Odd Triangular Graceful Labeling and Odd Triangular Graceful Number of Special Trees and Some Graphs

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Article Info
Page Number: 10982 - 10995
Publication Issue:
Vol 71 No. 4 (2022)

Abstract

A flourishing and application-oriented area of research in graph theory is graph labeling. Researchers have been studying graph labeling since the mid-20th century. A new variation of graceful labeling was introduced in 2001 called *triangular graceful labeling* which builds on Rosa's 1967 β *-valuation* and Goloumb's *graceful labeling*. In our recent study, we noticed a variation of triangular graceful labeling introduced in [4] in 2020. It is called *odd triangular graceful labeling.* We use throughout the paper in short form as OTGL . In our study , we establish odd triangular gracefulness of a few classes of tree graphs and found odd triangular graceful number of few families of graphs which we have introduced recently. A graph G' with $|V(G')| = \nu$and $|E(G')|=\varepsilon$ is called an OTG graph if there is an one-one function $\phi: V(G') \to \{0, 1, 2, 3 \dots T_{2\varepsilon-1}\}$ where $T_{2\varepsilon-1}$ denotes $(2\varepsilon\text{-}1)^{\text{th}}$ triangular number, (in general $T_k = \frac{k(k+1)}{2}$) such that the induced function on $E(G')$ namely $\overline{\phi}$ $: E(G') \to \{T_1, T_3, T_5 \dots T_{2\varepsilon-1}\}$ defined by $\overline{\phi}\ (xy) = |\phi(x) - \phi(y)|$ for every edge $xy\ of\ G'$ is bijective. The function ϕ is called an OTGL. Here, we establish odd triangular gracefulness of a few classes of tree graphs and found odd triangular graceful number introduced recently by us for few families of graphs.

Article History
Article Received: *15 September 2022*
Revised: *25 October 2022*
Accepted: *14 November 2022*
Publication: *21 December 2022*

Keywords: Odd triangular graceful trees, Odd triangular graceful number.

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1. Introduction

The graph theory has had a huge impact on the development of applied mathematics. There have been many fascinating results obtained in graph theory this decade, and they have also had applications in the fields of Chemistry, Biology, Mathematics, Computer Science and in

Engineering. Graph labeling is one of the active research areas within graph theory. The goal of this paper is to prove the existence of a particular type of graph labeling as well as to investigate if a particular graph can have such a labeling or how to get it to have one. A graph labeling is an assignment of integers to the vertices or edges or to both of a graph. There are certain conditions that must be met in order to assign integers to the vertices or edges of a graph. In 1967, Rosa [3] proposed a labeling scheme for graph G' known as β -valuation, which was later renamed graceful labeling by Golomb [2]. Throughout this paper we considered simple undirected graphs. Devaraj et al. [1] introduced an analog of graceful labeling called triangular graceful labeling in 2001. In 2020, OTGL was introduced in [4], a variant of triangular graceful labeling. In our earlier article established that a few classes of trees are odd triangular graceful. With this study, we have proven the odd triangular gracefulness of some graphs and found graphs that were not triangular graceful could become triangular graceful by introducing a novel concept called the triangular graceful number. Labeling of this type finds application in fields such as coding theory, X-ray crystallography, and communications networking.

2. Preliminaries

Definition 2.1: Triangular graceful [1]

A graph $G'$of order ν and size ε is called triangular graceful if there is an 1-1 function $\psi: V \to \{0 ,1 ,2 ,3,\ldots\ T_\varepsilon\}$ where $T_\varepsilon$is the ε^{th}triangular number $T_\varepsilon = \frac{\varepsilon(\varepsilon+1)}{2}$ such that induced function on $edge\ set$ namely $\overline{\psi}$:$E(G') \to \{T_1, T_2, T_3, \ldots . T_\varepsilon\}$defined by $\overline{\psi}\ (v_i v_j) = |\psi(v_i) - \psi(v_j)|$for every edge$v_i v_j \in E(G')$, is injective. The function $\psi$is called triangular graceful labeling.

