A Characterization of 2-Vertex Self Switching of Connected Unicyclic Graphs

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Abstract: —A graph $G'(V, E')$ is created from G by eliminating all edges between S and its complement $V - S$ and any non-edges between s and $V - S$ are added as edges for a simple graph *G(V, E)* and a non empty subset $S \subset V$. We write G^v for $G^{\{v\}}$ when *S={v}*, and the associated switching is referred to as vertex switching. $|S|$-vertex switching is another name for it. 2-vertex switching occurs when $|S|$ equals 2. If *B* is connected and maximal, a joint at σ in *G* is a subgraph of *G* that includes *G*[σ]. If *B* is connected, we refer to it as a c$-$***joint***; otherwise, we refer to it as a d$-$***joint***. An acyclic graph is one that has no cycles. The term "tree" refers to a linked acyclic network. In this article, we characterize 2-vertex self switching for connected unicyclic graphs.
AMS classification: 05C60, 05C05, 05C40.
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1. Introduction

For any graph *G(V, E)* with $|V(G)| = p$, the graph $G'(V, E')$ is defined as the graph generated from *G* by deleting all edges between σ and its counterpart, $V - \sigma$, and any nonedges between σ and $V - \sigma$ are added as edges where $\sigma \subseteq V$. Seidel [1, 5] defined switching, which is also known as $|\sigma|$ $-$vertex switching. When $|\sigma| = 2$, it is called as 2$-$vertex switching. A graph which contains exactly one cycle is called an unicylic graph. In [4] the concept of self vertex switchings were studied. A survey in two graphs and reconstruction of graphs were studied in [6]. Switching classes and Euler graphs were discussed in [2].

In 2008, the concept of branches and joints in graphs were introduced by Vilfred V et al., [7]. A joint at σ in G is a subgraph B of G that includes $G[\sigma]$ if $B - \sigma$ is connected and maximum. If B is connected, we refer to it as a ***c-joint***; otherwise, we refer to it as a ***djoint***. **B** is a ***total joint*** if $B = \sigma + (B - \sigma)$. In [3] C. Jayasekaran, et al., analysed the graphs for 2–vertex switching of joints.

For the graph G in Figure 1.1, G^σ, $G[\sigma]$ and G–σ are shown in Figures 1.2 to 1.4 respectively, where σ = $\{u, v\}$. Figures 1.5, 1.6 and 1.7 show the $c - joints$ $d - joint$ and the $total\ joint,$ respectively.

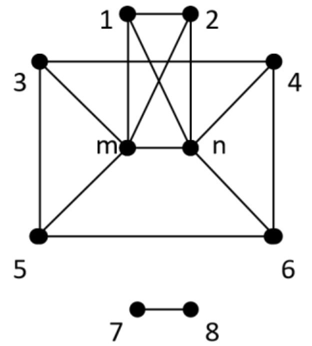


Figure. 1.1. G

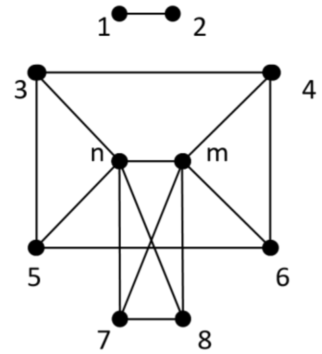


Figure. 1.2. G^{σ}

Figure. 1.3. $G[\sigma]$

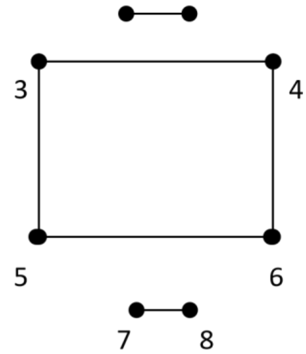


Figure. 1.4. $G - \sigma$

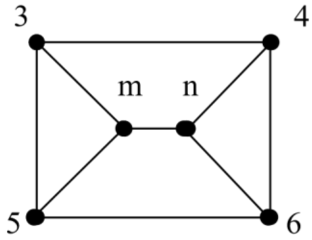


Figure 1.5. c-joint

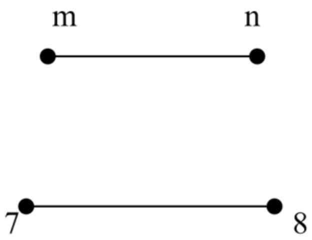


Figure 1.6. d-joint

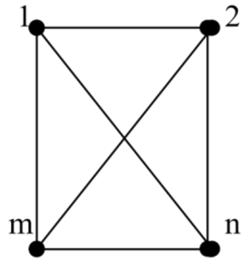


Figure 1.7. Total joint

Consider the following theorems, which will be used in the next section.

