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Transient Analysis of an M/M/1 Queue with Sleep Modes, Startup Time, Disaster, Repair and Its Application to Wireless Sensor Networks

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Abstract

An M/M/1 queueing system with vacation, threshold policy, start up time, waiting server, system disaster and repair is considered. This work is motivated by the energy management of Wireless Sensor Networks (WSNs). In this model, the system switches to various states namely busy state, idle state, shutdown state, inactive state, wake-up state and failure state. The disaster can occur at state state of the system except at failure state and subsequent repair is considered. The disaster can be considered as a jamming signal is WSNs. For this investigated system, a closed form of expression is obtained for both transient and steady- state probabilities. Furthermore, performance measures such as mean, variance and probability that the server is in various stages of power management modes are computed. Finally, graphical illustrations are made to understand the effect of the parameters on the performance of the system.

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1 Introduction

WSNs composed of several spatially distributed sensors called nodes that work together to monitor a specific purpose such as pollution levels, temperature, sound, vibration, pressure, and so on. Energy is regarded as a limited resource for a sensor node, particularly when deployed in a unfriendly region, and once depleted, it is extremely difficult to provide supplant energy. The main objective is to manage energy in such a waythat no node runs out of energy and the network remains operational indefinitely. Hence, it is essential for a sensor node to have an effective energy management policy for the lim- ited energy source, as well as to manage the application requirements in accordance with the available energy source. Energy management in WSNs can be considered as a set of rules for managing various energy supply mechanisms and then consuming the provided energy efficiently in a sensor node. In order to avoid energy deficiency in a network, an efficient power management between supply and load is required. Hence, a power saving scheme to extend the lifetime of WSNs has become an important research topic. Therehave been numerous analytical approaches developed on the subject. Excellent surveys on the PSMs are provided Machado and Tekinay[13] and Anastasi et.al. [1]. In this research paper, the PSM of WSN is analysed using an M/M/1 queueing with vacation,threshold policy and system disasters.

Vacation queueing systems are critical in analysing the power management of computer and communication systems. Several authors have applied various vacation policies to analyse different PSMs [See Dimitriou [7], Misra and Goswami [14], Sampath *et al*. [17] and Ren *et al*. [16] and references therein]. In these models, the server frequently switches between busy and vacation states. To overcome frequent switching, the system designer prefer N-policy scheme. In the N-policy scheme, the server is turned OFF or it stays idle when there is no data packet in the system and the server is turned ON when the system size reaches a predefined threshold value. The notion of an N-policy was first studied by Yadin and Naor [20]. Later, many researchers studied various queueing systems with N-policy in a different context [See Wang and Ke [19], Parthasarathy and Sudhesh [15] and references therein].

Several researchers have analysed the PSM of WSNs based on the vacation queueing system with N-policy. Jiang *et al*. [10] Jiang et al. studied the PSMs of WSNs using an M/M/1 queueing model and presented the steady state results. The simulated results of this research provide a potentially cost-effective approach to extending the lifetime ofthe sensor network. In this article the authors considered only two state namely busy and idle. Huang and Lee [8] studied the PSM of WSNs using M/G/1/K queueing model with an N-policy and presented the steady state results. In this research article, the authors considered three states namely busy, sleep and idle state. Blondia [4] presented the steady-state analysis of a WSNs using energy harvesting. In this paper the author considered two states namely transmit and vacation. Lee and Yang [11] analysed the PSM of WSNs using an Geo/G/1 queueing system with N-policy. Chen *et al*. [6] proposed an improved stochastic model for the WSNs which consists of three power-saving states namely shutdown, wake-up and inactive. The authors proposed that power saving can be achieved by decreasing the number of shutdown and wake-up processes. Jayarajan *et al*. [9] applied M/D/1 priority queueing model with threshold policy to study the PSM of sensor network and obtained the steady-state results. Ma *et al*. [12] studied the PSM of WSNs using an M/M/2 queueing model with threshold policy and presented the steady-state results using the matrix-geometry method.

In this article, we extend the research of Sudhesh and Shapique [18] by incorporating idle state, system disaster and repair. If the system encounter disaster, all the data packets are removed from the sensor node. Many researchers have performed extensive research on queues with system disasterin recent years due to their wide-ranging appli- cations in computer and communication systems [see [2], [3], [5] and references therein]. System disasters correspond to unreliable network connections in a WSNs, where data packets in sensor nodes are lost due to external attacks or jamming signals. Consider a WSN is deployed in an unfriendly environment like war zones for gathering intelligence in combat, tracking enemy troop movements, or measuring damage and casualties. An adversary may send out jamming signals in order to disrupt wireless communications. This jamming signals may interrupt with the radio frequencies of the sensor nodes. Packets transmitted over the frequency are discarded and may need to be re-transmitted.

From the literature survey, it is observed that most research on the mathematical modelling of WSNs has focussed mainly on the steady-state analysis of the system. Surprisingly the transient analysis of the system has not received as much attention. In many real-time applications, the system experiences a change and such changes can be measured by the transient analysis and not by steady-state analysis. The steady-state results cannot be used to determine the number of data packets waiting in the queue during the transmit state or in the vacation state at some time instant t. This motivates us to study the transient and steady-state analysis of the model.

The remainder of this paper is structured as follows. The model description is presented in Section 2. The transient probabilities of the system are presented in Section 3. The time dependent performance measures of the investigated system are presented in Section 4. The steady-state probabilities of the model is presented Section 5. The performance indices of the system in the steady-state are presented in section 6. The results obtained in Sections 3-6 are graphically illustrated in Section 7. The Conclusion and future work are presented in Section 8.

2 Description of the model

The model description of the WSN with start-up times and threshold policy subject to system disaster and repair is presented in this section.

1. The data frames join the queue according to a Poisson process with a rate of λ and it receive service with a rate of μ which follows an exponential distribution. The system switches to three types power saving modes namely shutdown state, inactive state and wake-up state. System disaster can occur at any state of the system.

2. After serving all the data frames in the busy state, the system switches to the shutdown state of duration T. The data frames may join the queue during this time, but the server will not resume service until the system accumulates k jobs.

3. At the end of shutdown period T, if the system reaches the threshold value k, then the system requires a start-up time which is exponentially distributed with the rate θ_1 to begin the service. To start-up, the system requires a change of state. The server switches from a shutdown state to an wake-up state which is exponentially distributed with a rate θ_2.

4. At the end of the shutdown period, if the system size is less than k, then the server switches to an inactive mode with the rate θ_2 which follows an exponential distribution. Data frames can enter the system during the inactive period. At this epoch when the system size reaches the threshold value k, the system switches to the wake-up state.

5. If system incurs disaster, all the data frames are removed from the system and server switches to the failure state with rate ζ which follows Poisson process. A repair process starts forthwith and the repair time of the system is exponentially distributed with mean η^{-1}.

6. The switches to the idle state at the end of repair process. If no customer arrive during the idle period, the sever immediately switches to the shutdown state of rate θ_3 which follows exponential distribution.

Let $\{H_1(t)), t \geq 0\}$ represent the status of the system at any time t and let $H_2(t)$ denotes the number of data frames in the system at any time t.

$$H_1(t) = \begin{cases} 0, & \text{the server is in busy state} \\ 1, & \text{the server is in wake-up state} \\ 2, & \text{the server is in shutdown state} \\ 3, & \text{the server is in inactive state} \end{cases}$$

Then, $Y(t) = \{H_1(t), H_2(t), t \geq 0\}$ represents a continuous time Markov chain with state space

$$\begin{aligned} S = &\{(0,n) : n = 1,2,3,...\} \cup \{(1,n) : n = k, k+1, k+2, ...\} \\ &\times \cup \{(2,n) : n = 0,1,2,...\} \cup \big\{(3,n) : n = 0,1,2,...k-1.\big\} \end{aligned}$$

Let

$$\begin{aligned} P_{i,n}(t) &= P\big\{H_1(t) = i, H_2(t) = n\big\}, i = 0\,; n = 1,2,3..., \\ P_{i,n}(t) &= P\big\{H_1(t) = i, H_2(t) = n\big\}, i = 1\,; n = k, k+1, k+2..., \\ P_{i,n}(t) &= P\big\{H_1(t) = i, H_2(t) = n\big\}, i = 2\,; n = 0,1,2,..., \\ P_{i,n}(t) &= P\big\{H_1(t) = i, H_2(t) = n\big\}, i = 3\,; n = 0,1,2,...k-1. \end{aligned}$$

Then $P_{i,n}(t)$ satisfies the following forward Kolmogorov equation.

$$P_F'(t) = -\eta P_F(t) + \zeta(1 - P_F(t)), \tag{2.1}$$
$$P_{0,0}'(t) = -(\lambda + \theta_3 + \zeta) P_{0,0}(t) + \eta P_F(t), \tag{2.2}$$
$$P_{0,n}'(t) = -(\lambda + \mu + \zeta) P_{0,n}(t) + \lambda P_{0,n-1}(t) + \mu P_{0,n+1}(t), n = 1,2,3,...,k-1, \tag{2.3}$$
$$P_{0,n}'(t) = -(\lambda + \mu + \zeta) P_{0,n}(t) + \lambda P_{0,n-1}(t) + \mu P_{0,n+1}(t) + \theta_1 P_{1,n}(t), n = k, k+1, k+2, .. \tag{2.4}$$
$$P_{1,k}'(t) = -(\lambda + \theta_1 + \zeta) P_{1,k}(t) + \theta_2 P_{2,k}(t) + \lambda P_{3,k-1}(t), \tag{2.5}$$
$$P_{1,n}'(t) = -(\lambda + \theta_1 + \zeta) P_{1,n}(t) + \theta_2 P_{2,n}(t) + \lambda P_{1,n-1}(t), n = k+1, k+2, k+3..., \tag{2.6}$$

$$P_{2,0}'(t) = -(\lambda + \theta_2 + \zeta) P_{2,0}(t) + \mu P_{0,1}(t) + \theta_3 P_{0,0}(t), \tag{2.7}$$

$$P_{2,n}'(t) = -(\lambda + \theta_2 + \zeta) P_{2,n}(t) + \lambda P_{2,n-1}(t), n = 1, 2, 3, ..., \tag{2.8}$$

$$P_{3,0}'(t) = -(\lambda + \zeta) P_{3,0}(t) + \theta_2 P_{2,0}(t), \tag{2.9}$$

$$P_{3,n}'(t) = -(\lambda + \zeta) P_{3,n}(t) + \lambda P_{3,n-1}(t) + \theta_2 P_{2,n}(t), n = 1, 2, ..., k-1. \tag{2.10}$$

It is assumed that the server is on inactive state initially. Therefore,

$$P_{i,n}(0) = \begin{cases} 1, & i = 3, n = 0 \\ 0, & \text{elsewhere.} \end{cases}$$

3 Transient analysis

This section presents the time-dependent probabilities $P_{i,n}(t)$, $i = 0, 1, 2, 3$.