Definition 2.2: Odd Triangular graceful [4]

A graph $G'$with $|V(G')| = \nu$ and $|E(G')|$=ε is called an odd triangular graceful graph if there is an 1-1 function $\phi: V \to \{0 ,1 ,2 ,3 \ldots T_{2\varepsilon-1}\}$ where $T_{2\varepsilon-1}$ denotes $(2\varepsilon\text{-}1)^{th}$triangular number, (in general $T_k = \frac{k(k+1)}{2}$) such that the induced function on $E(G')$ namely $\overline{\phi}$:$E(G') \to \{T_1, T_3, T_5 \ldots T_{2\varepsilon-1}\}$defined by $\overline{\phi}\ (xy) = |\phi(x) - \phi(y)|$for all edge $xy\ \ of\ G'$ is 1-1 and onto. The function ϕ is called an OTGL .

3. Existing Results

The following results are already proved in [4] and proved by us in our recent article [5]

Theorem 3.1: All paths $\boldsymbol{P_n}$ are OTG graphs.[4]

Theorem 3.2: All stars $\boldsymbol{K_{1,n}}$ are OTG graphs.[4]

Theorem 3.3: The double star ***S(n,m)*** is an OTG graph.[4]

Theorem 3.4: $\boldsymbol{Y_k}$ Trees are OTG.[5]

Theorem 3.5: Comb graph $P_n{}^{\Theta}K_1$ ($n\geq 2$) is odd triangular graceful.[5]

Theorem 3.6: A Coconut tree *CT(m,n)* consisting of path P_m with a leaf of **$K_{1,n}$ ($m\geq 2$, $n\geq 1$) is** odd triangular graceful .[5]

Theorem 3.7: $\boldsymbol{F_n}(\boldsymbol{n \geq 3})$ Trees are odd triangular graceful.

Observations 3.8 [5]:

(i) Graceful graphs need not be triangular graceful or OTG.

(ii) Some graphs are graceful, triangular graceful and OTG.

(iii) Some graphs are triangular graceful but not OTG..

4. Main Results

Definition 4.1:

Tree t_n t_n $(n\geq 1)$ is a tree constructed by adding n pendant edges to all pendant vertices of star graph$K_{1,2}$. The cardinality of vertex set $|V(t_n)| = 2n + 3$ and since it is a tree it will have (2n+2) edges.

Theorem 4.2:

$\boldsymbol{t_n}$ trees are odd triangular graceful.

Proof:

Let t_n *be the tree* , then $|V(t_n)| = 2n + 3$ and $|E(t_n)| = 2n + 2$.

Let$V(t_n) = \{v_0{}', v_1{}', v_2{}', w_1{}', w_2{}', \dots w_n{}', u_0{}', u_1{}', u_2{}', \dots u_{n-1}{}'\}$.

Define μ: $V(t_n) \rightarrow \{0,1,2, \dots T_{4n+3}\}$as follows: (where $T_k = \frac{k(k+1)}{2}$is the k^{th} triangular number)

$$\mu(v_0') = 0$$
$$\mu(v_1') = T_{4n+3} = 8n^2 + 14n + 6$$
$$\mu(v_2') = T_{4n+1} = 8n^2 + 6n + 1$$
$$\mu(w_i') = 8(i+1)n - 2i^2 + 3i + 5 \qquad (1 \leq i \leq n)$$
$$\mu(u_i') = 6n^2 + (4i+7)n - 2i^2 - i + 1 \;\; (0 \leq i \leq n-1)$$

Clearly the above function is one –one as the labels of vertex are different. Also the induced function $\bar{\mu}$ on the set edges is injective. Since the induced function gives the label on it's edges as $\overline{\mu}(xy) = |\mu(x) - \mu(y)|$ for every edge $xy \in E(t_n)$.Also checked the validity of above statements by writing python coding and executing it. The python coding and a sample output is given below:

```
n=int(input())
v0=0
v1=(8*n*n)+(14*n)+6
v2=(8*n*n)+(6*n)+1
w=[]
u=[]
w.append(0)
for i in range(1,n+1):
    w.append((8*(i+1)*n)-(2*i*i)+(3*i)+5)
for j in range(n):
    u.append((6*n*n)+((4*j)+7)*n-(2*j*j)-j+1)
print("v0: "+str(v0))
print("v1: "+str(v1))
print("v2: "+str(v2))
for i in range(1,n+1):
    print("w"+str(i)+": "+str(w[i]))
for j in range(n):
    print("u"+str(j)+": "+str(u[j]))
```

Output : when n=5

```
5
v0: 0
v1: 276
v2: 231
w1: 86
w2: 123
w3: 156
w4: 185
w5: 210
u0: 186
u1: 203
u2: 216
u3: 225
u4: 230
```

Hence the graph $G = t_n$ trees are odd triangular graceful.

