Relatively Prime Detour Domination Number of a Graph

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Abstract

The concept of relatively prime detour domination number of a graph is introduced in this paper. If a set S is a detour dominating set with at least two members, and a pair of vertices u and w such that $(deg(u), deg(w)) = 1$, it is said to be a relatively prime detour dominating set of G. The lowest cardinality of a relatively prime detour dominating set in G is known as the relatively prime detour domination number and it is denoted by $\gamma_{rpdn}(G)$.

Keywords : Domination number, Detour number, Detour domination number, Relatively prime domination number, Relatively prime detour domination number.

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1 Introduction

We refer to a finite undirected graph with many edges and no loops as $G = (V, E)$. We consider connected graphs that have two or more vertices. The order $|V|$ and size $|E|$ of G are denoted by p and q respectively. For graph theoretical terms, we refer to Harary[3]. The detour distance $D(x, y)$ for vertices x and y is the longest $x - y$ path within a connected graph. A path of length $x - y$[1] is a detour of length $D(x, y)$. Every vertex x in the set G, commonly known as a detour set, is on a detour connecting two vertices in that set. A minimum detour set is defined as any detour set of order $dn(G)$ [2].

It is considered to be a dominating set of the graph G if every vertex that is not in S is adjacent to at least one vertex that is in S. A γ-set of G is referred to be any order $\gamma(G)$ dominating set. The lowest order of G's dominating sets is represented by the domination number $\gamma(G)$ [4]. The idea of detour domination number of a graph was first established in [5,7]. Relatively prime dominating sets in graphs was a concept that Jayasekaran et al. introduced in [6]. In this paper, we indicate that γ_{dn}-set is a minimum detour dominating set. In this paper, we introduce the concept relatively prime detour domination number of a graph and find the number $\gamma_{rpdn}(G)$.

2 Preliminaries

Theorem 2.1 *[5] Every end vertex of a graph G belongs to every detour dominating set of G.*

3 Relatively Prime Detour Dominating Set

Definition 3.1 *A set $S \subseteq V$ is said be relatively prime detour dominating set of a graph G if it is a detour set and a dominating set with at least two elements and for every pair of vertices u and v such that $(deg(u), deg(v)) = 1$. The minimum cardinality of a relatively prime detour dominating set is called the relatively prime detour domination number of a graph G and is denoted by $\gamma_{rpdn}(G)$. If the relatively prime detour dominating set does not exist, then the relatively prime detour domination number is zero.*

Example 3.2 *For the graph G given in 3.1, $S = \{v_1, v_3, v_6\}$ is a detour dominating set and $(deg(v_1), deg(v_3)) = (3,1) = 1, (deg(v_1), deg(v_6)) = 1,\ (deg(v_3), deg(v_6)) = 1$. Therefore S is a minimum relatively prime detour dominating set and hence $\gamma_{rpdn}(G) = 3$.*

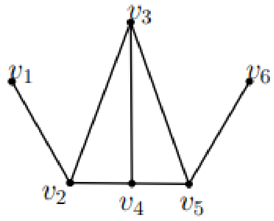

Figure 3.1

Theorem 3.3 *Every end vertex of a graph G belong to every relatively prime detour dominating set of G.*

Proof. Every relatively prime detour dominating set is a detour dominating set of G and so the result follows from Theorem 2.1.

Theorem 3.4 *Let G be a connected graph of order p. If $\gamma_{rpdn}(G)$ exists, then $dn(G) \leq \gamma_{rpdn}(G) \leq p$.*

Proof. Let G be a connected graph of order p such that $\gamma_{rpdn}(G)$ exists. Since every relatively prime detour dominating set is a detour set, $dn(G) \leq \gamma_{rpdn}(G)$. Also any relatively prime detour set can have at most p vertices and hence $\gamma_{rpdn}(G) \leq p$. Thus $dn(G) \leq \gamma_{rpdn}(G) \leq p$.