3.1 Evaluation of $P_{1,n}(t)$, $P_{2,n}(t)$ and $P_{3,n}(t)$

Let $\hat{P}_{i,n}(s)$ denote the Laplace transform of $P_{i,n}(t)$ for $i = 1, 2, 3; n = 0, 1, 2, ...$ Taking Laplace transform on equations (2.1) – (2.2) and (2.5) – (2.10), we get

$$s\hat{P}_F(s) = -\eta \hat{P}_F(s) + \zeta\left(\frac{1}{s} - \hat{P}_F(s)\right), \tag{3.1}$$

$$s\hat{P}_{0,0}(s) = -(\lambda + \theta_3 + \zeta) \hat{P}_{0,0}(s) + \eta \hat{P}_F(s), \tag{3.2}$$

$$s\hat{P}_{1,k}(s) = -(\lambda + \theta_1 + \zeta) \hat{P}_{1,k}(s) + \theta_2 \hat{P}_{2,k}(s) + \lambda \hat{P}_{3,k-1}(s), \tag{3.3}$$

$$s\hat{P}_{1,n}(s) = -(\lambda + \theta_1 + \zeta) \hat{P}_{1,n}(s) + \theta_2 \hat{P}_{2,n}(s) + \lambda \hat{P}_{1,n-1}(s), \tag{3.4}$$

$$s\hat{P}_{2,0}(s) = -(\lambda + \theta_2 + \zeta) \hat{P}_{2,0}(s) + \mu \hat{P}_{0,1}(s) + \theta_3 P_{0,0}(t), \tag{3.5}$$

$$s\hat{P}_{2,n}(s) = -(\lambda + \theta_2 + \zeta) \hat{P}_{2,n}(s) + \lambda \hat{P}_{2,n-1}(s), n = 1, 2, 3, ... \tag{3.6}$$

$$s\hat{P}_{3,0}(s) - 1 = -(\lambda + \zeta) \hat{P}_{3,0}(s) + \theta_2 \hat{P}_{2,0}(s) \tag{3.7}$$

$$s\hat{P}_{3,n}(s) = -(\lambda + \zeta) \hat{P}_{3,n}(s) + \lambda \hat{P}_{3,n-1}(s) + \theta_2 \hat{P}_{2,n}(s), n = 1, 2, ...k-1. \tag{3.8}$$

Let $\beta_1 = \lambda + \theta_1 + \zeta$, $\beta_2 = \lambda + \theta_2 + \zeta$, $\beta_3 = \lambda + \theta_3 + \zeta$, and $\beta = \lambda + \mu + \zeta$. Simplifying Equation (3.1), we get

$$\hat{P}_F(s) = \frac{\zeta}{(\eta+\zeta)}\left[\frac{1}{s} - \frac{1}{s+\eta+\zeta}\right] \tag{3.9}$$

Substituting Equation (3.9) in Equation (3.2), we obtain

$$\hat{P}_{0,0}(s) = \eta\zeta\hat{A}(s), \tag{3.10}$$

where

$$\hat{A}(s) = \left[\frac{1}{s(\eta+\zeta)\beta_3} + \frac{1}{(s+\beta_3)(\lambda+\theta_3-\eta)\beta_3} - \frac{1}{(s+\eta+\zeta)(\eta+\zeta)(\lambda+\theta_3-\eta)}\right].$$

Using equations (3.5) and (3.6), after some manipulation, we get

$$\hat{P}_{2,n}(s) = \frac{\lambda^n}{(s+\beta_2)^{n+1}}\left[\mu\hat{P}_{0,1}(s) + \theta_3\eta\zeta\hat{A}(s)\right], n = 0, 1, 2, ... \tag{3.11}$$

Using Equation (3.5) in Equation (3.7) and further using it in Equation (3.8), we get

$$\hat{P}_{3,n}(s) = \frac{\theta_2\lambda^n}{(s+\lambda+\zeta)(s+\beta_2)}\left[\left(\frac{1}{s+\lambda+\zeta}\right)^n + \sum_{i=1}^{n}\left(\frac{1}{s+\lambda+\zeta}\right)^{n-i}\left(\frac{1}{s+\beta_2}\right)^i\right]$$
$$\times\left[\mu\hat{P}_{0,1}(s) + \theta_3\eta\zeta\hat{A}(s)\right] + \frac{\lambda^n}{(s+\lambda+\zeta)^{n+1}}. \tag{3.12}$$

Using equations (3.9) and (3.10) in (3.4), we get

$$\hat{P}_{1,n}(s) = \frac{\lambda^n}{(s+\beta_1)^{n-k+1}(s+\lambda+\zeta)^k} + f_n(s)\left[\mu\hat{P}_{0,1}(s) + \theta_3\eta\zeta A(s)\right], n = k, k+1, k+2, k+3. \tag{3.13}$$

where

$$f_n(s) = \frac{\theta_2\lambda^n}{(s+\beta_1)(s+\beta_2)}\left[\sum_{i=k+1}^{n}\left(\frac{1}{s+\beta_1}\right)^{n-i}\left(\frac{1}{s+\beta_2}\right)^{i} + \left(\frac{1}{s+\beta_1}\right)^{n-k}\right.$$
$$\left.\times\left\{\frac{1}{(s+\beta_2)^k} + \left(\frac{1}{s+\lambda+\zeta}\right)^k + \sum_{j=1}^{k-1}\left(\frac{1}{s+\lambda+\zeta}\right)^{k-j}\left(\frac{1}{s+\beta_2}\right)^{j}\right\}\right].$$

Inversion on equations (3.9) – (3.13) respectively yields

$$P_F(t) = \frac{\zeta}{\eta+\zeta}\left[1-\exp\{-(\eta+\zeta)t\}\right], \tag{3.14}$$

$$P_{0,0}(t) = \eta\zeta A(t), \tag{3.15}$$

$$P_{2,n}(t) = \frac{\lambda^n t^n}{n!}\exp\{-(\beta_2)t\} * \left[\mu P_{0,1}(t) + \theta_3\eta\zeta A(t)\right], n = 0,1,2,..., \tag{3.16}$$

$$P_{3,n}(t) = \frac{\lambda^n t^n}{n!}\exp\{-(\lambda+\zeta)t\} + \theta_2\lambda^n\left[\frac{\exp\{-(\lambda+\zeta)t\}}{\beta_2-\lambda-\zeta} + \frac{\exp\{-\beta_2 t\}}{\lambda+\zeta-\beta_2}\right]$$
$$* \left[\frac{\exp\{-(\lambda+\zeta)t\}t^{n-1}}{(n-1)!} + \sum_{i=1}^{n}\frac{\exp\{-(\lambda+\zeta)t\}t^{n-i-1}}{(n-i-1)!} * \frac{\exp\{-\beta_2 t\}t^{i-1}}{(i-1)!}\right]$$
$$* \left[\mu P_{0,1}(t) + \theta_3\eta\zeta A(t)\right], n = 1,2,3,...k-1. \tag{3.17}$$

.

$$P_{1,n}(t) = \frac{\lambda^n t^{n-k}}{(n-k)!}\exp\{-\beta_1 t\} * \frac{t^{k-1}}{(k-1)!}\exp\{-(\lambda+\zeta)t\} + \frac{f_n(t)}{\theta_1}$$
$$* \left[\mu P_{0,1}(t) + \theta_3\eta\zeta A(t)\right], n = k+1, k+2, k+3, ..., \tag{3.18}$$

where

$$f_n(t) = \frac{\theta_2\lambda^n}{\beta_1-\beta_2}\left[\exp\{-\beta_1 t\} - \exp\{-\beta_2 t\}\right] * \left[\sum_{i=k+1}^{n}\frac{t^{n-i-1}}{(n-i-1)!}\exp\{-\beta_1 t\}\right.$$
$$* \frac{t^{i-1}}{(i-1)!}\exp\{-\beta_2 t\} + \frac{t^{n-k-1}}{(n-1)!}\exp\{-\beta_1 t\} * \left\{\frac{t^{k-1}}{(k-1)!}\exp\{-\beta_2 t\}\right.$$
$$+ \frac{t^{k-1}}{(k-1)!}\exp\{-(\lambda+\zeta)t\} + \sum_{j=1}^{k-1}\frac{t^{k-j-1}}{(k-j-1)!}\exp\{-(\lambda+\zeta)t\}$$
$$\left.\left.* \frac{t^{j-1}}{(j-1)!}\exp\{-\beta_2 t\}\right\}\right]$$

and

$$A(t) = \left[\frac{1}{\beta_3(\eta+\zeta)} + \frac{\exp\{-\beta_3 t\}}{\beta_3(\lambda+\theta_3-\eta)} - \frac{\exp\{-\beta_5 t\}}{\beta_5(\lambda+\theta_3-\eta)}\right].$$

Thus, we expressed $P_{1,n}(t)$, $P_{2,n}(t)$ and $P_{3,n}(t)$ in-terms of $P_{0,1}(t)$. The expression for $P_{0,1}(t)$ is given in Equation (3.29).