Figure 1 below shows a OTGL of the t_5 tree:

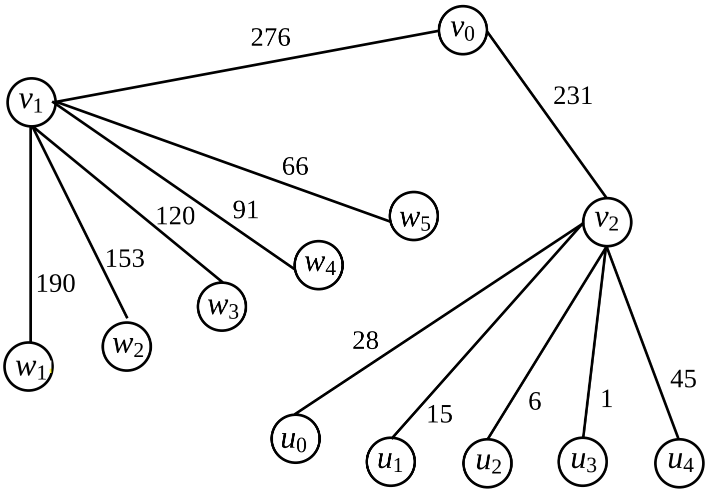


Figure 1

We have shown in the figure above the edge labels that are determined by following the vertex labels defined in the theorem, shown in figure 2

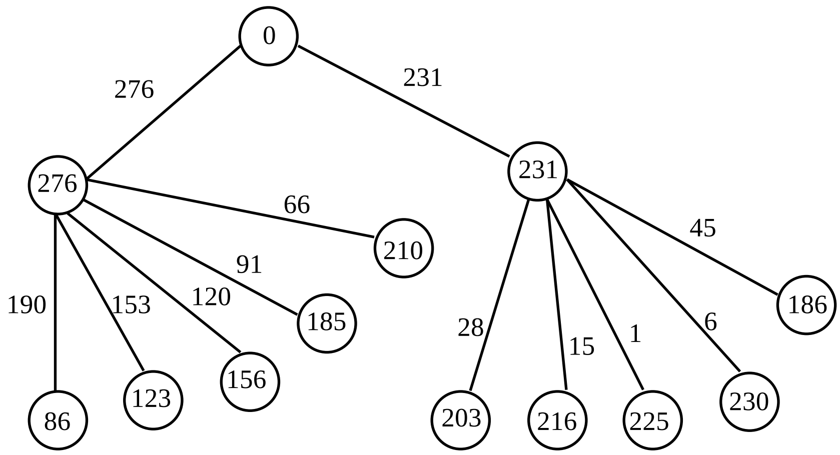


Figure 2

Definition 4.3: Multi-star graph

The graph $K_{1,n,n}(n \geq 1)$ having *(2n+1)* vertices and *2n* vertices as shown below in figure 3 is called a multi star graph.

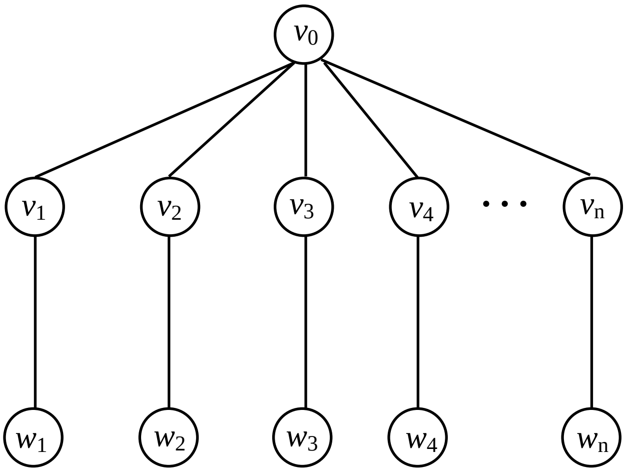


Figure 3: Multi star graph $K_{1,n,n}$

Also, the graphs $K_{1,n,n,n}(n \geq 1)$ (n-3 times) and in general $K_{1,\underline{n,n,n,..n}}$ (n≥ 1) (where n some m times) are multi- star graphs with 3n+1 vertices and 3n edges and (mn+1) vertices and mn edges respectively.

