The Upper Connected Square Free Detour Number of a Graph

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Abstract

For any two vertices u and v in a connected graph $G = (V, E)$, the u - v path P is called a u - v square free path if no four vertices of P induce a square. The square free detour distance is the length of a longest u - v square free path in G. A u - v path of length is called a u - $v$square free detour. A subset S of V is called a square freedetour set if every vertex of G lies on a u - v square free detour joining a pair of vertices of S. The square free detour of G is the minimum order of its square free detour sets. A square free detour set S of G is called a minimal square free detour set if no proper subset of S is a square free detour set of G. The upper square free detour numberof G is the maximum cardinality of a minimal square free detour set of G. We introduce the upper connected square free detour number and determine the upper connected square free detour number of certain classes of graphs. Further, we investigate the bounds for it and characterize the graphs which realize these bounds. We show that there is no "Intermediate Value Theorem" for minimal connected square free detour sets.

Keywords: upper square free detour number; minimal square free detour set; minimal connected square free detour set; upper connected square free detour number.

1 Introduction

By a graph$G = (V, E)$, we mean a finite undirected connected simple graph. For basic definitions and terminologies, we refer to Chartrand et al. [6]. The concept of geodetic number was introduced by Harary et al. [1], [7]. For any vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of the shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic. A set $S \subseteq V$ is called geodetic set of G if every vertex of G lies on a geodesic joining a pair of vertices of S. The geodetic number $g(G)$ of G is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is called a geodetic basis of G. The concept of detour number was introduced by Chartrand et al. [4], [5]. The detour distance$D(u, v)$is the length of the longest $u - v$ path in G.A $u - v$ path of length $D(u, v)$ is called a u − v detour. A set $S \subseteq V$ is called detour set of G if every vertex of G lies on a detour joining a pair of vertices of S. The detour number $dn(G)$ of G is the minimum order of its detour sets and any detour set of order $dn(G)$ is called a detour basis of G. The concept of connected detour number was introduced by Santhakumaran and Athisayanathan [13]. The triangle free detour concept was studied by Sethu Ramalingam et al. [10], [11], [16]. The square free detour eccentricity $e_{\square f}(u)$ of a vertex u in G is the maximum square free detour distance from$u$ to a vertex of G. The square free detour radius, $R_{\square f}$ of G is the minimum square free detour eccentricity among the vertices of G, while the square free detour diameter, $D_{\square f}$of G is the maximum square free detour eccentricity among the vertices of G. A set $S \subseteq V$ is called a square free detour set of G if every vertex of G lies on a square free detour joining a pair of vertices of S. The square free detour number $dn_{\square f}(G)$ of G is the minimum order of its triangle free detour sets and any square free detour set of order $dn_{\square f}(G)$ is called a square free detour basis of G. The square free detour number of a graph was introduced and studied by Priscilla Pacifica et al. [8], [12]. In this article, we introduce the upper connected square free detour number of a connected graph G and investigate certain results related to upper connected square free detour sets.

The significant concept that pervades the graph theory is that of distance. Detour distance is used in Hamiltonicity problems, connectivity and convexity problems in graphs, molecular problems in Chemistry and Channel assignment problems. The upper connected square free detour concept is widely applied in communication networks and radio technologies.

2 Preliminaries

Theorem 2.1 [8] Let $G = (V,\ E)$be a complete graph $K_n (n \geq 2)$ or a cycle $C_n (n \geq 3, n \neq 4)$. Then a set $S \subseteq V$ is a connected square free detour basis of G if and only if S consists of any two adjacent vertices of G.

Theorem 2.2 [12] Let $G = (V,\ E)$be an even cycle C_n of order $n \geq 4$. Then a set $S \subseteq V$ is a square free detour basis of G if and only if S consists of any two antipodal vertices of G.

Theorem 2.3 [12]Let $G = (V,\ E)$be a complete bipartite graph $K_{m,n} (2 \leq m \leq n)$ with partitions X and Y where $|X| = m, |Y| = n$. Then a set $S \subseteq V$ is a square free detour basis of G if and only if $S = X$.