3.2 Evaluation of $P_{0,n}(t)$

The busy-state probability $P_{0,n}(t)$; $n = 1, 2, 3, ...$ is obtained using equations (2.3)-(2.4) by applying the generating function defined as follows.
Let

$$G(z,t) = \sum_{n=1}^{\infty} P_{0,n}(t) z^n, G(z,0) = 0.$$

Using equations (2.3)-(2.4), we get

$$\frac{\partial}{\partial t} G(z,t) = \left[-(\lambda+\mu+\zeta) + \frac{\mu}{z} + \lambda z\right] G(z,t) + \lambda z P_{0,0}(t) - \mu P_{0,1}(t) + \theta_1 \sum_{n=k}^{\infty} P_{1,n}(t) z^n$$

On solving,

$$G(z,t) = \int_0^t \left[\theta_1 \sum_{m=k}^{\infty} P_{1,m}(w) z^m - \mu P_{0,1}(w) + \lambda z P_{0,0}(w)\right] \exp\left\{\left(-\beta + \lambda z + \frac{\mu}{z}\right)(t-w)\right\} dw \tag{3.19}$$

where $\beta = \lambda + \mu + \zeta$. Let $\kappa = 2\sqrt{\lambda\mu}$ and $\nu = \sqrt{\lambda\mu^{-1}}$, then

$$\exp\left[\left(\lambda z + \frac{\mu}{z}\right)(t-w)\right] = \sum_{n=-\infty}^{\infty} (\nu z)^n I_n(\kappa(t-w)). \tag{3.20}$$

where $I_m(t)$ represents the modified Bessel function of the first kind of order m. Applying (3.20) in Equation (3.19) and equating the coefficient of z^n on both sides for $n = 1, 2, 3...$

$$P_{0,n}(t) = \int_0^t \left[\theta_1 \sum_{m=k}^{\infty} P_{1,m}(w) \nu^{n-m} I_{n-m}(.) - \mu P_{0,1}(w) \nu^n I_n(.) + \lambda P_{0,0}(w) \nu^{n-1} I_{n-1}(.)\right] \times \exp\{-\beta(t-w)\} dw. \tag{3.21}$$

Equating the coefficients of z^{-n} on both sides of Equation (3.19) for $n = 1, 2, 3...$ and using $I_{-n}(.) = I_n(.)$, we get

$$0 = \int_0^t \left[\theta_1 \sum_{m=k}^{\infty} P_{1,m}(w) \nu^{n-m} I_{n+m}(.) - \mu P_{0,1}(w) \nu^n I_n(.) + \lambda P_{0,0}(w) \nu^{n-1} I_{n+1}(.)\right] \times \exp\{-\beta(t-w)\} dw. \tag{3.22}$$

Multiplying ν^{2n} on both sides of Equation (3.22) and subtracting it from Equation (3.21) for $n = 1, 2, 3...$, we arrive

$$P_{0,n}(t) = \theta_1 \int_0^t \sum_{m=k}^{\infty} P_{1,m}(w) \exp\{-\beta(t-w)\} \nu^{n-m} \{I_{n-m}(.) - I_{n+m}(.)\} dw$$
$$+ \lambda \int_0^t P_{0,0}(w) \exp\{-\beta(t-w)\} \nu^{n-1} \{I_{n-1}(.) - I_{n+1}(.)\} dw \tag{3.23}$$

Taking Laplace transform on Equation (3.23), we obtain

$$\hat{P}_{0,n}(s) = \frac{\theta_1}{\sqrt{d^2-\kappa^2}} \sum_{m=k}^{\infty} \hat{P}_{1,m}(s) \nu^{n-m} \left\{\hat{\chi}(s)^{n-m} - \hat{\chi}(s)^{n+m}\right\} - \frac{2\lambda}{\kappa} \hat{P}_{0,0}(s) \nu^{n-1} \hat{\chi}(s)^n \tag{3.24}$$

where

$$\hat{\chi}(s) = \frac{d - \sqrt{d^2-\kappa^2}}{\kappa} \text{ and } d = s + \lambda + \mu.$$

Using equations (3.10) and (3.13) in Equation (3.24), we obtain

$$\hat{P}_{0,n}(s) = \frac{\theta_1}{\sqrt{d^2-\kappa^2}} \sum_{m=k}^{\infty} \nu^{n-m} \left\{\hat{\chi}(s)^{n-m} - \hat{\chi}(s)^{n+m}\right\}$$
$$\times \left[\frac{\lambda^m}{(s+\beta_1)^{m-k+1}(s+\lambda+\zeta)^k} + f_m(s) \left\{\mu \hat{P}_{0,1}(s) + \theta_3 \eta \zeta A(s)\right\}\right] - \frac{2\lambda}{\kappa} \eta \zeta A(s) \nu^{n-1} \hat{\chi}(s)^n$$

On inversion,

$$P_{0,n}(t) = \sum_{m=k}^{\infty} \nu^{n-m} \{I_{n-m}(\kappa t) - I_{n+m}(\kappa t)\} \exp\{-\beta t\}$$
$$* \left[\theta_1 \lambda^m \frac{\exp\{-\beta_1 t\} t^{m-k}}{(m-k)!} * \frac{t^{k-1} \exp\{-(\lambda+\zeta)t\}}{(k-1)!} + \left\{\mu P_{0,1}(t) + \theta_3 \eta \zeta A(t)\right\} * f_m(t)\right]$$
$$- \lambda \eta \zeta \nu^{n+1} A(t) * \{I_{n-1}(\kappa t) - I_{n+1}(\kappa t)\} \exp\{-\beta t\}. \tag{3.25}$$

3.3 Evaluation of $P_{0,1}(t)$

Setting $n = 1$ in Equation (3.23), we obtain

$$P_{0,1}(t) = 2\theta_1 \int_0^t \sum_{m=k}^{\infty} P_{1,m}(w) \exp\{-\beta(t-w)\} m\nu^{1-m} \frac{I_m \kappa(t-w)}{\kappa(t-w)} dw + 2\lambda \int_0^t P_{0,0}(w) \exp\{-\beta(t-w)\} \frac{I_1(\kappa(t-w))}{\kappa(t-w)} dw$$

On inversion

$$\hat{P}_{0,1}(s) = 2\theta_1 \sum_{m=k}^{\infty} \nu^{1-m} \hat{P}_{1,m}(s) \frac{\kappa^{m-1}}{\left(d+\sqrt{d^2-\kappa^2}\right)^m} + 2\lambda \hat{P}_{0,0}(s) \frac{1}{\left(d+\sqrt{d^2-\kappa^2}\right)}. \quad (3.26)$$

Substituting equations (3.10) and (3.13) in Equation (3.26) after some manipulation, we get

$$\hat{P}_{0,1}(s) = \left[2\theta_1 \sum_{m=k}^{\infty} \nu^{1-m} \frac{\kappa^{m-1}}{\left(d+\sqrt{d^2-\kappa^2}\right)^m} \left\{\frac{\lambda^m}{(s+\beta_1)^{m-k+1}(s+\lambda+\zeta)^k} + \theta_3\eta\zeta\hat{A}(s)\hat{f}(s)\right\} + 2\lambda\eta\zeta\hat{A}(s)\frac{1}{\left(d+\sqrt{d^2-\kappa^2}\right)}\right] \sum_{h=0}^{\infty} (\hat{g}(s))^h, \quad (3.27)$$

where

$$g(s) = 2\mu\theta_1 \hat{f}(s) \sum_{m=k}^{\infty} \nu^{1-m} \frac{\kappa^{m-1}}{\left(d+\sqrt{d^2-\kappa^2}\right)^m} \quad (3.28)$$

Inversion on (3.27) gives

$$P_{0,1}(t) = \left[\theta_1 \sum_{m=k}^{\infty} \nu^{1-m} \{I_{1-m}(\kappa t) - I_{1+m}(\kappa t)\} \exp\{-\beta t\} * \left\{\lambda^m \frac{\exp\{-\beta_1 t\} t^{m-k}}{(m-k)!} * \frac{\exp\{-(\lambda+\zeta)t\} t^{k-1}}{(k-1)!} + \theta_3\eta\zeta A(t) * f(t)\right\} + \lambda\eta\zeta A(t) * \{I_0(\kappa t) - I_2(\kappa t) \exp(-\beta t)\}\right] * \sum_{h=0}^{\infty} (g(t))^{*h}, \quad (3.29)$$

where

$$g(t) = \mu f(t) * \sum_{m=k}^{\infty} \nu^{1-m} [I_{1-m}(\kappa t) - I_{1+m}(\kappa t)] \exp\{-\beta t\}$$

and '$*h$' denotes h-fold convolution. Thus we have obtained an explicit expression for $P_{0,1}(t)$.

4 Performance measures

The mean and variance of the system size are presented in this section.