Theorem 4.4:

The multi-star graphs $K_{1,n,n}(n \geq 1)$ are Odd triangular graceful.

Proof:

Let $V(K_{1,n,n}) = \{v_0', v_1', v_2', \dots v_n', w_1', w_2', \dots w_n'\}$

Define $\rho: V(K_{1,n,n})) \rightarrow \{0,1,2,3,\dots,T_{4n-1}\}$ as follows: (where $T_k = \frac{k(k+1)}{2}$ is the k^{th} triangular number)

$\rho(v_0') = 0$

$\rho(v_i') = T_{4n-(2i-1)} = 8n^2 - 8ni + 6n + 2i^2 - 3i + 1 \quad (1 \leq i \leq n)$

$\rho(w_i) = 6n^2 - (4i-3)n$ $(1 \leq i \leq n)$

Due to the distinct vertex labels, the function above is clearly one -one.

Furthermore, the function $\overline{\rho}$ induced on the set edges is injective. Since the induced function gives the label on its edges as $\overline{\rho}(xy) = |\rho(x) - \rho(y)|$ for every edge $xy \in K_{1,n,n}$.

When the induced function gives the label on its edges, it is injective.

Hence, $K_{1,n,n}$admits OTGL, making $K_{1,n,n}$ odd triangular graceful.

The above statements are validated by executing the following python coding. Python coding and sample output are displayed below:

```
n=int(input())
v0=0
v=[]
w=[]
v.append(0)
w.append(0)
for i in range(1,n+1):
    v.append((8*n*n)-(8*n*i)+(6*n)+(2*i*i)-(3*i)+1)
for j in range(1,n+1):
    w.append((6*n*n)-((4*j)-3)*n)
print("v0: "+str(v0))
for i in range(1,n+1):
    print("v"+str(i)+": "+str(v[i]))
for j in range(1,n+1):
    print("w"+str(j)+": "+str(w[j]))
```

Sample output when n=3:

```
3
v0: 0
v1: 66
v2: 45
v3: 28
w1: 51
w2: 39
w3: 27
```

In fact, we checked for many values of n like 50,100 etc.

Figure 4 below shows a triangular graceful labeling of the multi-star graph $K_{1,3,3}$

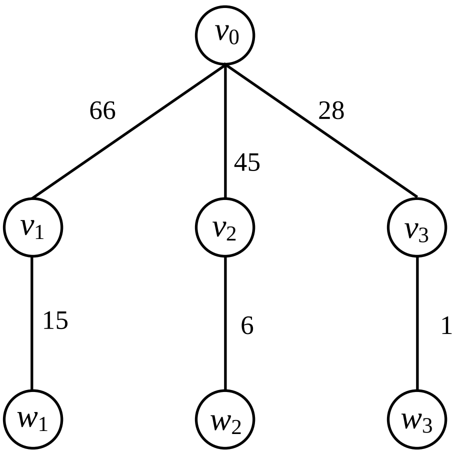


Figure 4

we have shown in the figure above the edge labels that are determined by following the vertex labels defined in the theorem, shown in figure 5

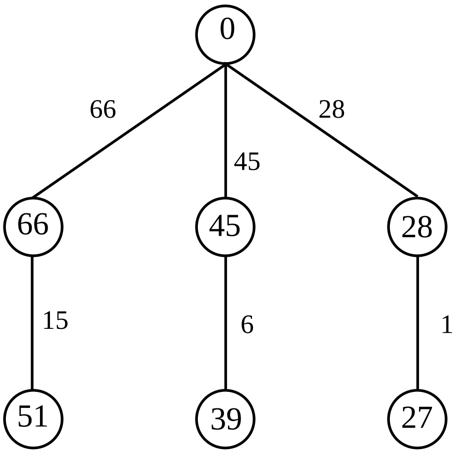


Figure 5

Theorem 4.5:

The Multi-star Graphs $K_{1,n,n,n}(n \geq 1)$ are odd triangular graceful.