Theorem 2.4 [12]Let $G = (V, E)$ be a wheel $W_n = K_1 + C_{n-1} (n \geq 10)$. Then a set $S \subseteq V$ is a square free detour basis of G if and only if S consists of

(i) the vertex of K_1 withany two adjacent vertices, when n is odd

(ii) the vertex of K_1 two antipodal vertices of C_{n-1}, when n is even.

Theorem 2.5 [8]All theend-vertices and cut-vertices of a connected graph G belong to every connected square free detour set of G.

3 Upper connected Square free detour number of a graph

Definition 3.1A connected square free detour set S in a connected graph G is called a minimal connected square free detour setof G if no proper subset of S is a connected square free detour set of G. Theupper connected square free detour number $cdn_{\square f}^{+}(G)$of G is the maximum cardinality of a minimal connected square free detour set of G.

Example 3.2 For the graph G given in Figure 3.1, $S_1 = \{v_1, v_2, v_7\}, S_2 = \{v_1, v_4, v_7\}$ and $S_3 = \{v_1, v_6, v_7\}$ are the minimal connected square free detour sets of G so that $cdn_{\square f}^{+}(G) = 3$.

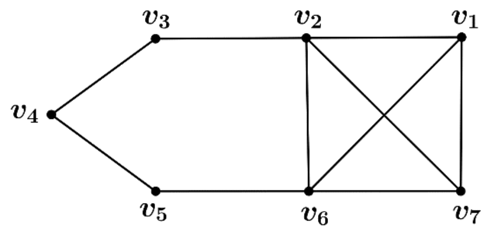

Figure 3.1: G

Remark 3.3 Every minimum connected square free detour set is a minimal connected square free detour set, but the converse is not true. For the graph G given in Figure 3.1, $S_2 = \{v_1, v_4, v_7\}$ is a minimal connected square free detour set of G but not a minimum connected square free detour set of G.

Theorem 3.4 For any connected graph $G, cdn_{\square f}(G) \leq cdn_{\square f}^{+}(G)$.

Proof. Let S be any connected square free detour basis of G. Then S is also a minimal connected square free detour set of G and hence the result follows.

Corollary 3.5 Let G be any connected graph. If $cdn_{\square f}(G) = n,$ then$cdn_{\square f}^{+}(G) = n$.

Remark 3.6 The bound in Theorem 3.4 is sharp. For the path P_3, $cdn_{\square f}(P_3) = cdn_{\square f}^{+}(P_3) = 3$. Also, for the graph G given in Figure 3.1, $cdn_{\square f}(G) = 2$ and $cdn_{\square f}^{+}(G) = 3$ and so$cdn_{\square f}(G) < cdn_{\square f}^{+}(G)$.

Now, we proceed to determine $cdn_{\square f}^{+}(G)$ for some classes of graphs.

Theorem 3.7 Let Gbe the complete graphK_n $(n \geq 2)$or the cycleC_n $(n \geq 3, \mathrm{n} \neq 4)$. Then a set S is a minimal connected square free detour set of G if and only if S consists of two adjacent vertices of G.

Proof. Let $G = K_n$or $G = C_n (n \geq 3, \mathrm{n} \neq 4)$. If S consists of any two adjacent vertices of G, then by Theorem 2.1,S is a connected square free detour set of G so that S is minimal. Conversely, assume that $S \subseteq V$is a minimal connected square free detour set of G with $|S| \geq 3$. Since $G[S]$ is connected, there exists a subset $S_1 = \{x, y\}$ of S such that x and y are adjacent vertices in G. Then by Theorem 2.1, S_1 is a connected square free detour set of G so that S is not a minimal connected square free detour set of G, which is a contradiction.

Theorem 3.8 Let G be the cycle C_4. Then a set S is a minimal connected square free detour set of G if and only if S consists of any three vertices of G.

