Iterative Methods for Solving Nonlinear Equations

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Abstract
Solving such problems often involves the Newton technique. Nonlinear equations are the go-to solution for most issues in mathematics and engineering. Analysis of numerical data is a method of solving mathematical issues. As a result of the stimulating and fruitful research being done in this field, we propose and examine a novel iterative strategy for resolving nonlinear equations. The numerous methodologies, which are employed by researchers to produce higher order iterative methods such as functional approximation, sampling, composition, geometric and Adomian approaches, are given. the iterative techniques for finding multiple roots several iterative methods to solve systems of nonlinear equations are developed.

Keywords: Iterative Methods, Nonlinear Equations, Method, Mathematical Questions And Iterative Techniques

1. INTRODUCTION

The issue of determining the values of the independent variable x such that a given function f(x)=0 is one of the first numerical approximation problems and one of the most fundamental in numerical analysis. Solving such problems often involves the Newton technique. You may learn more about the background of this approach in. A number of writers have, in recent years, investigated strategies for addressing the nonlinear equations. Recent findings in numerical integration have highlighted the superiority of quadrature formula error boundaries over those of their equivalent approximation polynomial counterparts. The topic of whether or not these quadrature formulae may be used to derive techniques for solving nonlinear equations is an obvious one to ask. The relationship between quadrature formulae and nonlinear equations is well-established; for evidence, seeNonlinear equations are the go-to solution for most issues in mathematics and engineering. Numerical approaches for solving nonlinear equations have been the focus of much research and development by scientists and engineers. Many other iterative algorithms using different techniques, such as Taylor series, decomposition, quadrature formulae, homotopy, etc., have been suggested for this purpose; for examples, see and references therein. Convergence to these techniques is quadratic, cubic, or higher.

It is commonly recognised that nonlinear equations may be used to define a large class of linear and nonlinear problems that emerge in several areas of mathematics, including physical, biomedical, regional optimization, ecology, economics, and engineering sciences. We propose and critically examine a novel iterative approach for solving nonlinear equations, and we are inspired and motivated to do so by the continuing research activity in this field.Analyzing numerical data in mathematics entails using standard arithmetic operations such addition, subtraction, multiplication, division, and comparison to arrive at answers to mathematical questions. Numerical analysis and computers are inextricably linked due to the nature of the operations being performed by the former.

Since the advent of powerful digital computers, numerical approaches have become more important in the field of science and engineeringby the ongoing research activities in this area, wesuggest and analyze a new iterative method for solving nonlinear equations. To derive these iterativemethods, we show that the nonlinear function can

2. LITERATURE REVIEW

Obadah Said Solaiman et.al (2021) In this paper, we offer a novel iterative method for solving nonlinear equations. The first two stages of the new scheme are based on the authors' sixth-order modified Halley's approach, and the last stage is a Newton step, with appropriate approximations for the initial derivatives that have formed in the new scheme. Mathematica code is used to show that the new approach converges to the ninth order. Three functions and a first derivative need to be evaluated at each iteration in the provided approach. According to the Kung-Traub conjecture, this system is the best possible solution. The performance of the suggested approach is evaluated against that of other optimum methods of the same order on a set of test nonlinear problems.

GUL SANA et.al (2020) In order to find the solutions to nonlinear equations, we provide a series of iterative methods of the third and fourth order that use the quadrature formula and a decomposition method. Furthermore, we checked the convergence of the proposed iterative algorithms under a wide range of constraints. We demonstrate the correctness, efficiency, and usability of our methods using a battery of numerical examples.

Rubén villafuerte et.al (2019) In this study, an iterative Newton-type approach of three steps and fourth order is employed to solve the nonlinear equations that describe the load flow in electric power systems. With the suggested approach (N-1) non-linear equations are written and solved iteratively to compute the Voltage in each node of an electrical system. This study presents the theoretical underpinnings of the methodology and explains why it is effective. Results from IEEE test systems are compared with those obtained using the suggested technique, with an error of 0.5% at most. The results show that the suggested approach is a viable option for resolving load fluxes in electrical systems.