Proof:

consider $K_{1,n,n,n}(n \geq 1)$, then $|V(K_{1,n,n,n})| = 3n + 1$ and $\left|E(K_{1,n,n,n})\right| = 3n$.

Let $V(K_{1,n,n,n}) = \{v'_0, v'_1, v'_2, \ldots v'_n, w'_1, w'_2, \ldots, w'_n, u'_0, u'_1, \ldots, u'_{n-1}\}$

Define $\tau: V(K_{1,n,n,n}) \rightarrow \{0,1,2,3, \ldots, T_{6n-1}\}$ as follows : (where $T_k = \frac{k(k+1)}{2}$is the k^{th} triangular number)

$$\tau(v'_0) = 0$$

$$\tau(v'_i) = 18n^2 - (12i - 9)n + (i - 1)(2i - 1) \quad (1 \leq i \leq n)$$

$$\tau(w'_i) = 10n^2 - (4i - 3)n \quad (1 \leq i \leq n)$$

$$\tau(u'_i) = 8n^2 - (2i^2 + i) \quad (0 \leq i \leq n)$$

The distinct vertex labels indicate that the function above is one -one.

Furthermore, the induced function on the set edges $\bar{\tau}$ on the set edges is injective. Since the induced function gives the label on its edges as $\bar{\tau}(xy) = |\tau(x) - \tau(y)|$ for every edge $xy \in E(K_{1,n,n,n})$

When the induced function gives the label on its edges, it is injective.

Hence, $K_{1,n,n,n}$ admits OTGL, making $K_{1,n,n,n}$ OTG.

Here again , we used python coding to check the above statement truthfulness. Python coding and sample output are given below :

```
n=int(input())
v0=0
v=[]
w=[]
u=[]
v.append(0)
w.append(0)
for i in range(1,n+1):
    v.append((18*n*n)-((12*i)-9)*n+(i-1)*(2*i-1))
for j in range(1,n+1):
    w.append((10*n*n)-((4*j)-3)*n)
for k in range(n):
    u.append((8*n*n)-((2*k*k)+k))
print("v0: "+str(v0))
for i in range(1,n+1):
    print("v"+str(i)+": "+str(v[i]))
for j in range(1,n+1):
    print("w"+str(j)+": "+str(w[j]))
for k in range(0,n):
    print("u"+str(k)+": "+str(u[k]))
```

Output :when n=3

```
3
v0: 0
v1: 153
v2: 120
v3: 91
w1: 87
w2: 75
w3: 63
u0: 72
u1: 69
u2: 62
```

The following figure 6 shows the Odd triangular Graceful labeling of $K_{1,3,3,3}$

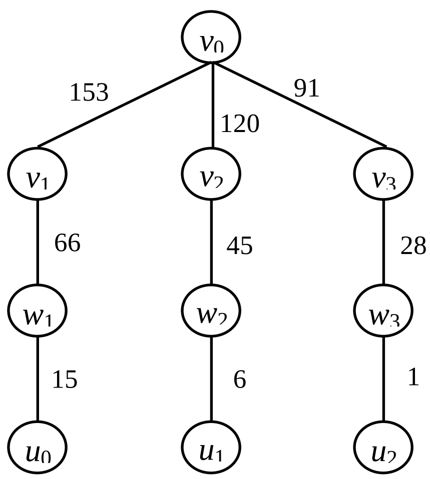


Figure 6

Theorem 4.6:

The Multi-star Graphs $K_{1,n,n,n,n...n}(n \geq 1, m\ \ times\ n)$ are odd triangular graceful.

In the same manner as the previous two theorems, the above theorem can be proved.

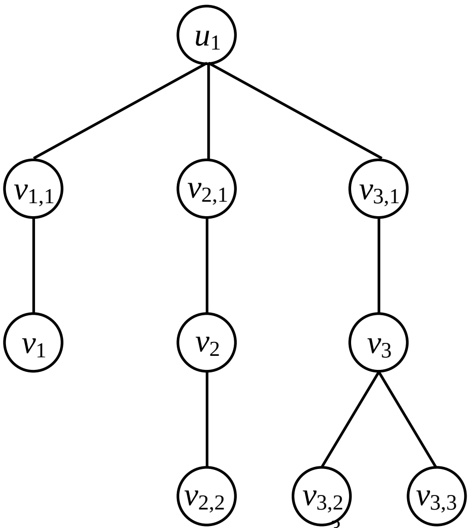
Here, we are introducing a new definition called odd triangular graceful number of a graph G.