Proof. Let $G = C_4$, where $V(C_4) = \{w, x, y, z\}$. If $S' = \{w, y\}$ is a set of two antipodal vertices of G, then by Theorem 2.2, S' is a square free detour basis of G but S is not connected. Therefore, assume $S = S' \cup \{a\}$, where $a \in \{x, z\}$. Therefore, $|S| = 3$. Thus S a is a connected square free detour basis of G, which is minimal. Conversely, assume that $S \subseteq V$ is a minimal connected square free detour set of G with $|S| \geq 4$. Since $G[S]$ is connected, there exists a subset S_1 of S such that S_1 consists of any three vertices in G. Then by Theorem 2.1, S_1 is a square free detour set of G that is obviously connected so that S is not a minimal connected square free detour set of G, which is a contradiction. Hence S consists of any three vertices of G.

Theorem 3.9 Let $G = (V, E)$ be a complete bipartite graph $K_{m,n} (2 \leq m \leq n)$ with partitions $|X| = m$ and $|Y| = n$. Then a set of $S \subseteq V$ is a minimal connected square free detour set of G if and only if S consists of n vertices of Y and exactly one vertex of X.

Proof. Let $G = K_{m,n} (2 \leq m \leq n)$ be a complete bipartite graph $K_{m,n} (2 \leq m \leq n)$ with bipartite sets X and Y. Let $X = \{x_1, x_2, x_3, \dots, x_m\}$ and $Y = \{y_1, y_2, y_3, \dots, y_n\}$. Let S^* be a set of n elements of Y and $v \in V$. Then every v lies on the square free detour $y_i\ v\ y_j$ of length 2 for any distinct i and $j (1 \leq i, j \leq n)$. Thus S^* is a square free detour basis of G. Since the bipartite set Y consists of n non-adjacent vertices, the set S^* is not connected. It is clear that $S = S^* \cup \{x : x \in X\}$ is a connected square free detour set of G with $|S| = n + 1$. Now we claim that S is minimal. If S_1 is any subset of S, then by Theorem 2.3, S_1 is not a square free detour set of G. Hence S is a minimal connected square free detour set of G.

Conversely, assume that S is a minimal connected square free detour set of G. Let us consider that $|S| \geq n + 2$. Since $G[S]$ is connected there exists a detour path containing the vertices of X and Y alternatively which induces a square. This results in a contradiction. Hence $|S| = n + 1$ and hence the minimal connected square free detour set S consists of n vertices of Y and exactly one vertex of X.

Theorem 3.10 Let $G = (V, E)$ be a wheel $W_n = K_1 + C_{n-1}$ $(n \geq 7)$. Then a set $S \subseteq V$ is a minimal connected square free detour set of G if and only if S consists of the hub of the wheel withthree independent non-antipodal vertices of C_{n-1}.

Proof. Let G be a wheel $W_n = K_1 + C_{n-1} (n \geq 7)$. Let x_0 be the hub of the wheel and $V(C_{n-1}) = \{x_1, x_3, \dots, x_{n-1}\}$. Let $S^* = \{x_i, x_j, x_k : 1 \leq i, j, k \leq n - 1; i \neq j \neq k\}$ be a set of three non-antipodal vertices of C_{n-1}. Let $S = S^* \cup \{x_0\}$, then as in the first part of the Theorem H, S is a square free detour set of G. Now, we claim that S is minimal. If $S^{**} = \{x_p, x_q, x_r\}$ is any subset of S, then by Theorem C, S^{**} is not a square free detour set of G so that S is a minimal square free detour set of G.

Conversely, assume that $S \subseteq V$ is a minimal square free detour set of G. If $|S| = 3, S$ consists of three vertices of G. Let $|S| \geq 4$. Then by Theorem C, any subset $S' = \{x_i, x_j, x_k : 1 \leq i, j, k \leq n - 1; i \neq j \neq k\}$ of S is a square free detour set of G so that $S$is not a minimal square free detour set of G, which is a contradiction. Thus S consists of the hub of the wheel with any three independent non-antipodal vertices of C_{n-1}.