4.1 Mean

Let $\Omega(t)$ denote the expected system size at time t. For $t > 0$, we have

$$\Omega(t) = E\left[X\left(t\right)\right] = \sum_{n=1}^{\infty} nP_{0,n}\left(t\right) + \sum_{n=k}^{\infty} nP_{1,n}\left(t\right) + \sum_{n=1}^{\infty} nP_{2,n}\left(t\right) + \sum_{n=1}^{k-1} nP_{3,n}(t).$$

Then using Equations (2.1)-(2.9), we get

$$\Omega'\left(t\right) = \lambda\left(1 - P_F\left(t\right)\right) - \mu \sum_{n=1}^{\infty} P_{0,n}\left(t\right) - \zeta \left\{ \sum_{n=k}^{\infty} nP_{1,n}\left(t\right) + \sum_{n=1}^{\infty} nP_{2,n}\left(t\right) + \sum_{n=1}^{k-1} nP_{3,n}\left(t\right) \right\}$$

The above equation gives

$$\begin{aligned} \Omega\left(t\right) = \lambda t - \int_0^t P_F\left(y\right)dy - \mu \sum_{n=1}^{\infty} \int_0^t P_{0,n}\left(y\right)dy \\ - \zeta \int_0^t \left\{ \sum_{n=k}^{\infty} nP_{1,n}\left(y\right) + \sum_{n=1}^{\infty} nP_{2,n}\left(y\right) + \sum_{n=1}^{k-1} nP_{3,n}\left(y\right) \right\} dy \end{aligned}$$

4.2 Variance

Let $V(t)$ denote the variance system size at time t. For $t > 0$,

$$V(t) = E[X^2(t)] - (E[X(t)])^2,$$

where

$$E\left[X^2(t)\right] = \sum_{n=1}^{\infty} n^2 P_{0,n}(t) + \sum_{n=k}^{\infty} n^2 P_{1,n}(t) + \sum_{n=1}^{\infty} n^2 P_{2,n}(t) + \sum_{n=1}^{k-1} n^2 P_{3,n}(t).$$

Using Equations (2.1)-(2.9), we obtain

$$\frac{d}{dt} E\left[X^2(t)\right] = \lambda\left(1 - P_F(t)\right) + 2\lambda\Omega(t) - \mu \sum_{n=1}^{\infty} (2n-1) P_{b,n}(t) - \zeta\left\{\sum_{n=k}^{\infty} n^2 P_{1,n}(t) + \sum_{n=1}^{\infty} n^2 P_{2,n}(t) + \sum_{n=0}^{k-1} n^2 P_{3,n}(t)\right\}$$

Then

$$E\left(X^2(t)\right) = \lambda t - \int_0^t P_F(y) dy + 2\lambda \int_0^t \Omega(y)\, dy - \mu \sum_{n=1}^{\infty} (2n-1) \int_0^t P_{0,n}(y) dy - \zeta \int_0^t \left\{\sum_{n=k}^{\infty} n^2 P_{1,n}(y) + \sum_{n=1}^{\infty} n^2 P_{2,n}(y) + \sum_{n=1}^{k-1} n^2 P_{3,n}(y)\right\} dy.$$

4.3 Probability that the server is in power-saving modes

Let $P_{i,\bullet}(t)\,; i = 1, 2, 3$ denote the probability that the server is on wake-up state, shutdown state and inactive state respectively, then

$$\hat{P}_{1,\bullet}(s) = \sum_{n=k}^{\infty} \hat{P}_{1,n}(s),$$

$$\hat{P}_{2,\bullet}(s) = \sum_{n=0}^{\infty} \hat{P}_{2,n}(s),$$

$$\hat{P}_{3,\bullet}(s) = \sum_{n=0}^{k-1} \hat{P}_{3,n}(s).$$

Using Equations (3.10), (3.7) and (3.8) in the above expression and taking inversion, respectively yield

$$P_{1,\bullet}(t)=\frac{\lambda^k t^{k-1}\exp\{-(\lambda+\zeta)t\}*\exp\{-(\beta_1-\lambda)t\}}{(k-1)!}+\frac{\theta_2\{\exp(-\beta_2 t)-\exp(-\beta_1 t)\}}{\beta_1-\beta_2}$$
$$*\left[\frac{\lambda^{k+1}t^{k-1}\exp(-\beta_2 t)}{(k-1)!(\beta_2-\beta_1)}*(\delta'(t)+\beta_1\delta(t))*\exp\{-(\beta_1-\lambda)t\}-\exp\{-(\beta_2-\lambda)t\}\right.$$
$$*\left\{\frac{t^{k-1}\exp(-\beta_2 t)}{(k-1)!}+\frac{t^{k-1}\exp\{-(\lambda+\zeta)\}t}{(k-1)!}+\sum_{j=1}^{k-j}\frac{t^{k-j-1}\exp\{-(\lambda+\zeta)t\}}{(k-j-1)!}\right.$$
$$\left.\left.*\frac{t^{j-1}\exp(-\beta_2 t)}{(j-1)!}\right\}*\frac{\lambda^k t^{k-1}}{(k-1)!}\exp(-\beta_1 t)*(\delta(t)+\lambda\exp\{-(\beta_1-\lambda)t\})\right]$$
$$*\{\mu P_{0,1}(t)+\theta_3\eta\zeta A(t)\}$$

$$P_{2,\bullet}(t)=[\mu P_{0,1}(t)+\theta_3\eta\zeta A(t)]*\exp\{-(\beta_2-\lambda)t\},$$

4.4 Probability that the server is in busy state

Let $P_{b,\bullet}(t)$ denote the probability that the server is on busy state, then

$$\hat{P}_{b,\bullet}(s)=\sum_{n=1}^{\infty}\hat{P}_{b,n}(s).$$

Using Equation (3.19) and taking inversion, we get

$$P_{b,\bullet}(t)=\theta_1\sum_{n=1}^{\infty}\sum_{m=k}^{\infty}P_{1,m}(t)\nu^{n-m}*[I_{n-m}(\kappa t)-I_{n+m}(\kappa t)]\exp(-\beta t).$$

4.5 Probability that the server is either in busy state or wake-up state or shutdown state or inactive state

Let $P(t)$ denote the probability that the server is either in busy state or in wake-up state or in shutdown state or in inactive state, then

$$P(t)=\sum_{n=1}^{\infty}P_{0,n}(t)+\sum_{n=k}^{\infty}P_{1,n}(t)+\sum_{n=0}^{\infty}P_{2,n}(t)+\sum_{n=0}^{k-1}P_{3,k-1}(t).$$

$$P_{3,\bullet}(t) = \exp(-\zeta t) * \left\{\delta(t) - \frac{\lambda^k t^{k-1}}{(k-1)!}\exp\{-(\lambda+\zeta)t\}\right\} + [\exp\{-(\lambda+\zeta)t\} - \exp\{-\beta_2 t\}]$$
$$* \{\delta(t) + \exp(-\zeta t)\} * \left\{\delta(t) - \frac{\lambda^k t^{k-1}}{(k-1)!}\exp\{-(\lambda+\zeta)t\}\right\} + \left\{\frac{\delta'(t) + (\lambda+\zeta)\delta(t)}{\theta_2}\right\}$$
$$* \left[\lambda\exp(-\zeta t) * \left\{\delta(t) - \frac{\lambda^{k-1}t^{k-2}}{(k-2)!}\exp\{-(\lambda+\zeta)t\}\right\} - \lambda\exp(-\theta_2 t)\right.$$
$$\left. * \left\{\delta(t) - \frac{\lambda^{k-1}t^{k-2}}{(k-2)!}\exp\{-(\lambda+\theta_2)t\}\right\}\right] * \{\mu P_{0,1}(t) + \theta_3\eta\zeta A(t)\}.$$

where $\delta(t)$ represents Dirac delta function.

5 Steady-state probabilities

The system size probabilities of the wake-up state, the shutdown state and the inactive state are presented in this section.
Let $\{\pi_{k,n}; k = v_1, v_2, c, b, n \geq 0\}$ represent the steady-state probability distributions for the model considered. Applying $\lim_{s\to 0} s\hat{P}_{i,n} = \pi_{i,n}$ on equations (2.1)-(2.10), we get

$$\eta\pi_F = \zeta(1-\pi_F), \tag{5.1}$$
$$(\lambda+\theta_3+\zeta)\pi_{0,0} = \eta\pi_F, \tag{5.2}$$
$$(\lambda+\mu+\zeta)\pi_{0,n} = \lambda\pi_{0,n-1} + \mu\pi_{0,n+1}, n = 1,2,3,...,k\text{–}1, \tag{5.3}$$
$$(\lambda+\mu+\zeta)\pi_{0,n} = \lambda\pi_{0,n-1} + \mu\pi_{0,n+1} + \theta_1\pi_{1,n}, n = k, k+1, k+2, ..., \tag{5.4}$$
$$(\lambda+\theta_1+\zeta)\pi_{1,k} = \theta_2\pi_{2,k} + \lambda\pi_{3,k-1}, \tag{5.5}$$
$$(\lambda+\theta_1+\zeta)\pi_{1,n} = \theta_2\pi_{2,n} + \lambda\pi_{1,n-1}, n = k+1, k+2, k+3...., \tag{5.6}$$
$$(\lambda+\theta_2+\zeta)\pi_{2,0} = \mu\pi_{0,1} + \theta_3\pi_{0,0}, \tag{5.7}$$
$$(\lambda+\theta_2+\zeta)\pi_{2,n} = \lambda\pi_{2,n-1}, n = 1,2,3,..., \tag{5.8}$$
$$(\lambda+\zeta)\pi_{3,0} = \theta_2\pi_{2,0}, \tag{5.9}$$
$$(\lambda+\zeta)\pi_{3,n} = \lambda\pi_{3,n-1} + \theta_2\pi_{2,n}, n = 1,2,...,k-1 \tag{5.10}$$

From equations (5.1) and (5.2), we get

$$\pi_F = \frac{\zeta}{(\eta+\zeta)}, \tag{5.11}$$

$$\pi_{0,0} = \frac{\eta\zeta}{(\lambda+\theta_3+\zeta)(\eta+\zeta)}. \tag{5.12}$$