Remark 3.5 *For the path P_2, $dn(G) = \gamma_{rpdn}(G) = p = 2$. Hence all the inequalities in Theorem 3.4 become sharp. Now consider the graph in figure 3.1. Here $S = \{v_1, v_6\}$ is a minimum detour set of G and so $dn(G) = 2$. The set $S_1 = \{v_1, v_3, v_6\}$ is a minimum relatively prime detour dominating set of G and so $\gamma_{rpdn}(G) = 3$ and number of elements $p = 6$. Thus all the inequalities in Theorem 3.4 become strict.*

Theorem 3.6 *If G is a path graph P_p, then* $\gamma_{rpdn}(P_p) = \begin{cases} 2 & if\ \ 2 \leq p \leq 4 \\ 3 & if\ \ 4 < p \leq 7 \\ 0 & otherwise \end{cases}$

Proof. Let $v_1 v_2 \ldots v_p$ be the path P_p. We consider the following three cases.
Case 1 . $2 \leq p \leq 4$

Let S be a detour dominating set of P_p. By Theorem 3.3 ,, $\{v_1, v_p\} \subseteq S$. Clearly $\{v_1, v_p\}$ itself is a minimum detour dominating set of P_p and $\left(deg(v_1), deg(v_p)\right) = 1$, $S = \{v_1, v_p\}$ is a minimum relatively prime detour dominating set and hence $\gamma_{rpdn}\left(P_p\right) = 2$.

Case 2. $5 \leq p \leq 7$

If $n = 5,6$, then $\{v_1, v_3, v_p\}$ is a minimum detour dominating set. Also $\left(deg(v_1), deg(v_3)\right) = (1,2) = 1$, $\left(deg(v_1), deg(v_p)\right) = (1,1) = 1$ and $\left(deg(v_3), deg(v_p)\right) = (2,1) = 1$. This implies that $\{v_1, v_3, v_p\}$ is a relatively prime minimum detour dominating set and hence $\gamma_{rpdn}\left(P_p\right) = 3$.

If $p = 7$, then $\{v_1, v_4, v_7\}$ is a minimum detour dominating set. Also $\left(deg(v_1), deg(v_4)\right)$= $\left(deg(v_4), deg(v_p)\right) = \left(deg(v_1), deg(v_p)\right) = 1$. Therefore $\{v_1, v_4, v_7\}$ is a minimum relatively prime detour dominating set and hence $\gamma_{rpdn}\left(P_p\right) = 3$.

Case 3. $p \geq 8$

Clearly any dominating set contains at least two internal vertices $v_i, v_j, 3 \leq i \neq j \leq n-2$ and $\left(deg(v_i), deg(v_j)\right) = 2$ which implies that $\gamma_{rpdn}\left(P_p\right) = 0$.

The theorem follows from above three cases.

Theorem 3.7 *If G is a star graph $K_{1,p-1} (p \geq 3)$, then $\gamma_{rpdn}\left(K_{1,p-1}\right) = p - 1$.*

Proof. Let G be the star $K_{1,p-1}$ with $V\left(K_{1,p-1}\right) = \{v, v_i : 1 \leq i \leq p-1\}$ and $E\left(K_{1,p-1}\right) = \{vv_i : 1 \leq i \leq p-1\}$. Let S be a detour dominating set of $K_{1,p-1}$. By Theorem 3.3, $\{v_1, v_2, \dots, v_{p-1}\} \subseteq S$. Clearly, $\{v_1, v_2, \dots, v_{p-1}\}$ itself is a minimum detour dominating set of $K_{1,p-1}$ and $\left(deg(v_i), deg(v_j)\right) = 1$ for $1 \leq i \neq j \leq p-1$, $S = \{v_1, v_2, \dots, v_{p-1}\}$ is a minimum relatively prime detour dominating set and hence $\gamma_{rpdn}\left(K_{1,p-1}\right) = p - 1$.