S. Parimala et.al (2018) In this work, we get modified versions of Ostrowski's techniques for resolving nonlinear equations of order eight and six, respectively. Both of the suggested approaches outperform Newton's method in terms of efficiency (1.516% and 1.566%, respectively) (1.414). Also, the numerical examples show that the suggested approaches need less iterations than Newton's method. Only a handful of different approaches are compared to the suggested approaches, and all of them need as many as or more iterations. The effectiveness of the new approaches is shown with the help of few instances.

3. METHODOLOGY

Various methods have been used by the researchers in order to produce I.M. Some effective methods for creating I.M. are listed below:

Functional Approximation

Approximating functions is a common task in numerical computing. An appropriate approximation may reduce the need for extensive computational effort. Interpolation is a crucial method for estimating functions in numerical analysis. Many functions are now only partially understood, and this incompleteness stems from the fact that they are only known experimentally at some stages. For approximating such functions, interpolation provides a practical solution. It all comes down to trying to make the polynomial curve fit. The fact that they may be used to consistently approximate continuous functions is one of their main advantages. It is possible to find a polynomial that is as "near" to a given function as needed, given that the function is defined and continuous over a closed limited interval. This finding has been formalized in the following theorem:

Theorem The approximation theorem of Weierstrass Let's assume f is well-defined and continuous on the interval [a,b]. For each $\varepsilon > 0$, If there is a polynomial P(x) with the characteristic

$$f(x)-P(x)| < \varepsilon, \text{ for all } x \text{ in } [a,b].$$

The derivatives and indefinite integrals of polynomials are themselves polynomials, making them both easily calculable and useful in the approximation of functions. Polynomials are useful for approximating continuous functions for these reasons. The degree one polynomial, or linear polynomial, is used as an approximation in the Newton, secant, and Regula-Falsi procedures. Pade approximation is another method for approximating f(x), and it uses a rational function. In this case, we're interested in approximations of the type Pm(x)/Qk(x), where Pm(x) and Qk(x) are polynomials of degrees m and k, respectively. Many researchers in the field of functional approximation have recently developed higher-order approaches for locating the simple root of f(x) = 0. The hyperbolic approximation of f(x) was studied by Amat et al.

$$ay2 +bxy+y+cx+d = 0$$

When approximating a quadratic curve, Chun , and Sharma used the following form.

$$x\ 2 +ay2 +bx+cy+d = 0.$$

Amat et al., Chun, and Sharma produced one-parameter families of third-order algorithms by enforcing the tangency criteria and iterating from the x-axis intersection.

By using the quadratic equation, Jiang et al. were able to derive a second family of third-order algorithms with a single tuning parameter.

$$x\ 2 +ay+bx+cxy+d = 0$$

applying and utilising the x-intersection axis's as the following iteration's starting point. By approximating f(x) by the curve, Popovski was able to develop a third-order family of multipoint iteration algorithms without memory.

$$y(x) = a+b(x-c)^{d}$$

, and under certain circumstances. With this curve we can calculate the following iteration, xk+1.

Sampling Approach

The sampling method provides another another option for producing I.M. This method involves taking samples of the function or its derivative at strategic locations. In order to efficiently solve the problem, Traub demonstrated how to build multipoint iterative functions. Traub focused specifically on two related technique families. In the first family, f is assessed at some number of points while f ′ is evaluated at just one, but in the second family, the opposite is true: f ′ is evaluated at some number of points while f is evaluated at just one. He provided a definition for the prototypical family as

$$x_{k+1} = x_k - \sum_{i=1}^{p-1} a_i^p \omega_i^p(x_k),$$

Where

$$\omega_i^p(x_k) = f\left[x_k + \sum_{j=1}^{i-1} b_{i,j}^p \omega_j^p(x_k)\right] / f'(x_k), \ \ i > 1,$$

$$\omega_1^p(x_k) = f(x_k)/f'(x_k) \ \text{ and } \ a_i^p, \ b_{i,j}^p$$

are constants and the second family as

$$x_{k+1} = x_k - \sum_{i=1}^{p-1} c_i^p \Omega_i^p(x_k),$$

Where

$$\Omega_i^p(x_k) = f(x_k)/f'\left[x_k + \sum_{j=1}^{i-1} d_{i,j}^p \Omega_j^p(x_k)\right], \ \ i > 1,$$