Using equations (5.7) and (5.8), we obtain

$$\pi_{2,n} = \frac{\lambda^n}{{\beta_2}^{n+1}}\left[\mu\pi_{0,1} + \frac{\eta\zeta\theta_3}{\beta_3(\eta+\zeta)}\right], n = 1,2,3,.... \tag{5.13}$$

Using equations (5.9) and (5.10), we get

$$\pi_{3,n} = \frac{\theta_2}{(\lambda+\zeta)\beta_2}\left[\left(\frac{\lambda}{\lambda+\zeta}\right)^n + \sum_{i=1}^{n}\left(\frac{\lambda}{\lambda+\zeta}\right)^{n-i}\left(\frac{\lambda}{\beta_2}\right)^i\right]\left[\mu\pi_{0,1} + \frac{\theta_3\eta\zeta}{\beta_3(\eta+\zeta)}\right], n = 0,1,2,...,k-1. \tag{5.14}$$

Applying the results (5.13) and (5.14) in Equation (5.6), we get

$$\pi_{1,n} = f_n(0)\mu\pi_{0,1}, n = k+1, k+2, k+3, ... \tag{5.15}$$

where

$$f_n(0) = \frac{\theta_2\lambda^n}{\beta_1\beta_2}\left[\sum_{j=k+1}^{n}\left(\frac{1}{\beta_1}\right)^{n-j}\left(\frac{1}{\beta_2}\right)^j + \left(\frac{1}{\beta_1}\right)^{n-k}\left\{\frac{1}{{\beta_2}^k} + \left(\frac{1}{\lambda+\zeta}\right)^k + \sum_{i=1}^{k-1}\left(\frac{1}{\lambda+\zeta}\right)^{k-i}\left(\frac{1}{\beta_2}\right)^i\right\}\right].$$

The probabilities $\pi_{i,n}, i = 1,2,3$ are expressed in-terms of $\pi_{0,1}$. To get an explicit expression for $\pi_{0,1}$, we define a generating function as follows:

$$G_0(z) = \sum_{n=1}^{\infty}\pi_{0,n}z^n, \tag{5.16}$$

Using equations (5.3) and (5.4), we obtain

$$G_0(z)\left\{\lambda z^2 - (\lambda+\mu+\zeta)z + \mu\right\} = \mu z\pi_{0,1} - \lambda z^2\pi_{0,0} - \theta_1 z\sum_{n=k}^{\infty}\pi_{1,n}z^n. \tag{5.18}$$

Applying the results (5.12) and (5.15) in the above expression, we get

$$G_0(z)(z-r_1)(z-r_2)=\mu z\pi_{0,1}-\frac{\lambda\eta\zeta}{(\lambda+\theta_3+\zeta)(\eta+\zeta)}z^2-\theta_1\mu z\pi_{0,1}\sum_{n=k}^{\infty}f_n(0)z^n. \quad (5.19)$$

where

$$r_1=\frac{\lambda+\mu+\zeta+\sqrt{(\lambda+\mu+\zeta)^2-4\lambda\mu}}{2\lambda},$$

$$r_1=\frac{\lambda+\mu+\zeta-\sqrt{(\lambda+\mu+\zeta)^2-4\lambda\mu}}{2\lambda}.$$

Setting $z=r_2$, we get

$$\pi_{0,1}=\frac{\lambda\eta\zeta r_2}{\mu(\lambda+\theta_3+\zeta)(\eta+\zeta)}\left[1-\theta_1\sum_{n=k}^{\infty}{r_2}^n f_n(0)\right]^{-1},\left|\theta_1\sum_{n=k}^{\infty}{r_2}^n f_n(0)\right|<1.$$

Applying the results in (5.11) – (5.15) Equation (5.1), we get

$$\sum_{n=1}^{\infty}\pi_{0,n}=\frac{\eta}{\eta+\zeta}-\frac{\eta\zeta}{(\lambda+\theta_3+\zeta)(\eta+\zeta)}-\sum_{n=k}^{\infty}f_n(0)\mu\pi_{0,1}-\left[\frac{\beta_2}{\beta_2-\lambda}+\frac{\theta_2}{(\lambda+\zeta)\beta_2}\right.$$
$$\left.\times\sum_{n=0}^{k-1}\left\{\left(\frac{\lambda}{\lambda+\zeta}\right)^n+\sum_{i=1}^{n}\left(\frac{\lambda}{\lambda+\zeta}\right)^{n-i}\left(\frac{\lambda}{\beta_2}\right)^i\right\}\left\{\mu\pi_{0,1}+\frac{\eta\zeta\theta_3}{\beta_3(\eta+\zeta)}\right\}\right]. \quad (5.20)$$

5.1 Probability that the server is on wake-up state, shutdown state and inactive state

We define generating functions as follows.

$$G_1(z)=\sum_{n=k}^{\infty}\pi_{1,n}z^n,$$

$$G_2(z)=\sum_{n=0}^{\infty}\pi_{2,n}z^n,$$

$$G_3(z)=\sum_{n=0}^{k-1}\pi_{3,n}z^n.$$

Multiplying suitable powers of z on equations (5.5) and (5.6), and summing, we obtain

$$G_1(z)=\frac{\theta_2\sum\limits_{n=k}^{\infty}\pi_{2,n}z^n+\lambda z^k\pi_{3,k-1}}{(\lambda+\theta_1+\zeta)-\lambda z}. \tag{5.21}$$

Applying the results (5.13) and (5.14) in Equation (5.21) and setting $z=1$, we obtain

$$G_1(1)=\left[\frac{\theta_2}{\beta_2-\lambda}\left(\frac{\lambda}{\beta_2}\right)^k+\frac{\lambda\theta_2}{(\lambda+\zeta)\,\beta_2}\left\{\left(\frac{\lambda}{\lambda+\zeta}\right)^{k-1}+\sum_{i=1}^{k-1}\left(\frac{\lambda}{\lambda+\zeta}\right)^{k-1-i}\left(\frac{\lambda}{\beta_2}\right)^i\right\}\right]$$
$$\frac{1}{(\theta_1+\zeta)}\left\{\mu\pi_{0,1}+\frac{\eta\zeta\theta_3}{\beta_3\,(\eta+\zeta)}\right\} \tag{5.22}$$

Multiplying suitable powers of z on equations (5.7) and (5.8) and summing, we obtain

$$G_2(z)=\frac{\mu\pi_{0,1}+\theta_3\pi_{0,0}}{(\lambda+\theta_2+\zeta-\lambda z)}. \tag{5.23}$$

Applying the results (5.12) in Equation (5.23) and setting $z=1$, we obtain

$$G_2(1)=\frac{1}{\theta_2+\zeta}\left[\mu\pi_{0,1}+\frac{\theta_3\eta\zeta}{(\lambda+\theta_3+\zeta)\,(\eta+\zeta)}\right]. \tag{5.24}$$

Multiplying suitable powers of z on equations (5.9) and (5.10), and summing, we obtain

$$G_3(z)=\frac{\theta_2\sum\limits_{n=0}^{k-1}\pi_{2,n}z^n}{(\lambda+\zeta-\lambda z)}. \tag{5.25}$$

Using the result (5.13) in Equation (5.25) and setting $z=1$, we get

$$G_3(1)=\frac{\theta_2}{\zeta\,(\beta_2-\lambda)}\left\{1-\left(\frac{\lambda}{\beta_2}\right)^{k-1}\right\}\left\{\mu\pi_{0,1}+\frac{\eta\zeta\theta_3}{\beta_3\,(\eta+\zeta)}\right\}. \tag{5.26}$$

Equations (5.22),(5.24) and (5.26) denotes the probability that the server is on wake-up state, shutdown state and inactive state respectively

6 Performance measures

This section presents expected system size in the steady-state. Let $E[N_0]$, $E[N_1]$, $E[N_2]$ and $E[N_3]$ be the mean number of events in the busy, wake-up, shutdown and inactive states respectively and let $E[N_s]$ denote the expected system size. Then,

$$E\,[N_s]=E\,[N_0]+E\,[N_1]+E\,[N_2]+E[N_3].$$

6.1 Expected system size in the busy state

The mean number of events in the the busy state is given by

$$E(N_0) = \lim_{z\to 1} G_0'(z).$$

Differentiating Equation (5.18) and setting $z = 1$, after some algebraic manipulation, we get

$$\begin{aligned} E(N_0) = \frac{1}{\zeta}&\left\{\frac{\lambda\eta\zeta}{(\lambda+\theta_3+\zeta)(\eta+\zeta)} - \theta_1\mu\sum_{n=k}^{\infty} n f_n(0)\,\pi_{0,1}\right\} \\ &+ \frac{\{\lambda-\mu\}}{\zeta^2}\left\{\frac{\lambda\eta\zeta}{(\lambda+\theta_3+\zeta)(\eta+\zeta)} + \theta_1\mu\sum_{n=k}^{\infty} f_n(0)\,\pi_{0,1} - \mu\pi_{0,1}\right\}. \end{aligned} \tag{6.1}$$