Theorem 1.1. [3] Let G be a graph of order p and let $\sigma = \{u, v\}$ be a subset of $V(G)$ with $|V(G)| \geq 5$ such that $uv \notin E(G)$. If B is a c-joint at σ in G, then B^{σ} is a c-joint and unicyclic if and only if $|V(B)| \geq 5$ and one of the following holds:

(i) $B - \sigma$ is connected, acyclic and $\{d_B(u), d_B(v)\} = \{|V(B)| - 3, |V(B)| - 4\}$.

(ii) $B - \sigma$ is connected, unicyclic and $d_B(u) = d_B(v) = |V(B)| - 3$.

Theorem 1.2. [3] Let G be a graph of order p and let $\sigma = \{u, v\}$ be a subset of $V(G)$ such that $uv \in E(G)$. If B is a c-joint at σ in G, then B^{σ} is a c-joint and unicyclic if and only if $|V(B)| \geq 5$ and one of the following holds

(i) $B - \sigma$ is connected, acyclic and either $d_B(u) = d_B(v) = |V(B)| - 2$ or $\{d_B(u), d_B(v)\} = \{|V(B)| - 3, |V(B)| - 1\}$.

(ii) $B - \sigma$ is connected, unicyclic and $\{d_B(u), d_B(v)\} = \{|V(B)| - 2, |V(B)| - 1\}$.

Main Results 2. 2-VERTEX SELF SWITCHING OF CONNECTED UNICYCLIC GRAPHS

Theorem 2.1. Let G be a connected unicyclic graph of order $p \geq 4$ and let σ = $\{u, v\}$ be a non−empty subset of $V(G)$ such that $G - \sigma$ is connected. Then G has a 2- vertex self switching at σ in G if and only if for $uv \notin E(G)$, G is either $C_{3(u)}(0, 0, P_3)$ or $C_{3(u)}(0, P_2, P_2)$ or $C_{4(u)}(0, 0, P_2, 0)$ with $d_G(v) = 1$ and for $uv \in E(G)$, G is either $C_{3(u)}(P_2, 0, 0)$ or C_4 or

$C_{3(u)}(u)(0,0, P_2)$.

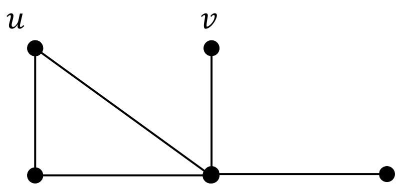
Proof. Let G be a connected unicyclic graph. Let $\sigma = \{u, v\}$ be a 2-vertex self switching of G. Then $G \cong G^\sigma$.

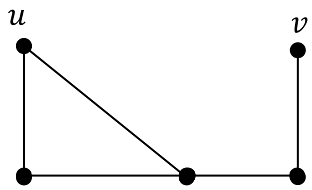
Case 1. $uv \notin E(G)$

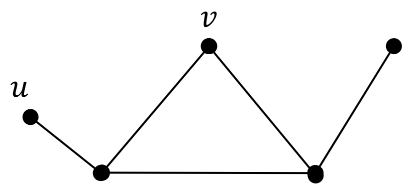
$G \cong G^\sigma$ implies that G^σ is connected and unicyclic. By Theorem 1.1, $p \geq 5$ and either $G - \sigma$ is connected, acyclic and $\{d_G(u), d_G(v)\} = \{|V(G)| - 3, V(G)| - 4\}$ or $G - \sigma$ is connected, unicyclic and $d_G(u) = d_G(v) = |V(G)| - 3$.