$$\Omega_1^p(x_k) = f(x_k)/f'(x_k) \ \text{ and } \ c_i^p, \ d_{i,j}^p$$

Similar to Gaussian quadrature integration formulae and Runge-Kutta techniques for integrating ordinary differential equations, such methods have a number of free parameters that can be used to impose desirable constraints on the methods, such as requiring a certain order of asymptotic convergence for simple zeros of arbitrary function f. Traub gave special attention to the abbreviated version of first family

$$x_{k+1} = x_k - a_1\omega_1(x_k) - a_2\omega_2(x_k) - a_3\omega_3(x_k),$$

Where

$$\omega_1(x_k) = f(x_k)/f'(x_k), \ \ \omega_2(x_k) = f(x_k + \alpha\omega_1(x_k))/f'(x_k),$$
$$\omega_3(x_k) = f(x_k + \alpha_1\omega_1(x_k) + \alpha_2\omega_2(x_k))/f'(x_k).$$

For

$$a_3 = 0, \ a_1 = (\alpha^2 - \alpha - 1)/\alpha^2 \ \text{ and } \ a_2 = 1/\alpha^2,$$

meanings iterative family of the third order defined as

$$x_{k+1} = x_k - \frac{\alpha^2 - \alpha - 1}{\alpha^2}\frac{f(x_k)}{f'(x_k)} - \frac{1}{\alpha^2}\frac{f(x_k + \alpha f(x_k)/f'(x_k))}{f'(x_k)}, \ \ \alpha \neq 0.$$

Methods may be constructed for a variety of values of. Using (1.6.6), we see that when = 1,

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{f(x_k - f(x_k)/f'(x_k))}{f'(x_k)}.$$

Also, by adding an extra evaluation to the phrase a33(xk), it is feasible to achieve fourth order procedures. Traub also gave some thought to a shortened version of the second-family

$$x_{k+1} = x_k - c_1\Omega_1(x_k) - c_2\Omega_2(x_k) - c_3\Omega_3(x_k),$$

Where

$$\Omega_1(x_k) = f(x_k)/f'(x_k), \Omega_2(x_k) = f(x_k)/f'(x_k + \beta\Omega_1(x_k)),$$
$$\Omega_3(x_k) = f(x_k)/f'(x_k + \beta_1\Omega_1(x_k) + \beta_2\Omega_2(x_k)).$$

Composition Approach

Recently, the composition approach has been used to construct numerous higher order techniques. The modified Newton approach , in which f ′ is assessed at every other step, was one option that Traub examined. Put another way, let's say

$$w_k = x_k - \frac{f(x_k)}{f'(x_k)}, \qquad x_{k+1} = w_k - \frac{f(w_k)}{f'(x_k)}.$$

Combining the two equations given above yields

$$x_{k+1} = x_k - \frac{f(x_k) + f(x_k - f(x_k)/f'(x_k))}{f'(x_k)},$$

It is Potra-iterative Ptak's approach to the third order. ´

Using a combination of Newton's technique and the secant method, Traub proposed yet another iterative approach . The procedure is outlined as

$$w_k = x_k - \frac{f(x_k)}{f'(x_k)}, \qquad x_{k+1} = w_k - \left[\frac{w_k - x_k}{f(w_k) - f(x_k)}\right] f(w_k).$$

When the above equations are combined, we get

$$x_{k+1} = x_k - \frac{f(x_k)}{f(x_k) - f(x_k - f(x_k)/f'(x_k))} \frac{f(x_k)}{f'(x_k)}.$$

The aforementioned technique, known as the Newton-secant approach, converges cubically. Also, similar results are achieved in [126]. By combining the formulae in, Kou et al. developed a family of iterative algorithms, which they formulated as

$$x_{k+1} = x_k - \theta\frac{f(x_k) + f(x_k - f(x_k)/f'(x_k))}{f'(x_k)} - (1-\theta)\frac{f(x_k)}{f(x_k) - f(x_k - f(x_k)/f'(x_k))}\frac{f(x_k)}{f'(x_k)}.$$

Except in the special situation when = 1, all members of this family converge with order four.