Theorem 3.11 Let $G = (V, E)$ be a wheel $W_n = K_1 + C_{n-1}(n = 4, 5, 6)$. Then a set $S \subseteq V$ is a minimal connected square free detour set of G if and only if S consists of

(i) any two adjacent vertices of $C_3$when $n = 4$

(ii) any two antipodal vertices of $C_4$with the hub of the wheel when $n = 5$

(iii) any two adjacent vertices of $C_5$with the hub of the wheel when $n = 6$.

Proof. This follows from Theorems 2.5 and 3.9.

Corollary 3.12 Let $G = (V, E)$be a connected graph.

(a) If G is the path P_n or the tree T with n vertices, then$cdn^{+}_{\square f}(G)$= n.
(b) If G is the complete graph K_n, then$cdn^{+}_{\square f}(G)$ = 2.
(c) If G is the complete bipartite graph$K_{m,n}(2 \leq m \leq n)$, then $cdn^{+}_{\square f}(G) = n + 1$.
(d) If G is the cycle $C_n(n \geq 3)$, then$cdn^{+}_{\square f}(G) = \begin{cases} 3, & n = 4 \\ 2 & otherwise \end{cases}$
(e) If G is the wheel $W_n(n \geq 4)$, then $cdn^{+}_{\square f}(G) = \begin{cases} 2, & n = 4 \\ 3, & n = 5, 6 \\ 4, & n \geq 7 \end{cases}$

Proof. (a)This follows from Theorems 2.5.
(b) This follows from Theorem 3.7.
(c) This follows from Theorem 3.9.
(d) This follows from Theorems 3.7 and 3.8.
(e) This follows from Theorems 3.10 and 3.11.

Theorem 3.13 For every pair α, β of positive integers with $5 \leq \alpha \leq \beta$, there exists a connected graph G with$cdn_{\square f}(G) = \alpha$ and $cdn^{+}_{\square f}(G) = \beta$.

Proof. Let 5≤ $\alpha = \beta$. Then by Theorem 2.5 and Corollary 3.12(a), $cdn_{\square f}(T) = cdn^{+}(T) = \alpha$ for any tree Twithα vertices, Let $5 \leq \alpha < \beta$. Let H be the graph obtained from the cycle $C_{\beta-\alpha+4}$: $x_1, x_2, \ldots, x_{\beta-\alpha+4}, x_1$ of order$\beta - \alpha + 4$ by adding$\alpha - 3$ newvertices $y_1, y_2, \ldots, y_{\alpha-3}$ and joining $y_1$to x_1 and each $y_i(2 \leq i \leq \alpha - 3)$ to $x_{\beta-\alpha+3}$ of $C_{\beta-\alpha+4}$. Let G be a graph derived from H by adding z_0 and joining to $y_1$to $x_1$and $x_{\beta-\alpha+3}$. The graph G is connected of order $\beta + 2$ and is shown in Figure 3.2.Let $A = \{x_2, x_3, \ldots, x_{\beta-\alpha+2}\}$, $B = \{y_1, y_2, \ldots, y_{\alpha-3}, x_1, x_{\beta-\alpha+3}\}$and $C = \{z_0, x_{\beta-\alpha+4}\}$.

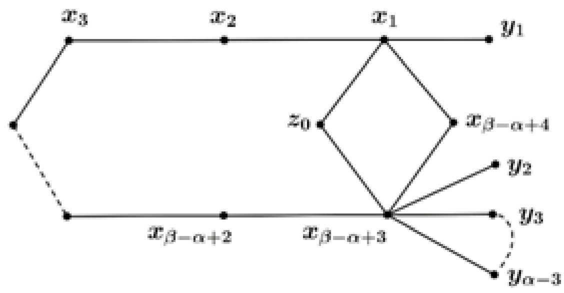

Figure 3.2: G

First, we show that $cdn_{\square f}(G) = \alpha$. By Theorem 2.5, every connected square free detour set of G contains B. Clearly B is not a connected square free detour set of G and so $cdn_{\square f}(G) \geq |B| + 1 = \alpha$. On the other hand, it is clear that the set $S = B \cup \{z\}$ where $z \in C$, is a connected square free detour set of G and so $cdn_{\square f}(G) \leq \alpha$. Therefore $cdn_{\square f}(G) = \alpha$.