6.2 Expected system size in the wake-up state

Let $E[N_1]$ be the mean number of events in the system during wake-up mode. Then

$$E[N_1] = \sum_{n=k}^{\infty} n\pi_{1,n}.$$

Differentiating Equation (5.21) and setting $z = 1$, we obtain

$$\begin{aligned} E(N_1) = \frac{\theta_2}{(\theta_1+\zeta)}&\left[\left(\frac{\lambda}{\beta_2}\right)^k\left\{\frac{k}{\beta_2-\lambda} + \frac{\lambda}{(\beta_2-\lambda)^2} + \frac{\lambda}{(\theta_1+\zeta)(\beta_2-\lambda)}\right\}\right. \\ &\left. + \frac{\lambda}{(\lambda+\zeta)}\left\{k + \frac{\lambda}{(\theta_1+\zeta)}\right\}\left\{\left(\frac{\lambda}{\lambda+\zeta}\right)^{k-1} + \sum_{i=1}^{k-1}\left(\frac{\lambda}{\lambda+\zeta}\right)^{k-1-i}\left(\frac{\lambda}{\beta_2}\right)^i\right\}\right] \\ &\times\left\{\mu\pi_{0,1} + \frac{\theta_3\eta\zeta}{\beta_3(\eta+\zeta)}\right\}. \end{aligned} \tag{6.2}$$

6.3 Expected system size in the shutdown state

Let $E[N_2]$ be the mean number of events in the system during shutdown mode. Then,

$$E[N_2] = \sum_{n=0}^{\infty} n\pi_{2,n}.$$

Differentiating Equation (5.23) and setting $z = 1$, we get

$$E(N_2) = \frac{\lambda}{(\theta_2+\zeta)^2}\left\{\mu\pi_{0,1} + \frac{\theta_3\eta\zeta}{(\lambda+\theta_3+\zeta)(\eta+\zeta)}\right\}. \tag{6.3}$$

6.4 Expected system size in the inactive state

Let $E[N_2]$ be the mean number of events in the system during inactive mode. Then,

$$E\left[N_3\right] = \sum_{n=0}^{k-1} n\pi_{3,n}.$$

Using the result (5.13), we obtain

$$E\left(N_3\right) = \frac{\theta_2}{\zeta\left(\lambda-\beta_2\right)}\left[\frac{\beta_2}{\lambda-\beta_2}\left\{(k-1)\left(\frac{\lambda}{\beta_2}\right)^{k+1} - k\left(\frac{\lambda}{\beta_2}\right)^{k} + \left(\frac{\lambda}{\beta_2}\right)\right\} - \frac{\lambda}{\zeta}\left\{1-\left(\frac{\lambda}{\beta_2}\right)^{k-1}\right\}\right]$$
$$\times\left\{\mu\pi_{0,1} + \frac{\eta\zeta\theta_3}{\beta_3\left(\eta+\zeta\right)}\right\}. \tag{6.4}$$

6.5 Expected number of events waiting in the system

Let $E[W_s]$ denote the mean number of events waiting in the system. Then,

$$E\left[W_s\right] = \frac{E\left[N_s\right]}{\lambda}.$$

The mean number of events waiting in the queue is given by

$$E\left[W_q\right] = \sum_{n=k}^{\infty} n\pi_{1,n} + \sum_{n=0}^{\infty} n\pi_{2,n} + \sum_{n=0}^{k-1} n\pi_{3,n} + \sum_{n=1}^{\infty}(n-1)\,\pi_{0,n}.$$

7 Numerical illustrations

This section presents the numerical illustrations of the system size probabilities obtained in sections $3-5$.

7.1 Numerical illustrations of the time-dependent probabilities

To plot the graph, the parameter values are chosen as follows: $\lambda = 1$, $\mu = 2$, $\theta_1 = 0.2$, $\theta_2 = 0.3$, $\theta_3 = 0.4$, $\eta = 0.4$, $\zeta = 0.01$ and $k = 5$.

Figure 1 presents the failure state probabilities $P_F(t)$. It is observed that as the disaster rate ζ increases the probability that the server in the failure state also increases. Figures $2-5$ demonstrates the behaviour of the shutdown, inactive, wake-up and busy state respectively. It is observed that as time t increases, the curve is also increases to some extend and then the curve decreases and attain the steady state. Figures 6 and 7 presents mean and variance of the system. It is evident from the graph that as disaster rate increases, the system size decreases. The probability curves of $P_{3,0}(t)$ start at 1 and decrease as t increases and attains the steady-state. The renaming curves of $P_{3,n}(t)$ increase to certain extent as t increases and attains the steady-state.

Figures $8-10$ presents the system size probabilities of shutdown, inactive and wake-up state in the steady-state environment. The graphs are plotted against n for varying disaster rates ζ. Figures $11-13$ illustrates expected system size in wake-up, shutdown and inactive states respectively. It is observed that as the disaster rate increases, the system size decreases.

8 Conclusion and future work

The Power saving mechanism of WSNs with jamming attack is analysed in this article. To study the investigated system, we chose an M/M/1 queueing model with vacation, set-up time, threshold policy, system disaster and repair. The transient and steady-state

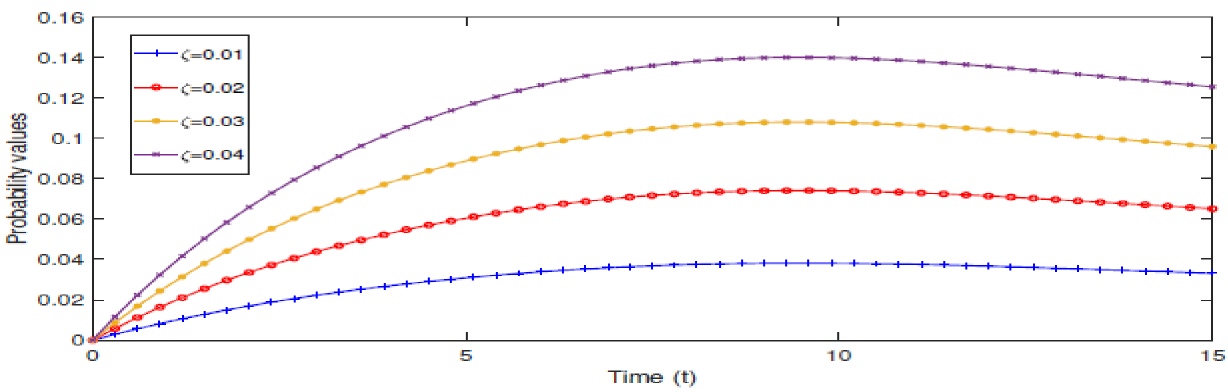

Figure 1: Probabilities of the failure state $P_F(t)$.

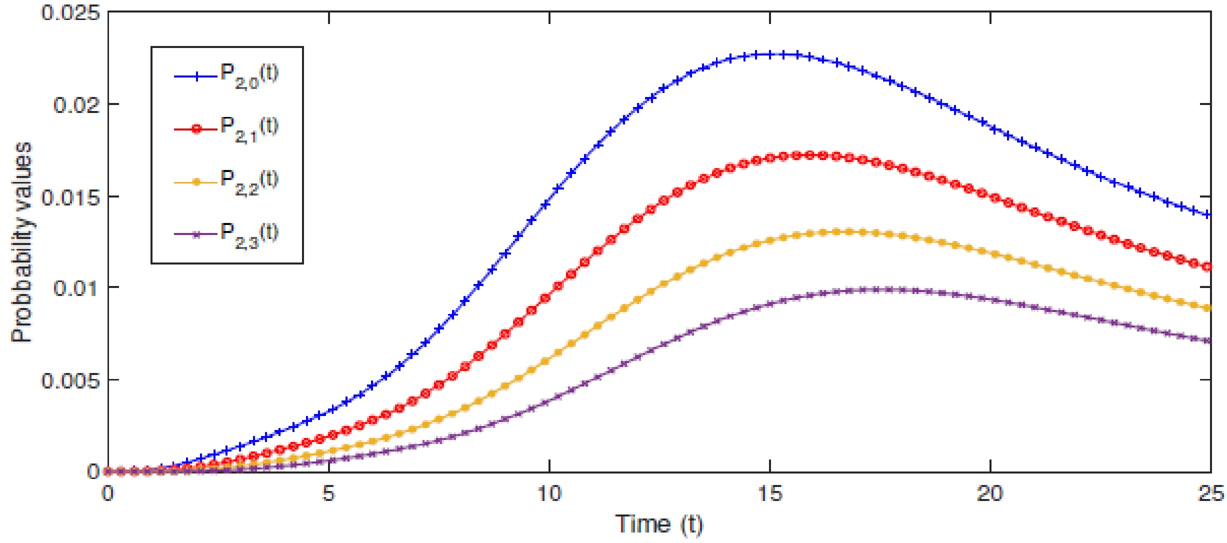

Figure 2: Probabilities of the shutdown $P_{2,n}(t)$.

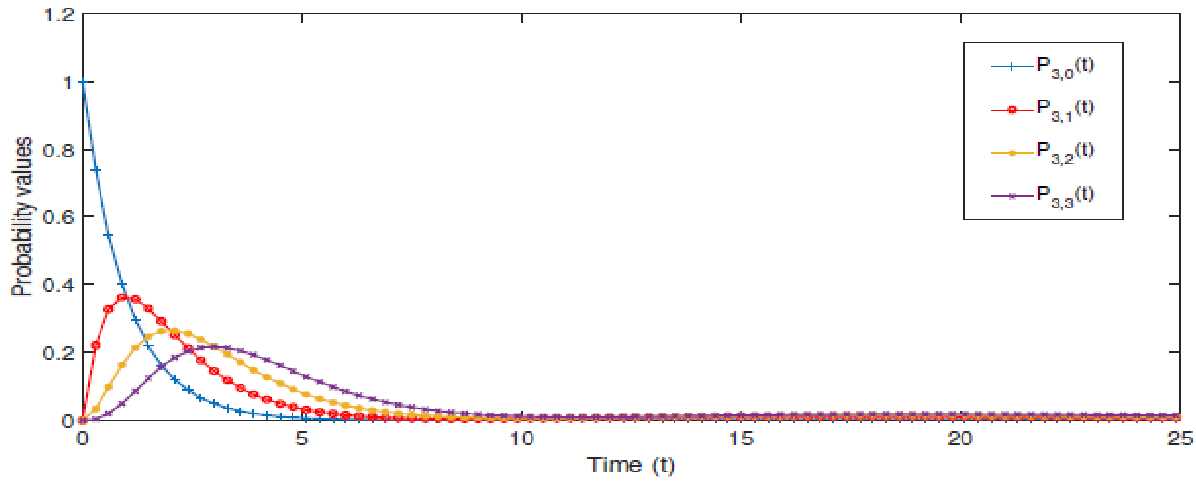

Figure 3: Probabilities of the inactive state $P_{3,n}(t)$.