Geometrical approach

The geometry used to explain classical approaches is well-known. If you know the point (x_k), where the function f(x_k) is defined, then you can use the Newton technique to locate the point where the tangent line to f(x_k) meets the x-axis. It was proposed by Traub that the geometric interpretation of the fourth order Ostrowski method [110] is the point where the line passing through (w_k, f(w_k)) and (1 2 (w_k + x_k), 1 2 f(x_k)) intersects the x-axis, with the latter point bisecting the segment of the tangent line at (x_k, f(x_k)) between (x_k, f(x_k)) (x_k). Specifically, describes the Ostrowski technique as

$$w_k = x_k - \frac{f(x_k)}{f'(x_k)},$$

$$x_{k+1} = w_k - \frac{f(x_k)}{f(x_k) - 2f(w_k)}\frac{f(w_k)}{f'(x_k)}.$$

Chun's geometrical development of a family of order three: Imagine that is a second-order iteration function. Let , be another function fulfilling (x)+ (x) 2 = x, i.e. x_k is the point that bisects the segment between [(x_k),0] and [(x_k),0] and choosing the next iteration as the intersection with the x-axis as the line through [(x_k), f((x_k))] which is parallel to the The formula for the iterative family of the third order is

$$x_{k+1} = \bar{\phi}(x_k) - \frac{f(\bar{\phi}(x_k))}{f'(x_k)}.$$

Adomian Approach

When working with issues in practical mathematics, the decomposition approach based on Adomian polynomials is often used. For this, we model the solution as an infinite series that eventually converges to the true answer. To illustrate, assume that r is a root of the nonlinear equation f(x) = 0 and that X is the interval that contains r. Supposing f is a function in C 2 (X), then. Next, convert f(x) = 0 to the standard form by writing it as:

$$x = c + N(x),$$

the constant c and the nonlinear function N. The Adomian decomposition technique involves a series calculation to get the answer, as

$$x = \sum_{n=0}^{\infty} x_n$$

and breaking down the nonlinear function into

$$N(x) = \sum_{n=0}^{\infty} A_n,$$

where An are functions dependent on x0,x1,...,xn and are referred to as Adomian's polynomials. In addition, the formula may be used to produce instances of the letter A.

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i x_i\right)\right]_{\lambda=0}, \quad n = 0,1,2,\ldots$$

Adomian polynomials up to a certain degree are provided by

$$A_0 = N(x_0),$$

$$A_1 = x_1 N'(x_0),$$

$$A_2 = x_2 N'(x_0) + \frac{1}{2}x_1^2 N''(x_0).$$

Canonical form (1.6.25) is obtained by plugging in values for x and N(x). $\sum_{n=0}^{\infty} x_n = c + \sum_{n=0}^{\infty} A_n.$

Polynomials may be created through a comparison of both sides. Several scholars have looked at the question of whether or not the decomposition series (1.6.26) converges. Recent work by Abbasbandy and Chun uses this strategy to construct iterative techniques of orders 3 and 4.

4. DATA ANALYSIS

To solve the nonlinear equation f(x) = 0, where f(x) is the continuously differentiable function, we use I.M. to identify a multiple root r of multiplicity m, that is, f (j) (r) = 0, j = 0, 1,m1 and f (m) (r) = 0. There are several higher-order techniques that may be used to find simple roots. However, there aren't a lot of recognized ways that can deal with the scenario with numerous roots. It is well knowledge that the Newton method's quadratic convergence breaks down when there are more than two roots. Schroder shown that similar convergence may be attained using the modified Newton approach.

$$x_{k+1} = x_k - m\frac{f(x_k)}{f'(x_k)}, \quad k = 0,1,2,3,\ldots$$

where m is the root multiplicity that you know. Some changes have been suggested and examined to enhance the order of convergence of the Newton method for multiple roots, but they depend on knowing the multiplicity m in advance. Methods like those created by Neta ,Osada , and Victory et al are only a few examples. The approaches established by Neta et al. and Li et al. are fourth-order convergent, whereas the others are third-order. On top of that, the approaches by Li et al. in and several methods by the same authors in [95] show optimum fourth order convergence.

In this paper, we suggest an I.M. using established one-point and multi-point I.M. techniques for calculating multiple roots. as a substitute for basic roots. In section 5.2, we describe different third order approaches with and without the knowledge of second derivative, summing up the work of the chapter. Both the Chebyshev-Halley and Kou-Li-Wang approaches have been generalised to form the basis of the suggested methodologies.