Theorem 3.8 *If G is a bistar graph $B_{m,n}$, then $\gamma_{rpdn}\left(B_{m,n}\right) = m + n$.*

Proof. Let $G = B_{m,n}$ with $V\left(B_{m,n}\right) = \{v, v_i, u, u_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E\left(B_{m,n}\right) = \{uv, uu_j, vv_i : 1 \leq i \leq m, 1 \leq j \leq n\}$ and so $|V(B_{m,n})| = m + n + 2$. Let S be a detour dominating set of $B_{m,n}$. By Theorem 3.3, $\{v_i, u_j : 1 \leq i \leq m, 1 \leq j \leq n\} \subseteq S$. Clearly $\{v_1, v_2, \dots, v_m, u_1, u_2, \dots, u_n\}$ itself is a minimum detour dominating set of $B_{m,n}$, and $\left(deg(v_i), deg(v_j)\right) = \left(deg(u_x), deg(u_y)\right) = \left(deg(v_i), deg(u_x)\right) = 1$ where $1 \leq i \neq j \leq m, 1 \leq x \neq y \leq n$, $S = \{v_1, v_2, \dots, v_m, u_1, u_2, \dots, u_n\}$ is a relatively prime detour dominating set and hence $\gamma_{rpdn}\left(B_{m,n}\right) = m + n$.

Theorem 3.9 *If G is the complete graph $K_p (p \geq 2)$, then*

$$\gamma_{rpdn}\left(K_p\right) = \begin{cases} 2 & if p = 2 \\ 0 & if p > 2 \end{cases}.$$

Proof. Any two vertices in $K_p,\ p \geq 2$ is adjacent. Hence a minimum detour dominating set is $\{v_i, v_j\}$, $1 \leq i \neq j \leq p$ and $\left(deg(v_i), deg(v_j)\right) = (p-1, p-1) = p-1$. If $p = 2$, then $\{v_1, v_2\}$ is the relatively prime detour dominating set and hence $\gamma_{rpdn}\left(K_p\right) = 2$ and if $p > 2$, then $\{v_i, v_j\}$ is not a relatively prime detour dominating set and hence $\gamma_{rpdn}\left(K_p\right) = 0$.

Theorem 3.10 *If G is the complete bipartite graph $K_{m,n}$, then*

$$\gamma_{rpdn}\left(K_{m,n}\right) = \begin{cases} 2 & if m = n = 1 \; and \; (m,n) = 1 \; where m,n \geq 2 \\ n & if m = 1, n \geq 2 \\ m & if n = 1, m \geq 2 \\ 0 & if \quad (m,n) \neq 1 \; and m,n \geq 2. \end{cases}$$

Proof. Let $V_1 \cup V_2$ be the bipartition of $V(K_{m,n})$ where $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$ and so $|V(K_{m,n})| = m + n$.
Case 1. $m = n = 1$
The graph $K_{m,n}$ is P_2 . By Theorem 3.6, $\gamma_{rpdn}\left(K_{m,n}\right) = 2$.
Case 2. $m = 1$ and $n \geq 2$
The graph is $K_{1,n}$. By Theorem 3.7, $\gamma_{rpdn}\left(K_{1,n}\right) = n$.
Case 3. $n = 1$ and $m \geq 2$
The graph is $K_{m,1}$. Also $K_{m,1} \simeq K_{1,m}$. By Theorem 3.7, $\gamma_{rpdn}\left(K_{m,1}\right) = m$.
Case 4. $m = n \geq 2$
A minimum detour dominating set is $\{u_i, v_j\}$ where $1 \leq i, j \leq n$ and $\left(deg(u_i), deg(v_j)\right)$ = $m \geq 2$. This implies that $K_{m,n}$ has no relatively prime detour dominating set and hence $\gamma_{rpdn}\left(K_{m,n}\right) = 0$.
Case 5. $m \neq n$ and $m,n \geq 2$
A minimum detour dominating set is $\{u_i, v_j\}$ and $\left(deg(u_i), deg(v_j)\right) = (m,n)$. Clearly $S = \{u_i, v_j\}$ is a minimum relatively prime detour dominating set if $(m,n) = 1$ and not a relatively prime detour dominatind set if $(m,n) \neq 1$. Hence $\gamma_{rpdn}\left(K_{m,n}\right) = 2$ if $(m,n) = 1$ and $\gamma_{rpdn}\left(K_{m,n}\right) = 0$ if $(m,n) \neq 1$

The theorem follows from cases 1, 2, 3 , 4 and 5.