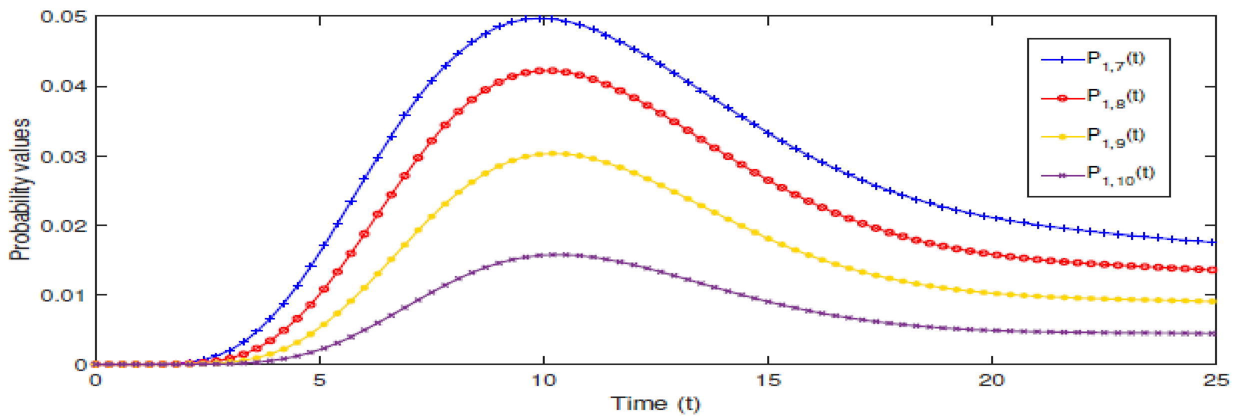

Figure 4: Probabilities of the wake-up $P_{1,n}(t)$.

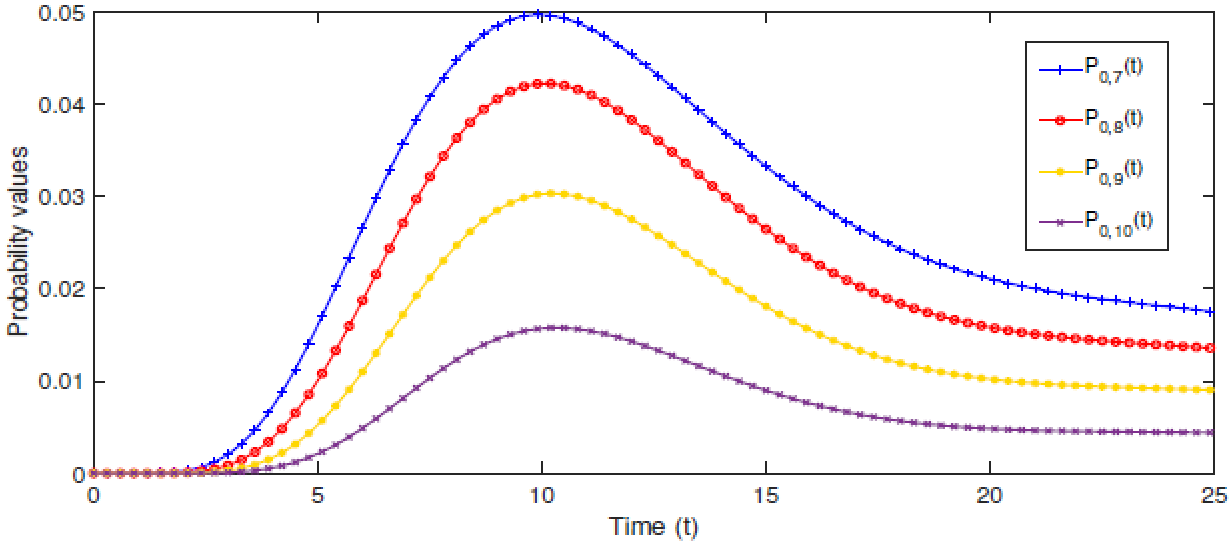

Figure 5: Probabilities of the busy state $P_{0,n}(t)$.

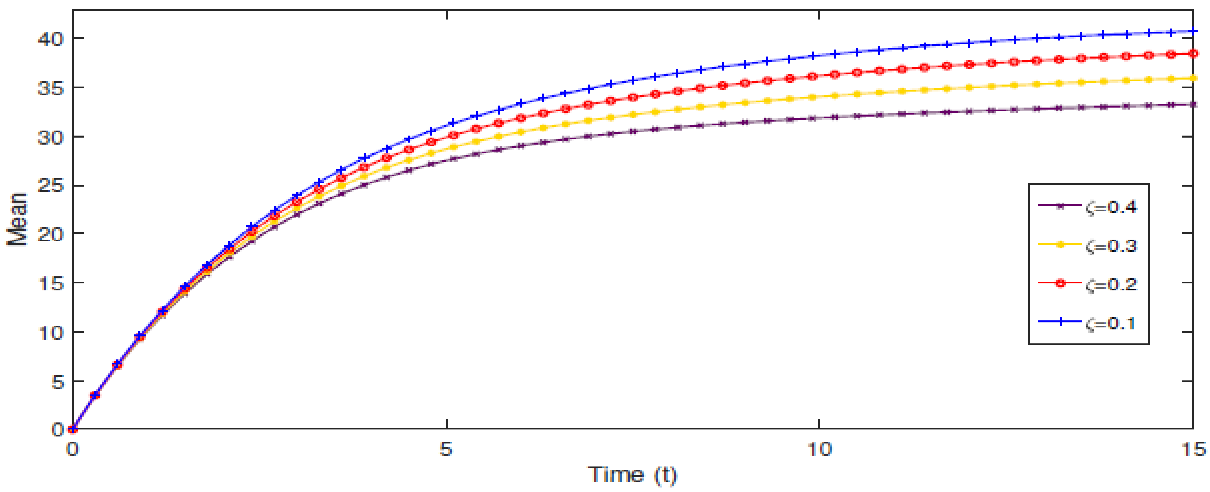

Figure 6: Mean system size

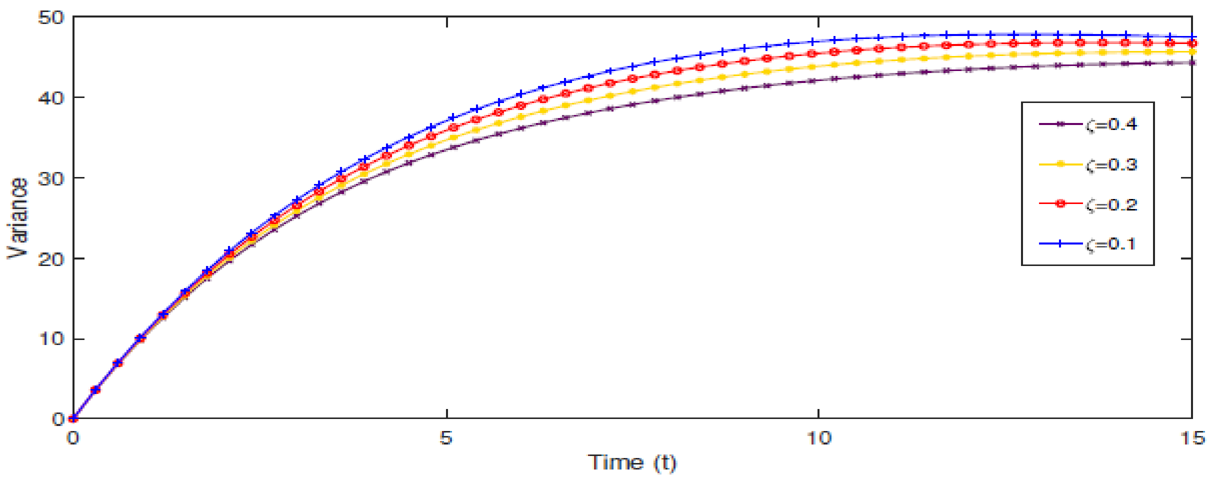

Figure 7: Variance of the system.

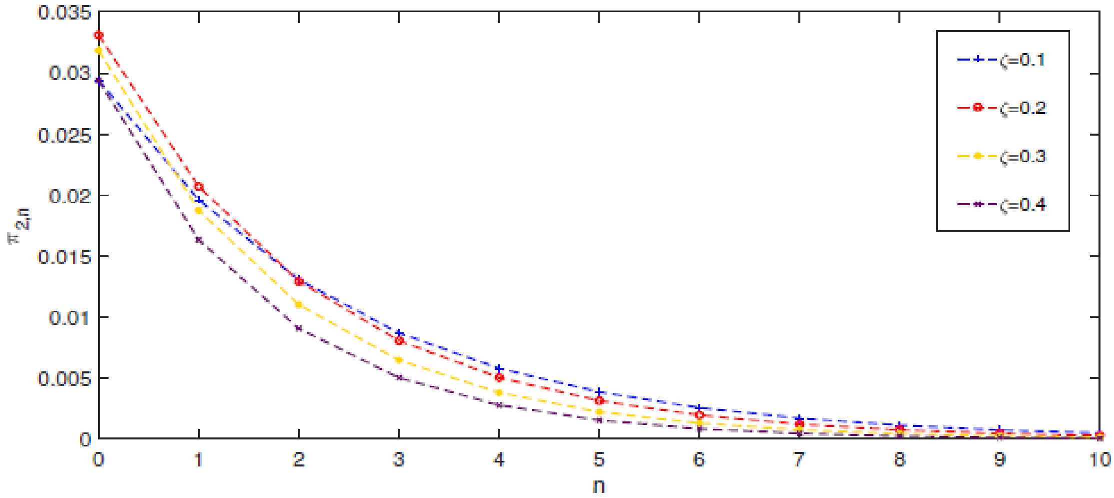

Figure 8: Steady-state probabilities of the shutdown state $\pi_{2,n}$ against n for varying ζ.