Third Order Methods for Multiple Roots

Here, we offer families of third-order techniques by generalising the Chebyshev-Halley and Kou-Li-Wang approaches to multiple roots. In all classes of procedures, three function evaluations are needed for each iteration. All three values, f, f′, and f″, are used in these assessments. After that, we create techniques that are similar to the Chebyshev-Halley and Kou-Li-Wang ones, but don't need a second derivative. Each entire step corresponds to one evaluation of the function f and two evaluations of the first derivative f ′, hence both of these variations are of the third order.

The Methods with Second Derivative

We then create third-order techniques that step through three iterations of f, f′, and f″.

Modified Chebyshev-Halley method

The formula for the Chebyshev-Halley family of techniques (2.1.2) is as follows:

$$x_{k+1} = x_k - \left[1 + \frac{\mathcal{L}(x_k)}{1-\beta\mathcal{L}(x_k)}\right]\frac{f(x_k)}{f'(x_k)}, \quad \beta \in \mathbb{R},$$

Where

$$\mathcal{L}(x_k) = \frac{f(x_k)f''(x_k)}{2f'(x_k)^2}.$$

As a family, we will refer to them as CHM. For simple roots, this family fulfils the error equation at the third order.

$$e_{k+1} = [(2-\beta)A_2^2 - A_3]e_k^3 + O(e_k^4),$$

the point at which ek = xk r. However, third order convergence breaks down in this family when there are more than two roots. It is possible to state the specific mistake that has occurred here as

$$e_{k+1} = \left[\frac{(m-1)\left((1-2m)+\beta(m-1)\right)}{m(\beta(m-1)-2m)}\right]e_k + \left[\frac{(m-1)(\beta-\beta^2+m(\beta-2)(\beta-3))}{m^2(\beta(m-1)-2m)^2}\right]B_1 e_k^2 + O(e_k^3),$$

where B1 = $\dfrac{1}{(m+1)}\dfrac{f^{(m+1)}(r)}{f^{(m)}(r)}.$

Except in the special situation when = (2m 1)/(m 1), m = 1, CHM converges linearly in all other cases. Our goal in the following is to extend the application of CHM to a more broad set of roots. As a result, the following structure is being considered:

$$x_{k+1} = x_k - \left[\delta + \frac{\gamma\mathcal{L}(x_k)}{1-\beta\mathcal{L}(x_k)}\right]\frac{f(x_k)}{f'(x_k)},$$

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$$x_{k+1} = x_k - \left[\delta + \frac{\gamma \mathcal{L}(x_k)}{1 - \beta \mathcal{L}(x_k)}\right] \frac{f(x_k)}{f'(x_k)},$$

where and L(xk) are the same as in and and are unknown values. We show the following theorem about the scheme in order to talk about its features:

Theorem The root r of a sufficiently differentiable function f: I R for an open interval I is a multiple root with multiplicity m. The iterative strategy specified by (5.2.2) exhibits third order convergence if and only if x0 is sufficiently near to r and = 1 4 ((m 1)2m) 2, = 1 4 ((m1) 2 2m(m3)).

Proof. If we denote the mistake at the kth iteration as ek, then ek = xk r. By extending Taylor's series on r, we get f(xk), f′(xk), and f′′(xk).

$$f(x_k) = \frac{f^{(m)}(r)}{m!} e_k^m [1 + B_1 e_k + B_2 e_k^2 + B_3 e_k^3 + B_4 e_k^4 + O(e_k^5)],$$

$$f'(x_k) = \frac{f^{(m)}(r)}{(m-1)!} e_k^{m-1} [1 + C_1 e_k + C_2 e_k^2 + C_3 e_k^3 + C_4 e_k^4 + O(e_k^5)],$$

$$f''(x_k) = \frac{f^{(m)}(r)}{(m-2)!} e_k^{m-2} [1 + D_1 e_k + D_2 e_k^2 + D_3 e_k^3 + D_4 e_k^4 + O(e_k^5)],$$

where Bi = $\frac{m!}{(m+i)!} \frac{f^{(m+i)}(r)}{f^{(m)}(r)}, \; C_i = \frac{(m-1)!}{(m+i-1)!} \frac{f^{(m+i)}(r)}{f^{(m)}(r)},$

$$D_i = \frac{(m-2)!}{(m+i-2)!} \frac{f^{(m+i)}(r)}{f^{(m)}(r)}, \quad i = 1, 2, 3, \ldots$$