Theorem 3.11 *If G is a wheel graph W_p, then*

$$\gamma_{rpdn}\left(W_p\right) = \begin{cases} 2 & if p \not\equiv 1 (mod\ 3) \\ 0 & otherwise \end{cases}.$$

Proof. Let $v_1 v_2 \ldots v_{p-1} v_1$ be the outer cycle C_{p-1} and v be the central vertex of W_p. Then $deg(v) = p - 1$ and $deg(v_i) = 3$ for each $i \in \{1, 2, \ldots, p-1\}$.

Consider, $S = \{v, v_i\}$, $1 \leq i \leq p-1$. Then S is a minimum detour dominating set of W_p. Now $(deg(v), deg(v_i)) = (p-1, 3) = 1$ if and only if $p - 1$ is not a multiple of 3 if and only if $p \not\equiv 1 (mod\ 3)$. Hence $\gamma_{rpdn}\left(W_p\right) = 2$ if and only if $p \not\equiv 1 (mod\ 3)$.

Theorem 3.12 *If G is a helm graph H_n, then $\gamma_{rpdn}\left(H_n\right) = n$.*

Proof. Let $v_1v_2\ldots v_{n-1}v_1$ be the cycle C_n. Add a vertex v which is adjacent to $v_i, 1 \leq i \leq n-1$. The resultant graph is the wheel W_n. For $1 \leq i \leq n-1$, add u_i which is adjacent to v_i. The resultant graph G is the helm graph H_n and so $|V(H_n)| = 2n-1$. Then $deg(v) = n-1$, $deg(v_i) = 4$ and $deg(u_i) = 1$ for each $i = 1,2,\ldots,n-1$.

Let $S = \{v, u_1, u_2, \ldots, u_{n-1}\}$. Then S is a minimum detour dominating set of H_n. Also $(deg(v), deg(u_i)) = (n-1,1) = 1$ for each $i = 1,2,\ldots,n-1$ and $\left(deg(u_i), deg(u_j)\right) = 1$ for $1 \leq i \neq j \leq n-1$. This implies that $\{v, u_1, u_2, \ldots, u_{n-1}\}$ is a relatively prime detour dominating set and hence $\gamma_{rpdn}(H_n) = n$.

Theorem 3.13 *If G is a fan graph $F_{1,n}$, then* $\gamma_{rpdn}\left(F_{1,n}\right) = \begin{cases} 2 & if \;\; n \; is \; odd \\ 0 & if \;\; n \; is \; even \end{cases}$.

Proof. Let $v_1v_2\ldots v_n$ be a path P_n and let v be the K_1. The join $K_1 + P_n$ is the fan graph $F_{1,n}$ and so $|V(F_{1,n})| = n+1$. Clearly $deg(v) = n$, $deg(v_1) = deg(v_n) = 2$ and $deg(v_i) = 3$ for $2 \leq i \leq n-1$.

Let S be a minimum detour dominating set of $F_{1,n}$. Then S is either $\{v, v_1\}$ or $\{v, v_n\}$.

Case 1. n is odd

Then $(deg(v), deg(v_1)) = (n, 2) = 1$ and $(deg(v), deg(v_n)) = (n, 2) = 1$. This implies that either $\{v, v_1\}$ or $\{v, v_n\}$ is a minimum relatively prime detour dominating set and hence $\gamma_{rpdn}\left(F_{1,n}\right) = 2$.

Case 2. n is even

Then $(deg(v), deg(v_1)) = (n, 2) = 2$ and $(deg(v), deg(v_n)) = (n, 2) = 2$. This implies that $F_{1,n}$ has no relatively prime detour dominating set and hence $\gamma_{rpdn}\left(F_{1,n}\right) = 0$.

