Radio Geometric Mean Graceful Labeling on Degree Splitting of Cycle Related Graphs

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Abstract
A Radio geometric mean graceful labeling of a connected graph G is a bijection μ from the vertex set V(G) to {1,2,3,...,|V(G)|} such that for any two distinct vertices u and v of G, d(u,v) + \left\lfloor \sqrt[\mu(u)\mu(v)] \right\rfloor \geq 1 + diam(G). A graph which admits radio geometric mean graceful labeling is called radio geometric mean graceful graph. In this paper, we introduced the radio geometric mean graceful labeling on degree splitting of graphs.

Keywords: labeling, radio geometric mean graceful labeling, degree splitting graph.

I. Introduction

Graphs described here is simple, undirected and connected graphs. Let V(G) and E(G) denotes the vertex set and edge set of a graph G respectively. A graph labeling is an assignment of integers to the vertices or edges both based on certain conditions. The concept of radio labeling was introduced by Chartrand et al[1] in 2001. S. Somasundaram and R.Ponraj introduced the notion of mean labeling of graphs [4]. R. Ponraj et al. introduced the concept of Radio mean labeling of graphs [5]. The notion of degree splitting of graphs was introduced by R.Ponraj and S. Somasundaram [7].C. David Raj, A. Subramanian and
K. Sunitha determined the radio mean labeling of double triangular snake graph and quadrilateral snake graph along with some path and cycle related graphs [6,9]. C. David Raj, M. Deva Saroja and Brindha Mary V.T introduced the radio mean labeling on splitting and degree splitting graphs [11]. C. David Raj and Brindha Mary.V.T determined radio mean graceful labeling on degree splitting of cycle related graphs [8] and even radio mean graceful labeling on degree splitting of snake related graphs [10]. S. Somasundaram, P.Vidhyarani and R. Ponraj introduced the concept of geometric mean labeling of graphs [12]. The concept of geometric mean labeling on degree splitting graphs was found by S. Somasundaram et al. [13]. Radio geometric graceful labeling was introduced by K.N. Meera [14]. We refer Gallian for more comprehensive survey [2]. We follow Harary [3] for some standard words, expressions and symbols. The notations DS(G) is the degree splitting of G, 

\[
d(u, v) = \text{distance between the vertices } u \text{ and } v \text{ and } \text{diam}(G) = \text{diameter of } G.
\]

**Definition 1:** A walk of a graph is an alternating sequence of vertices and edges \(v_0x_1, v_1x_2, ..., v_{n-1}, x_nv_n\) beginning and ending with vertices such that each edge \(x_i\) incident with \(v_{i-1}\) and \(v_i\). A walk with distinct vertices is called path. A path with \(n\) vertices is denoted by \(P_n\).

**Definition 2:** The distance \(d(u,v)\) from a vertex \(u\) to a vertex \(v\) in a connected graph \(G\) is the minimum of the length of the \(u-v\) paths in \(G\).

**Definition 3:** The eccentricity, \(e(V)\), of a vertex \(v\) in a connected graph \(G\) is the distance between \(v\) and a vertex farthest from \(v\) in \(G\).

**Definition 4:** The diameter, \(\text{diam}(G)\), of \(G\) is the greatest eccentricity among the vertices of \(G\).

**Definition 5:** The sum of two graphs \(G\) and \(H\), denoted by \(G+H\), is the graph obtained by taking disjoint copies of \(G\) and \(H\) and then adding every edge \(xy\), where \(x\) is a vertex in \(G\) and \(y\) is a vertex in \(H\).

**Definition 6:** A closed path is called a cycle of \(G\). A cycle on \(n\) vertices is denoted by \(C_n\).

**Definition 7:** An alternate triangular cycle \(A(C_{2n})\) is obtained from an even cycle \(C_{2n}\) with vertex set \(\{x_1, y_1, x_2, y_2, ..., x_n, y_n\}\) by joining \(x_i\) and \(y_i\) to a new vertex \(z_i\), \(1 \leq i \leq n\) i.e. every alternate edge of a cycle is replaced by \(C_3\).