Now, we show that $cdn^{+}_{\square f}(G) = \beta$. Let $S^* = A \cup B$. Then it is clear that $S^*$is a connected square free detour set of G . We show that S^* is a minimal connected square free detour set of G. Assume, to the contrary, that S^* is not a minimal connected square free detour set of G. Then there is a proper subset P of S^*such that P is a connected square free detour set of G. Since P is a proper subset of S^*, there exists a vertex $x \in S^*$ and $x \notin P$. By Theorem 2.5, every connected square free detour set contains B and so we must have $x = x_i \in A$ for some $i(2 \leq i \leq \beta - \alpha + 3)$. Then it is clear that $G[P]$is not connected and so P is not a connected square free detour set of G, which is a contradiction. Thus $S^*$is a minimal connected square free detour set of G and so $cdn^{+}_{\square f}(G) \geq |S^*| = \beta$. Now, if $cdn^{+}_{\square f}(G) > \beta$, then let S^+ be a minimal connected square free detour set of G with $|S^+| \geq \beta + 1$. Since G has $\beta + 2$ elements and S^* is a minimal connected square free detour set of G and so S^+ is not a minimal connected square free detour set of G, which is a contradiction. Hence $cdn^{+}_{\square f}(G) = \beta$.

Remark 3.14 The graph G in Figure 3.2 contains two minimal connected square free detour sets namely S and S^*. Hence this example shows that there is no "Intermediate Value Theorem" for minimal connected square free detour sets, that is, if t is an integer such that $cdn_{\square f}(G) < t < cdn^{+}_{\square f}(G)$, then there need not exist a minimal connected square free detour set of cardinality t in G.

Using the structure of the graph G constructed in the proof of Theorem 3.13, we can obtain a graph M_n of order n with $cdn_{\square f}(G)$= 5 and $cdn^{+}_{\square f}(G) = n - 2$ for all $n \geq 8$. Thus we have the following.

Theorem 3.15 There is an infinite sequence$\{M_n\}$of connected graphs$M_n$of order$n \geq 8$such that$cdn_{\square f}(M_n) = 5, cdn^{+}_{\square f}(M_n) = n - 2, \lim_{n\to\infty} \frac{cdn\,(M_n)}{n} = 0and\lim_{n\to\infty} \frac{cdn^+(M_n)}{n} = 1$.

Proof. Let M_n be the graph generated from the cycle $C_{n-3}: x_1, x_2, \ldots, x_{n-3}, x_1$ of order $n-3$ by adding three new vertices y_1, y_2, z_0 and joining y_1 to $x_1$and y_2 to x_{n-4} of C_{n-3} and z_0 to both x_1 and x_{n-4}. Then M_n is a connected graph of order n and is shown in Figure 3.3.

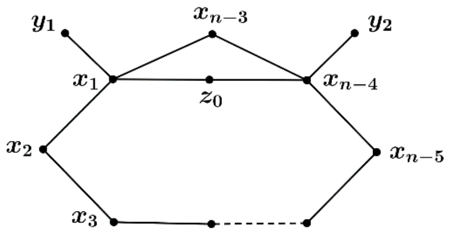

Figure 3.3: M_n

Let $A = \{x_2, x_3, \ldots, x_{n-5}\}, B = (y_1, y_2, x_1, x_{n-4}\}$ and $C = \{z_0, x_{n-2}\}$. By the proof of Theorem 3.13, it is clear that the graph M_n contains exactly two minimal connected square free detour sets namely S and S^*so that $cdn^+_{\square f}(M_n) = n-2$ and $cdn_{\square f}(M_n) = 5$. Hence the theorem follows.

4 Conclusion

In this article, we have introduced minimal connected square free detour sets and discussed some of its properties. Further, the upper connected square free detour number of some standard graphs are studied and the relations with connected square free detour number are derived. This concept can be developed to incorporate various detour distance considerations in a subsequent investigation.

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