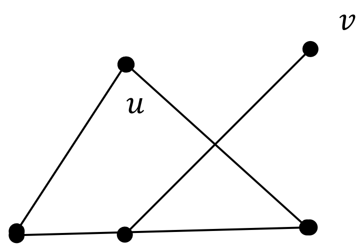
Subcase 1a. $G - \sigma$ is connected, acyclic and $\{d_G(u), d_G(v)\} = \{|V(G)| - 3, V(G)| - 4\}$ Let $d_G(u) = |V(G)| - 3$ and $d_G(v) = |V(G)| - 4$. If $|V(G)| = 4$, then $|V(G) - \sigma| = 2$. Since $G - \sigma$ is acyclic and connected, $G - \sigma = P_2$. Also $d_G(u) = 1$ and $d_G(v) = 0$ implies that $G = K_1 \cup P_3$, where K_1 is the vertex v, which is contradiction to G is unicyclic and connected. Hence we have $|V(G)| \geq 5$.

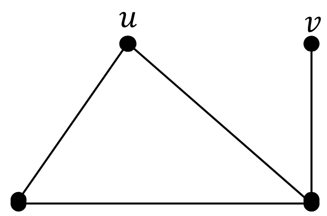
If $\left|V(G)\right| \geq 6$, then $d_G(u) \geq 3$, $d_G(v) \geq 2$ and $|V(G - \sigma)| \geq 4$. Then there exists at least three vertices say a, b and c in $G - \sigma$ such that u is adjacent to a, b and c. Since $G - \sigma$ is connected, there exist paths P_1: $a - b$, P_2: $b - c$ and P_3: $a - c$. Now the edges au, bu and cu and the paths $a - b$, $b - c$ and $a - c$, form at least three different cycles uP_1u, uP_2u and uP_3u in G, which is a contradiction to G is unicyclic. Therefore $|V(G)| = 5$. This implies that $d_G(u) = 2$ and $d_G(v) = 1$ and $|V(G) - \sigma| = 3$. Since G – σ is connected and acyclic, $G - \sigma = P_3$. The five non-isomorphic unicyclic graphs on 5 vertices with $d_G(u) = 2$ and $d_G(v) = 1$ are $C_{3(u)}(0,0,2P_2)$, $C_{3(u)}(0,0, P_3)$, C_3(u) $(0,P_2,P_2)$, $C_{4(u)}(0,0, P_2, 0)$ and $C_{4(u)}(0,0,0, P_2)$ which are given in figures 8 to 12.











Clearly, $C_{3(u)}(0,0,P_3)$, $C_{3(u)}(0,P_2,P_2)$ and $C_{4(u)}(0,0,P_2,0)$ are the graphs with 2-vertex self switchings at σ = $\{u, v\}$.

Subcase 1b. $G - \sigma$ is connected, unicyclic and $d_G(u) = d_G(v) = |V(G)| - 3$

If $|V(G)| = 4$, then $|V(G) - \sigma| = 2$. It implies that $G - \sigma$ is not unicyclic and hence $|V(G)| \geq 5$. If $|V(G) - \sigma| \geq 6$, then $d_G(u) = d_G(v) \geq 3$ implies that there exists at least three vertices x, y and z in $V(G) - \sigma$ such that ux, uy and uz are edges in G. Since $G - \sigma$ is unicyclic, let C_1 be the unique cycle in $G - \sigma$. Now the edge ux, $x - y$ path in $G - \sigma$ and the edge yu form a cycle different from C_1. This is a contradiction to $C_1$is the unique cycle in G.

Hence $|V(G - \sigma)| = 5$. Since $G - \sigma$ is unicyclic and $|V(G) - \sigma| = 3$, $G - \sigma = C_3 = K_3$. Clearly, $d_G(u) = d_G(v)$ = 5−3 = 2. This implies that u is adjacent to two vertices, say a and b in $V(G) - \sigma$ and hence au and bu are edges in G. Also there exists an $a - b$ path in $G - \sigma$ and hence in G. Now the edges au, bu and the path $a - b$, form a cycle C_2 different from C_1, which is a contradiction to G is unicyclic. Hence there is no connected unicyclic graph G such that $G - \sigma$ is unicyclic and $d_G(u) = d_G(v) = p - 3$.

Case 2. $uv \in E(G)$

Since G is connected and G^σ is both connected and unicyclic, by Theorem 1.2, either $G - \sigma$ is connected, acyclic and either $d_G(u) = d_G(v) = |V(G)| - 2$ or $\{d_G(\mathrm{u}), d_G(v)\}$ $\{|V(G)| - 3, |V(G)| - 1\}$, or $G - \sigma$ is connected, unicyclic and $\{d_G(u), d_G(v)\} = \{|V(G)| - 1, |V(G)| - 2\}$. We consider the following three subcases.