**Definition 8:** Let \(G=(V, E)\) be a graph with \(V = S_1 \cup S_2 \cup ... \cup S_i \cup T\), where each \(S_i\) is a set of vertices having at least two vertices and having the same degree and \(T= V - \cup S_i\). The degree
splitting graph of $G$ denoted by $DS(G)$ is obtained from $G$ by adding vertices $w_1, w_2, ..., w_t$ and joining $w_i$ to each vertex of $S_i (1 \leq i \leq t)$.

**Definition 9:** A Radio geometric mean graceful labeling of a connected graph $G$ is a bijection $\mu$ from the vertex set $V(G)$ to \{1,2,3,...,$|V(G)|$\} such that for any two distinct vertices $u$ and $v$ of $G$, $d(u,v) + \lceil \sqrt{\mu(u)\mu(v)} \rceil \geq 1 + diam(G)$. A graph which admits radio geometric mean graceful labeling is called radio geometric mean graceful graph.

### II. Main Result

**Theorem 2.1:** $DS(C_n), n \geq 3$ is a radio geometric mean graceful graph.

**Proof:**

Let $x_1, x_2, x_3, ..., x_n$ be the vertices of cycle $C_n$. Introduce a new vertex $x$ and join it with the vertices of $C_n$ to obtain $DS(C_n)$ whose vertex set is $V = \{x_i / 1 \leq i \leq n\} \cup \{x\}$.

Clearly the $diam(DS(C_n)) = \begin{cases} 1 & \text{if } n = 3 \\ 2 & \text{if } n > 3 \end{cases}$.

Define a bijection $\mu: V(DS(C_n)) \rightarrow \{1,2,3,\ldots, |V(DS(C_n))|\}$ by

$\mu(x_i) = i; \ 1 \leq i \leq n$; $\mu(x) = n + 1$.

To check the radio geometric mean graceful condition for $\mu$.

**Case 1:** $n=3$

Subcase (i): Examine the pair $(x_i, x_j), 1 \leq i \leq n - 1, i + 1 \leq j \leq n$;

$d(x_i, x_j) + \lceil \sqrt{\mu(x_i)\mu(x_j)} \rceil \geq 1 + \lceil \sqrt{(i)(j)} \rceil \geq 2 = 1 + diam(DS(C_n))$.

Subcase (ii): Examine the pair $(x_i, x), 1 \leq i \leq n$;

$d(x_i, x) + \lceil \sqrt{\mu(x_i)\mu(x)} \rceil \geq 1 + \lceil \sqrt{i(n + 1)} \rceil \geq 2$.

**Case 2:** $n > 3$

Subcase (i): Examine the pair $(x_i, x_j), 1 \leq i \leq n - 1, i + 1 \leq j \leq n$;

$d(x_i, x_j) + \lceil \sqrt{\mu(x_i)\mu(x_j)} \rceil = 1 + \lceil \sqrt{(i)(j)} \rceil \geq 3 = 1 + diam(DS(C_n))$. 

Vol. 70 No. 2 (2021)  
http://philstat.org.ph

657
Subcase (ii): Examine the pair \((x_i, x), 1 \leq i \leq n;\)

\[
d(x_i, x) + \left\lfloor \sqrt{\mu(x_i)\mu(x)} \right\rfloor = 1 + \left\lfloor \sqrt{(i)(n+1)} \right\rfloor \geq 3.
\]

Thus all the pair of vertices satisfies the radio geometric mean graceful condition.

Hence \(DS(C_n)\) is a radio geometric mean graceful graph.

**Example 2.2:**

![Radio Geometric mean graceful labeling of DS(C₆).](image)

**Theorem 2.3:** \(DS(C_n + K_1), n \geq 3\) is a radio geometric mean graceful graph.

**Proof:**

Let \(x_1, x_2, x_3, ..., x_n\) be the vertices of cycle \(C_n\). Let \(x\) be the vertex of \(K_1\). Joining each vertex of the cycle \(C_n\) with the vertex \(x\). The resulting graph is \(C_n + K_1\). Introduce a new vertex \(y\) and join it with the vertices of \(C_n + K_1\) graph of degree three. The new graph so obtained is \(DS(C_n + K_1)\) whose vertex set is \(V=\{x, x_i / 1 \leq i \leq n\} \cup \{y\}\).