From we get

$$\frac{f(x_k)}{f'(x_k)} = \frac{e_k}{m} - \frac{B_1 e_k^2}{m^2} + \frac{1}{m^3}\left[(m+1)B_1^2 - 2mB_2\right] e_k^3 - \frac{1}{m^4}\left[(m+1)^2 B_1^3 - m(3m+4)B_1 B_2 + 3m^2 B_3\right] e_k^4 + O(e_n^5)$$

Modified Kou-Li-Wang Method

Here, we take into account the Kou-Li-Wang technique for simple roots of the third order, defined as

$$x_{k+1} = x_k - \left[1 + \mathcal{L}(x_k) + \frac{\gamma \mathcal{L}(x_k)^2}{1 - \beta \mathcal{L}(x_k)}\right] \frac{f(x_k)}{f'(x_k)}, \quad \beta, \gamma \in \mathbb{R},$$

where L(xk) is given in (5.2.1). The error equation in this case is given by

$$e_{k+1} = [(2-\gamma)A_2^2 - A_3]e_k^3 + O(e_k^4).$$

Our goal now is to maximise the order of this technique for numerous roots, thus we'll be figuring out values for parameters and.

Extension Of Methods to Systems of Nonlinear Equations

This study is focused on finding an answer to a set of nonlinear equations.

$$\begin{aligned} f_1(x_1,x_2,\ldots,x_n) &= 0, \\ f_2(x_1,x_2,\ldots,x_n) &= 0, \\ &\vdots \\ f_n(x_1,x_2,\ldots,x_n) &= 0. \end{aligned}$$

There exists a solution to this set of equations which reads

F(x) = 0,

where F = (f1, f2,..., fn) t and x = (x1,x2,...,xn) t . The functions f1, f2,..., fn are the coordinate functions of F.

The quadratically convergent Newton method is one of the most well-known fixed point methods and one of the fundamental approaches to approximate a solution to, using a definition of

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}), \quad k = 0,1,\ldots$$

or we can write as

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \mathbf{J_F}(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),$$

where JF(x) is the Jacobian matrix of F(x), and JF(x) 1 is the inverse of that matrix. The first Frechetderivative,'F ′ (x), for a system of n equations with n unknowns is a matrix with n 2 evaluations, while the second Frechetderivative,'F ″ (x), has n 3 evaluations. Even if approaches such as Halley and Chebyshev [5, 56] converge cubically, they are less desirable from a computing perspective because of the high cost of the second derivative. Several variations on the traditional techniques that exclude the second derivative and are of increasing order have been proposed in the literature. By extending the cubically convergent approach to systems of equations, Traub has shown its usefulness. Both Neta and Neta-Victory provide algorithms that converge to at least the fourth order. Frontini et al.'s Homeier and Ozban's , and Ozban's approaches have been extended to systems of equations, and these developments are discussed in . Higher order approaches have recently been developed for systems of nonlinear equations without a second Frechet derivative by researchers including Babajee et al. Cordero et al., Darvishi et al, Hueso et al. [65, 66], Lin et al. [96], Nedzhibov Noor et al. and Chun

Basic Results and Notations

(First, we'll use the notation Ji j(x) to designate the I j) entry of the matrix JF(x), and Hi j(x) to designate the elements of JF(x) 1. (x). Then,

$$\sum_{j=1}^{n} H_{ij}(\mathbf{x}) J_{jk}(\mathbf{x}) = \delta_{ik},$$

where δik represents Kronecker's delta, which is defined as

$$\delta_{ik} = \begin{cases} 1 & \text{if } i = k; \\ 0 & \text{if } i \neq k. \end{cases}$$

In this example, we'll pretend that I and q are both constant and chosen at random. The logical conclusion is then

$$\sum_{j=1}^{n} H_{ij}(\mathbf{x}) \frac{\partial J_{jk}(\mathbf{x})}{\partial x_q} + \sum_{j=1}^{n} \frac{\partial H_{ij}(\mathbf{x})}{\partial x_q} J_{jk}(\mathbf{x}) = 0.$$