The theorem follows from cases 1 and 2.

Theorem 3.14 *For $n \geq 2$, $C_n \odot K_1$ then $\gamma_{rpdn}(C_n \odot K_1) = n$.*

Proof. Let $v_1v_2\ldots v_nv_1$ be the cycle C_n. For $1 \leq i \leq n$, add vertex u_i which is adjacent to v_i. The resultant graph is $C_n \odot K_1$ and so $|V(C_n \odot K_1)| = 2n$. Then $deg(u_i) = 1$ and $deg(v_i) = 3$ for $1 \leq i \leq n$. Let S be a detour dominating set of $C_n \odot K_1$. By Theorem 3.3, $\{u_1, u_2, \ldots, u_n\} \subseteq S$. Since $\{u_1, u_2, \ldots, u_n\}$ itself is a minimum detour dominating set of $C_n \odot K_1$, and $\left(deg(u_i), deg(u_j)\right) = (1,1) = 1$ for $1 \leq i \neq j \leq n$, $\{u_1, u_2, \ldots, u_n\}$ is a relatively prime detour dominating set and hence $\gamma_{rpdn}(G) = n$.

Theorem 3.15 *For $n \geq 2, \gamma_{rpdn}(P_n \odot K_1) = n$.*

Proof. Let $v_1v_2\ldots v_n$ be the path P_n. For $1 \leq i \leq n$, add vertex u_i which is adjacent to v_i. The resultant graph is $P_n \odot K_1$ and so $|V(P_n \odot K_1)| = 2n$. Then $deg(u_i) = 1$, $deg(v_1) = deg(v_n) = 2$ and $deg(v_j) = 3$ for $1 \leq i \leq n$ and $2 \leq j \leq n-1$. Let S be a detour dominating set of $P_n \odot K_1$. By Theorem 3.3, $\{u_1, u_2, \ldots, u_n\} \subseteq S$. Since $\{u_1, u_2, \ldots, u_n\}$ itself is a minimum detour dominating set of $P_n \odot K_1$, and $\left(deg(u_i), deg(u_j)\right) = 1$ for $1 \leq i \neq j \leq n$, $\{u_1, u_2, \ldots, u_n\}$ is the minimum relatively prime detour dominating set and hence $\gamma_{rpdn}(P_n \odot K1=n$.

Theorem 3.16 $\gamma_{rpdn}\left(K_{1,n-1} \odot K_1\right) = n$.

Proof. Let $v, v_1, v_2, \ldots, v_{n-1}$ be the vertices of $K_{1,n-1}$ with central vertex v. For $1 \leq i \leq n-1$, add u_i which is adjacent to v_1. Add vertex u which is adjacent to v. The resultant graph is $K_{1,n-1} \odot K_1 |(K_{1,n} \odot K_1)| = 2n$. Clearly $deg(u_i) = deg(u) = 1$, $deg(v_i) = 2$ and $deg(v) = n$ for $1 \leq i \leq n-1$. Let S be a minimum detour dominating set of $K_{1,n-1} \odot K_1$. By Theorem 3.3, $\{u, u_1, u_2, \ldots, u_{n-1}\} \subseteq S$. Since $\{u, u_1, u_2, \ldots, u_{n-1}\}$ itself is a minimum relatively prime detour dominating set of $K_{1,n-1} \odot K_1$ and $\left(deg(u_i), deg(u_j)\right) = \left(deg(u), deg(u_i)\right) = 1$ for $1 \leq i \neq j \leq n-1$, $\{u, u_1, u_2, \ldots, u_{n-1}\}$ is the minimum relatively prime detour dominating set and hence $\gamma_{rpdn}\left(K_{1,n-1} \odot K_1\right) = n$.

4 Conclusion

In this paper, we have found the relatively prime detour domination of some standard graphs like path graph, star graph, complete graph, wheel graph, helm graph.

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