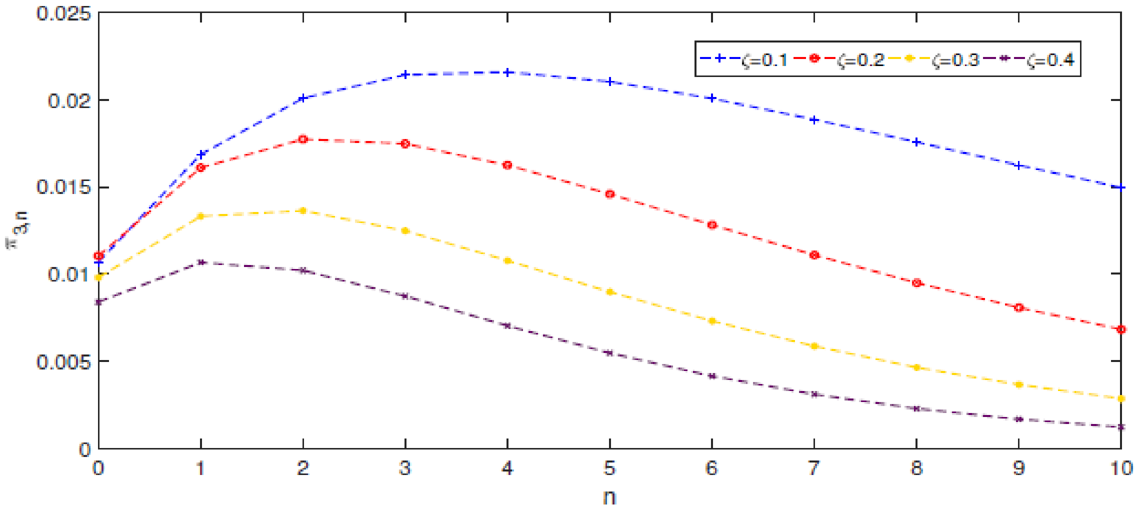

Figure 9: Steady-state probabilities of the inactive state $\pi_{3,n}$ against n for varying ζ.

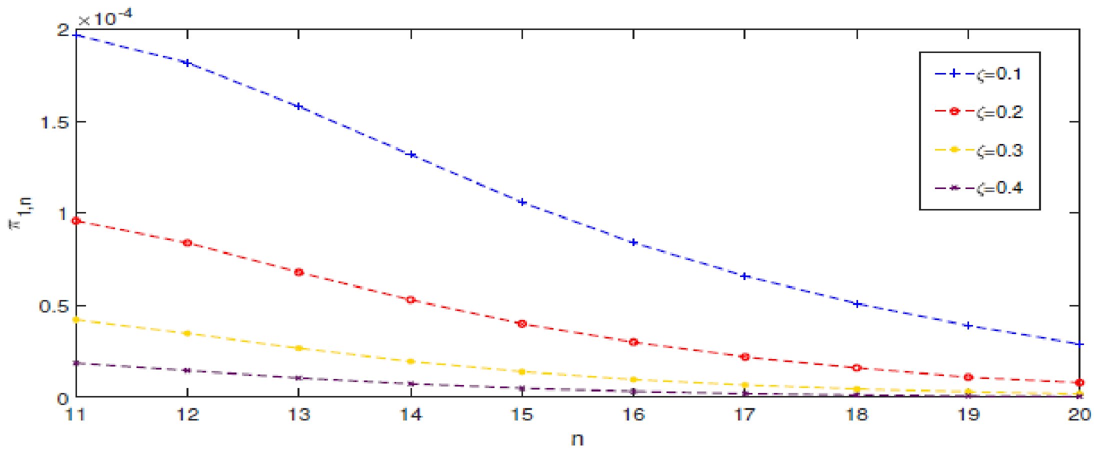

Figure 10: Steady-state probabilities of the wake-up state $\pi_{1,n}$ against n for varying ζ.

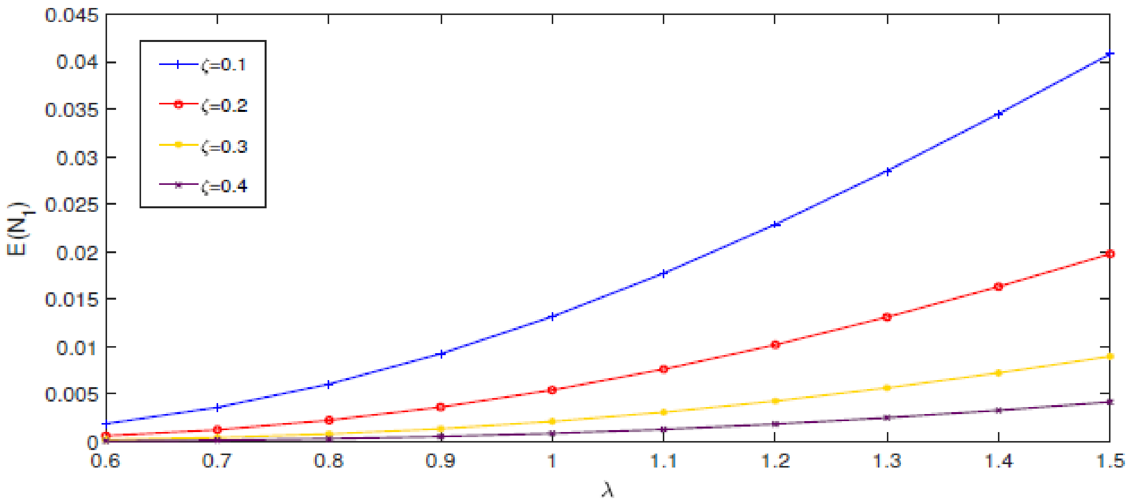

Figure 11: Expected system size in wake-up state against λ for varying disaster rates.

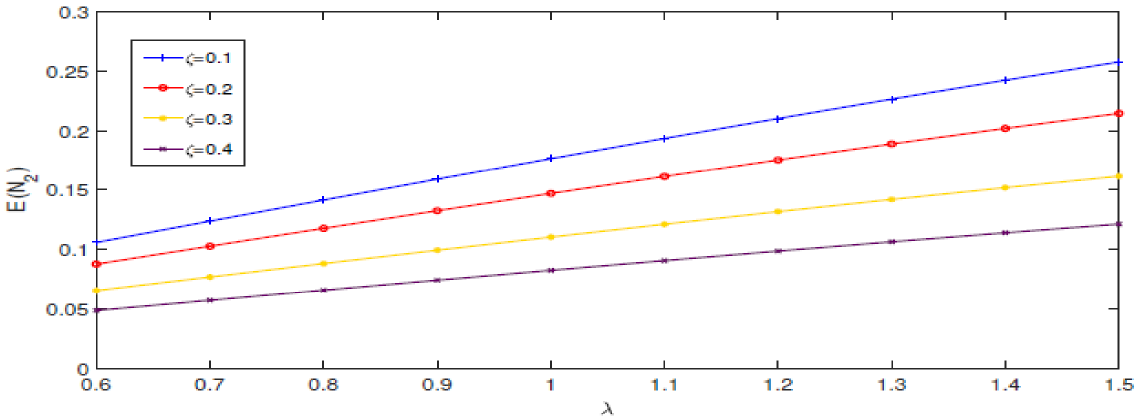

Figure 12: Expected system size in shutdown state against λ for varying disaster rates.

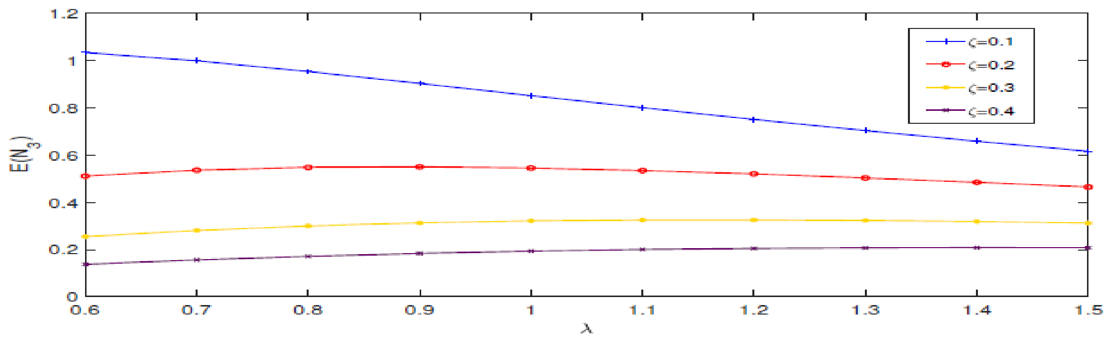

Figure 13: Expected system size in inactive state against λ for varying disaster rates.

system size probabilities of the system are obtained in a closed form. The performance indices such as mean, variance, probability that the system is in power-saving modes and mean power consumption are obtained. This work may be extended to an M/M/C queueing model with working vacation and close-down times.

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