The \(diam(DS(C_n + K_1)) = \begin{cases} 1 & \text{if } n = 3 \\ 2 & \text{if } n \geq 4 \end{cases}\).

Define a bijection \(\mu: V(DS(C_n + K_1)) \rightarrow \{1, 2, 3, ..., |V(DS(C_n + K_1))|\}\) by

\[
\mu(x_i) = i, 1 \leq i \leq n;
\]

\[
\mu(x) = n + 1;
\]
\( \mu(y) = n + 2. \)

To check the radio geometric mean graceful condition for \( \mu \).

**Case 1: \( n = 3 \)**

Subcase (i): Examine the pair \((x_i, x_j), 1 \leq i \leq n - 1, i + 1 \leq j \leq n;\)

\[
d(x_i, x_j) + [\sqrt{\mu(x_i)\mu(x_j)}] \geq 1 + [\sqrt{(i)(j)}] \geq 2 = 1 + \text{diam}(DS(C_n + K_1)).
\]

Subcase (ii): Examine the pair \((x_i, x), 1 \leq i \leq n;\)

\[
d(x_i, x) + [\sqrt{\mu(x_i)\mu(x)}] \geq 1 + [\sqrt{(i)(n+1)}] \geq 2.
\]

Subcase (iii): Examine the pair \((x_i, y), 1 \leq i \leq n;\)

\[
d(x_i, y) + [\sqrt{\mu(x_i)\mu(y)}] \geq 1 + [\sqrt{(i)(n+2)}] \geq 2.
\]

Subcase (iv): Examine the pair \((x, y);\)

\[
d(x, y) + [\sqrt{\mu(x)\mu(y)}] \geq 2 + [\sqrt{(n+1)(n+2)}] \geq 2.
\]

**Case 2: \( n \geq 4 \)**

Subcase (i): Examine the pair \((x_i, x_j), 1 \leq i \leq n - 1, i + 1 \leq j \leq n;\)

\[
d(x_i, x_j) + [\sqrt{\mu(x_i)\mu(x_j)}] \geq 1 + [\sqrt{(i)(j)}] \geq 3 = 1 + \text{diam}(DS(C_n + K_1)).
\]

Subcase (ii): Examine the pair \((x_i, x), 1 \leq i \leq n;\)

\[
d(x_i, x) + [\sqrt{\mu(x_i)\mu(x)}] \geq 1 + [\sqrt{(i)(n+1)}] \geq 3.
\]

Subcase (iii): Examine the pair \((x_i, y), 1 \leq i \leq n;\)

\[
d(x_i, y) + [\sqrt{\mu(x_i)\mu(y)}] \geq 1 + [\sqrt{(i)(n+2)}] \geq 3.
\]

Subcase (iv): Examine the pair \((x, y);\)

\[
d(x, y) + [\sqrt{\mu(x)\mu(y)}] \geq 2 + [\sqrt{(n+1)(n+2)}] \geq 3.
\]

Thus all the pair of vertices satisfies the radio geometric mean graceful condition.
Hence $DS(C_n + K_1)$ is a radio geometric mean graceful graph.

Example 2.4:

Fig 2. Radio Geometric Mean graceful labeling of $DS(C_n + k_1)$.

Theorem 2.5: $DS(A(C_{2n})), n \geq 3$ is a radio geometric mean graceful graph.

Proof:

Let $x_1, x_2, x_3, \ldots, x_{2n}$ be the vertices of even cycle $C_{2n}$. Join $x_{2i-1} - x_2i$ with $y_i, 1 \leq i \leq n$ to form $A(C_{2n})$. Introduce two new vertices $x$ and $y$ and join them with the vertices of $A(C_{2n})$ of degree three and two respectively. Then the resultant graph is $DS(A(C_{2n}))$ whose vertex set is $V = \{x_i, 1 \leq i \leq 2n, y_i, 1 \leq i \leq n\} \cup \{x, y\}$.

