G)=n$ which is a contradiction. Therefore $r<n-1$. Now the node 'u' and the nodes which are non-adjacent to 'u' are extreme nodes.
These extreme nodes form a minimum steiner dominating set which contains
$n-r-1+1=n-r$nodes. Hence $\gamma^{fs}(G)=n-r$. So $r=2$. Thus 'u' must be adjacent to exactly two nodes in U. Thus G can be decomposed into $\boldsymbol{P_3}$ and $\boldsymbol{K_{n-1}}$ as in fig 5.

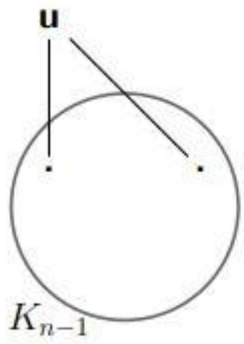

Fig 5

Now assume that G has no (n-1)-clique. It is known that $\chi(G) \le \Delta(G)+1$. Hence $\Delta(G) \ge \chi(G)-1=n-2$. So $\Delta(G) \ge n-3$. Let 's' be the node with maximum degree and hence adjacent to atleast $n-3$ nodes.

Suppose 's' is adjacent to exactly $n-3$ nodes, then let S be the neighborhood of 's' and x, y be the nodes which are not in S. If S is independent, then the nodes of S can be assigned '1'color. Also 's' is non-adjacent to x,y and hence are assigned the color of 's'. Hence G can be assigned '2' colors which is the minimum 2-coloring.

Hence $\chi(G) = n - 1 = 2$. So $n = 3$. 's' is adjacent to no node and hence this case is not possible. If S is not independent, then S can be assigned a minimum of r-coloring. Suppose x and y are non adjacent, then both must have neighbors in S. There is a minimum $r + 1$ coloring and hence $\chi(G) = n - 1 = r + 1$. So $r = n - 2$. There must be a $n - 2$-clique. Including the node's' we get a $n - 1$-clique which is a contradiction. If x and y are adjacent, either one of the nodes of x,y or both must have neighbors in S(say 't' disjoint neighbors), then there is a minimum of $r + 2$ coloring which gives $r = n - 3$ and hence there is a (n-3)-clique. Along with the node 's' there forms a (n-2)-clique. All these n-2 nodes except the 't'neighbors of x,y are extreme nodes. Hence there are n-2-t extreme nodes. These extreme nodes and x.y forms a minimum steiner dominating set. Thus $\gamma^{fs}(G) = n - 2 - t + 2 = n - t$. So $t = 2$. If both x, y are adjacent to exactly two nodes in S, then the components of G are $\boldsymbol{K_{n-2}}$ and 2- $\boldsymbol{C_3}$ components. If any one of the nodes x,y is adjacent to two nodes, then components are $\boldsymbol{K_{n-2}}$ and $\boldsymbol{K_{1,3}}$. If x,y each are adjacent to exactly one node in S, then the components are $\boldsymbol{K_{n-2}}$and$\boldsymbol{P_3}$.

If 's' is adjacent to exactly $n - 2$ nodes, then let the remaining node be 'x'. Here S has $n - 2$ nodes. If S is independent then G has minimum 2-coloring which leads to $n = 3$. But this is not possible. Hence S must be dependent and assume that S have minimum 'r' coloring and hence G has a minimum r+1 coloring.

Hence $\chi(G) = n - 1 = r + 1 \Rightarrow r = n - 2$. Hence there is a (n-2)-clique which along with the node 's' gives a (n-1)-clique and this is a contradiction.

If 's' is adjacent to (n-1) nodes, then the neighborhood of 's' has $n - 1$ nodes. If S is independent, then $\chi(G) = n - 1 = 2. \Rightarrow n = 3$ which is not the case. Suppose S is not independent assume that there are minimum r-colors assigned to the nodes of S. Then there is a r+1 coloring for G. Hence $\chi(G) = n - 1 = r + 1$. So $r = n - 2$. There must be a (n-2)-clique which in addition with the node 's' give rise to a (n-1)-clique which is a contradiction. Therefore the result holds.

3. Conclusion

The fuzzy Steiner domination have application background in various communication networks like telecommunication networks, radio stations message transmitting networks, channel networks, radio stations, transports, wireless mobile Ad-hoc networks.etc. In this article the authors characterized the fuzzy graphs with crisp nodes using steiner dominating number and chromatic number.

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