Definition 4.7: Standard Banana tree ($\boldsymbol{SB^n}$)

Chen W C, Lü H I and Yeh Y N defined Banana tree as follows:

Standard banana tree is constructed by using the family of stars $(K_{1,1}, K_{1,2}, K_{1,3}, \dots K_{1,n})$ by connecting one leaf of each star to a single root vertex that is different from all star vertices and is denoted by $\boldsymbol{SB^n}$**.**

The graph shown below is Standard banana tree $\boldsymbol{SB^3}$



igure 7: SB^3

Theorem 4.8:

The Standard banana tree SB^n is odd triangular graceful only for $1 \leq n \leq 24$.

Proof:

Consider the standard banana tree $SB^n(1 \leq n \leq 24)$. Then,

$|V(SB^n)| = 1 + n + \frac{n(n+1)}{2} = \frac{n^2+3n}{2} + 1.$

Since SB^n is a tree, number edges in it is $\frac{n^2+3n}{2}$.

Let $V(SB^n) = \{u'_1\} \cup v'_{i,1}/i = 1$ to n $\cup\ v'_i$/i = 1,2,3...n

$$\cup \{v'_{2,2}, v'_{3,2}, v'_{3,3}, v'_{4,2}, v'_{4,3}, v'_{4,4}, \dots v'_{n,2}, v'_{n,3}, \dots v'_{n,n}\}$$

Define $\theta: V(SB^n) \to \{0,1,2,3 \dots T_{n^2+3n-1}\}$by

$$\theta(u'_1) = 0$$

$$\theta(v'_{i,1}) = T_{n^2+3n-(2i-1)} \qquad (i = 1,2,3\ldots n)$$

$$\theta(v'_i) = T_{n^2+3n-(2i-1)} - T_{n^2+n-(2i-1)} \qquad (i = 1,2,3,\ldots n)$$

$$\theta(v'_{2,2}) = \theta(v'_2) - T_{n^2-n-1}$$

$$\theta(v'_3, k) = \theta(v'_k) - T_{n^2-n-(2k-1)} \quad k = 2,3$$

$$\theta(v'_{4,k}) = \theta(v'_4) - T_{n^2-n-(2k+3)} \qquad k = 2,3,4$$

In general $\theta(v'_{j,k}) = \theta(v'_j) - T_{j^2-j-(2k+j^2-3j-1)}$ $2 \le k \le j,$ j = 3,4,5…n

The distinct vertex labels indicate that the function above is one -one.

Furthermore, the induced function on the set edges $\bar{\theta}$ on the set edges is injective. Since the induced function gives the label on its edges as $\overline{\theta(xy)} = |\theta(x) - \theta(y)|$ for every edge $xy \in E(SB^n)$

When the induced function gives the label on its edges, it is injective.

We checked infectivity by writing the python code and noticed that it is valid when and it fails when n=25.Python code and a sample output is exhibited below :

```
n=int(input())
p=((n*n)+(3*n))/2+1
q=p-1
u1=0
rows, cols = (n+1, n+1)
v = []
ver=[]
vv=[[0]*cols]*rows
ver.append(0)
v.append(0)
for i in range(1,n+1):
    k=(n*n)+(3*n)-(2*i-1)
    T=int((k*(k+1))/2)
    v.append(T)
    k1=(n*n)+n-(2*i-1)
    T1=int((k1*(k1+1))/2)
    ver.append(T-T1)
    l=(n*n)-n-1
    T2=int((l*(l+1))/2)
v22=ver[2]-T2
for j in range(3,n+1):
    for k in range(2,j+1):
        s=(j*j)-j-((2*k)+(j*j)-(3*j)-1)
        x=int((s*(s+1))/2)
        vv[j][k]=ver[j]-x
print("u1: "+str(u1))
for i in range(1,n+1):
    print("V"+str(i)+",1: "+str(v[i]))
for i in range(1,n+1):
    print("V"+str(i)+": "+str(ver[i]))
print("V2,2: "+str(v22))
for j in range(3,n+1):
    for k in range(2,j+1):
        print("V"+str(j)+","+str(k)+": "+str(vv[j][k]))
```

Sample Output: when n=3

```
3
u1: 0
V1,1: 153
V2,1: 120
V3,1: 91
V1: 87
V2: 75
V3: 63
V2,2: 60
V3,2: 57
V3,3: 62
```

Hence, SB^n admits OTGL, making SB^n odd triangular graceful.