Clearly the $diam(DS(A(C_{2n}))) = 3$.

Define a bijection $\mu: V(DS(A(C_{2n}))) \rightarrow \{1, 2, 3, \ldots, |V(DS(A(C_{2n})))|\}$ by

$\mu(x_i) = n + i + 1; 1 \leq i \leq 2n;$

$\mu(y_i) = i; 1 \leq i \leq n;$
\[ \mu(x) = n + 1; \]
\[ \mu(y) = 3n + 2. \]

Now we verify the radio geometric mean graceful condition for \( \mu \).

Case (i): Verify the pair \((x_i, x_j), 1 \leq i \leq n - 1, \ i + 1 \leq j \leq 2n;\)
\[
d(x_i, x_j) + \left[ \sqrt{\mu(x_i)\mu(x_j)} \right] \geq 1 + \left[ \sqrt{(n + i + 1)(n + j + 1)} \right] \geq 4
\]
\[ = 1 + \text{diam}(\text{DS}(A(C_{2n}))). \]

Case (ii): Verify the pair \((x_i, y_j), 1 \leq i \leq 2n, 1 \leq j \leq n;\)
\[
d(x_i, y_j) + \left[ \sqrt{\mu(x_i)\mu(y_j)} \right] \geq 1 + \left[ \sqrt{(n + i + 1)(j)} \right] \geq 4.
\]

Case (iii): Verify the pair \((x_i, x), 1 \leq i \leq n;\)
\[
d(x_i, x) + \left[ \sqrt{\mu(x_i)\mu(x)} \right] \geq 1 + \left[ \sqrt{(n + i + 1)(n + 1)} \right] \geq 4.
\]

Case (iv): Verify the pair \((x_i, y), 1 \leq i \leq 2n;\)
\[
d(x_i, y) + \left[ \sqrt{\mu(x_i)\mu(y)} \right] \geq 1 + \left[ \sqrt{(n + i + 1)(3n + 2)} \right] \geq 4.
\]

Case (v): Verify the pair \((y_i, y_j), 1 \leq i \leq 2n, \ i + 1 \leq j \leq n;\)
\[
d(y_i, y_j) + \left[ \sqrt{\mu(y_i)\mu(y_j)} \right] \geq 2 + \left[ \sqrt{(i)(i + 1)} \right] \geq 4.
\]

Case (vi): Verify the pair \((y_i, x), 1 \leq i \leq n;\)
\[
d(y_i, x) + \left[ \sqrt{\mu(y_i)\mu(x)} \right] \geq 2 + \left[ \sqrt{(i)(n + 1)} \right] \geq 4.
\]

Case (vii): Verify the pair \((y_i, y), 1 \leq i \leq n;\)
\[
d(y_i, y) + \left[ \sqrt{\mu(y_i)\mu(y)} \right] \geq 2 + \left[ \sqrt{(i)(3n + 2)} \right] \geq 4.
\]

Case (viii): Verify the pair \((x, y);\)
\[
d(x, y) + \left[ \sqrt{\mu(x)\mu(y)} \right] \geq 3 + \left[ \sqrt{(n + 1)(3n + 2)} \right] \geq 4.
\]

Thus all the pair of vertices satisfies the radio geometric mean graceful condition.

Hence \( \text{DS} A((C_{2n})) \) is a radio geometric mean graceful graph.
Example 2.6:

Fig 3. Radio Geometric Mean graceful Labeling of $DS(C_2(4))$.

3.Conclusion

The labeling of graphs is an interesting and vast research area which is very useful and it is extended in various topics by several people. Radio geometric mean graceful labeling is discussed in this paper and some of the results are obtained. More results will be done in the further research article. This paper will help the beginners in research.

REFERENCES