Subcase 2a. $G - \sigma$ is connected, acyclic and $\{d_G(u), d_G(v)\} = \{|V(G)| - 1, |V(G)| - 3\}$ Without loss of generality, let $d_G(u) = |v(G)| - 1$ and $d_G(v) = |V(G)| - 3$. If $|V(G)| \geq 5$, then $d_G(u) \geq 4$ and $d_G(v) \geq 2$. This implies that there exist at least three vertices a, b, c in $V(G) - \sigma$ which are adjacent to u. Since $G - \sigma$ is connected, there exists paths $P_1: a - b$, $P_2: b - c$ and $P_3: a - c$ in $G - \sigma$. Now uP_1u, uP_2u and uP_3u form at least three cycles in G which is a contradiction to G is unicyclic. Hence $|V(G)| = 4$. The only unicyclic graph on 4 vertices with $d_G(u) = 3$ and $d_G(v) = 1$ is $C_{3(u)}(P_2, 0,0)$.

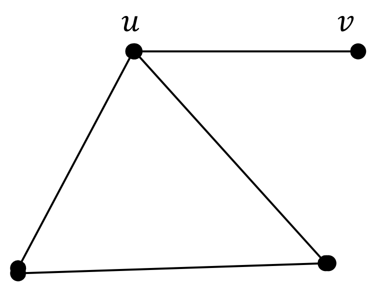


Figure 13 $C_{3(u)}(P_2, 0, 0)$

Subcase 2b. $G - \sigma$ is connected, acyclic and $d_B(u) = d_G(v) = |V(G)| - 2$

If $|V(G)| \geq 5$, then $d_G(u) = d_G(v) \geq 3$. Clearly, $V(G) - \sigma$ contains at least three vertices. Since $uv \in E(G)$, there exists at least two vertices say a and b in $V(G) - \sigma$ which are adjacent to u. Now $G - \sigma$ is connected implies that there exists an $a - b$ path in $G - \sigma$ and hence in G.

Now the edge ua, the $a-b$ path and the edge bu form a cycle without the vertex v. By a similar argument, we can find another cycle in G which contains the vertex v but not the vertex u which is a contradiction to G is unicyclic. Hence $|V(G)|=4$ and $d_G(u)=d_G(v)=2$. The only graphs on 4 vertices with $d_G(u)=d_G(v)=2$ are given in figures 14 and 15.

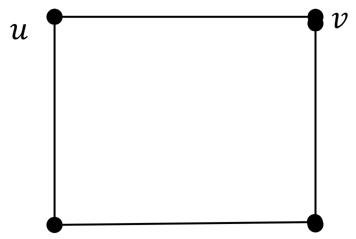


Figure 14. C_4

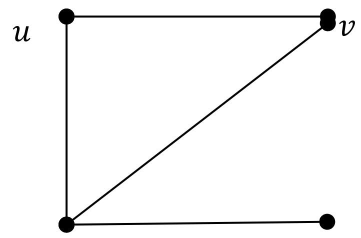


Figure15. $C_{3(u)}(0,0,P_2)$

Case 2c. $G-\sigma$ is connected, unicyclic and $\{d_G(u),d_G(v)\}=\{|V(G)|-1,|V(G)|-2\}$

Without loss of generality, let $d_G(u)=|V(G)|-1$ and $d_G(v)=|V(G)|-2$. If $|V(G)|\geq 5$, then $d_G(u)>4$. As in subcase 2a, G is not unicyclic. Hence $|V(G)|=4$. Now $|V(G)-\sigma|=2$ implies that $G-\sigma$ is not unicyclic. Hence there does not exist any graph with a 2-vertex self switching.

Thus from cases 1 and 2 we get, if $uv\notin E(G)$, then G is either $C_{3(u)}(0,0,P_3)$ or $C_{3(u)}(0,P_2,P_2)$ or $C_{4(u)}(0,0,P_2,0)$ with $d_G(v)=1$ and for $uv\in E(G)$, G is either $C_{3(u)}(P_2,0,0)$ or C_4 or $C_{3(u)}(u)(0,0,P_2)$.

Conversely, let G be the graph given in the statement. Clearly, for each graph G, $\sigma=\{u,v\}$ is a 2-vertex self switching of G.
Hence the theorem.

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