Definition 4.9: Odd Triangular graceful number

In the case of a simple graph G that is not odd triangular graceful, the *odd triangular graceful number* of G denoted by $O_g^t n(G)$ is defined as the number of minimum vertices removed from G in order to make the resulting graph odd triangular graceful.

Theorem 4.10:

The odd triangular graceful number of Cycles $\boldsymbol{C_n}$ **(**$(\boldsymbol{n} \geq \boldsymbol{3})$ **is** $O_g^t n(C_n) = 1$.

Proof:

As Cycles $C_n(n \geq 3)$ are not odd triangular graceful. when anyone vertex is removed from C_n it will result in a Path. i.e., $C_n \backslash \{v\}$ for any vertex v it becomes a path which is odd triangular graceful, as all paths are odd triangular graceful .Thus, minimum number of vertices removed from C_n to make it odd graceful graph is one.

Hence, $O_g^t(C_n) = 1$.

Definition 4.11:

The *generalized Theta graph* $\theta(p_1, p_2, \ldots p_n)$ consists of $n \geq 3$paired internally disjoint paths of length $p_1, p_2, \ldots p_n$ that shares a pair of common end vertices u_0 and v_0. The Theta graph $\theta(p_1, p_2, p_3)$ when all $p_i's$are of same length say, Paths of length three then it can be denoted by $\theta(3P_4)$.The graph $\theta(3P_4)$ is shown below in figure 7

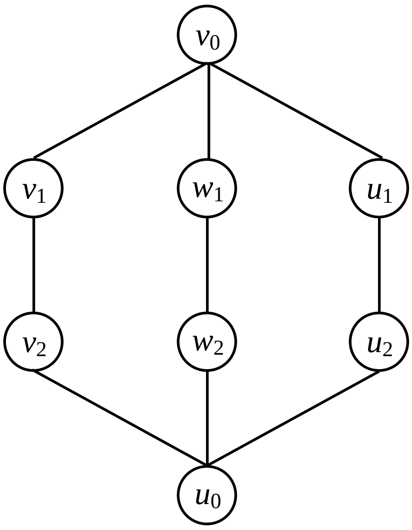


Figure 8

Theorem 4.12 The odd Triangular graceful number of the Theta graph $\theta(mP_n)$ is

$$O_g^t n(G) = 1.$$

Proof:

Theta graph $\theta(mP_n)$ is not odd triangular graceful as it does not admit

odd triangular graceful labeling . If one of the end vertices u_0 or v_0 is removed from the Theta graph ,it becomes a Multi-star graph which is odd triangular graceful by theorem 4.5 proved above.

Hence,the odd triangular graceful number of the theta graph is $O_g^t(\theta(mP_n) = 1$

5. Conclusion

As part of this study, we established the *odd triangular gracefulness* of several families of graphs, as well as defined a new concept called the *odd triangular graceful number* and determined the same for few families of graphs. Our study will continue to test the odd triangular gracefulness of all trees, and find odd triangular graceful numbers for families of graphs such as Lobster, planar graphs, cycle-related graphs, etc,.

References

[1] Devaraj .J**.,** A Study on Different Classes of Graphs and their Labeling, Ph.D Thesis, Kerala University (2002).

[2] Golomb S.W., How To Number a Graph in Graph Theory and Computing, R.C. Read, ed., Academic Press, New York (1972) 23 – 37.

[3] Rosa .A., On Certain Valuations of the Vertices of a Graph Theory of Graphs Gorden and Breach, N.Y. and Dunod Paris (1967) 349- 355.

[4] S. Selestin Lina and S. Asha , Odd triangular graceful labeling on simple graphs, Malaya Journal of Matematik, Vol. 8, No. 4, 1574-1576, 2020

[5] S.Manikandan and S.Mangaleswaran, Odd Triangular Graceful Labeling of Few classes of Trees (Presented in RETRIM'21 International Conference) Design Engineering, ISSN:0011-9342 | Year 2021 Issue 9| pages:5084-5096).