Thus, we have

$$\sum_{j=1}^{n} \frac{\partial H_{ij}(\mathbf{x})}{\partial x_q} J_{jk}(\mathbf{x}) = -\sum_{j=1}^{n} H_{ij}(\mathbf{x}) \frac{\partial J_{jk}(\mathbf{x})}{\partial x_q} = -\sum_{j=1}^{n} H_{ij}(\mathbf{x}) \frac{\partial^2 f_j(\mathbf{x})}{\partial x_k \partial x_q}.$$

Differentiating with respect to xr , r being arbitrary and fixed, we obtain

$$\sum_{j=1}^{n} \frac{\partial H_{ij}(\mathbf{x})}{\partial x_r} \frac{\partial J_{jk}(\mathbf{x})}{\partial x_q} + \sum_{j=1}^{n} H_{ij}(\mathbf{x}) \frac{\partial^2 J_{jk}(\mathbf{x})}{\partial x_r \partial x_q} + \sum_{j=1}^{n} \frac{\partial H_{ij}(\mathbf{x})}{\partial x_q} \frac{\partial J_{jk}(\mathbf{x})}{\partial x_r} + \sum_{j=1}^{n} \frac{\partial^2 H_{ij}(\mathbf{x})}{\partial x_r \partial x_q} J_{jk}(\mathbf{x}) = 0.$$

i.e.

$$\sum_{j=1}^{n} \frac{\partial^2 H_{ij}(\mathbf{x})}{\partial x_r \partial x_q} J_{jk}(\mathbf{x}) = -\left[\sum_{j=1}^{n} \frac{\partial H_{ij}(\mathbf{x})}{\partial x_r} \frac{\partial J_{jk}(\mathbf{x})}{\partial x_q} + \sum_{j=1}^{n} \frac{\partial H_{ij}(\mathbf{x})}{\partial x_q} \frac{\partial J_{jk}(\mathbf{x})}{\partial x_r} + \sum_{j=1}^{n} H_{ij}(\mathbf{x}) \frac{\partial^2 J_{jk}(\mathbf{x})}{\partial x_r \partial x_q} \right].$$

Numerical Experimentations

Here we compare the current approaches TMS1 3, TMS2 3, WNS, MJS5 and MJS6 and provide several instances to show how they function differently. The Newton method, abbreviated as NMS, and the Darvishi et al. -developed fourth-order method, abbreviated as DM4 and described as

$$\mathbf{y}^{(k)} = \mathbf{x}^{(k)} - \mathbf{J_F}(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),$$

$$\mathbf{z}^{(k)} = \mathbf{x}^{(k)} - \mathbf{J_F}(\mathbf{x}^{(k)})^{-1}(\mathbf{F}(\mathbf{x}^{(k)}) + \mathbf{F}(\mathbf{y}^{(k)})),$$

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \left[\frac{1}{6}\mathbf{J_F}(\mathbf{x}^{(k)}) + \frac{2}{3}\mathbf{J_F}\left(\frac{\mathbf{x}^{(k)}+\mathbf{z}^{(k)}}{2}\right) + \frac{1}{6}\mathbf{J_F}(\mathbf{z}^{(k)})\right]^{-1}\mathbf{F}(\mathbf{x}^{(k)}),$$

Cordero et al(.'s CM's) fourth-order approach (also written as CM4) is as follows.

$$\mathbf{y}^{(k)} = \mathbf{x}^{(k)} - \mathbf{J_F}(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),$$

$$\mathbf{x}^{(k+1)} = \mathbf{y}^{(k)} - [2\mathbf{J_F}(\mathbf{x}^{(k)})^{-1} - \mathbf{J_F}(\mathbf{x}^{(k)})^{-1}\mathbf{J_F}(\mathbf{y}^{(k)})\mathbf{J_F}(\mathbf{x}^{(k)})^{-1}]\mathbf{F}(\mathbf{y}^{(k)}),$$

Cordero and colleagues' sixth-order method, abbreviated CM6 and defined as

$$\begin{aligned}
\mathbf{y}^{(k)} &= \mathbf{x}^{(k)} - \frac{2}{3}\mathbf{J_F}(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),\\
\mathbf{z}^{(k)} &= \mathbf{x}^{(k)} - \left[-2\mathbf{J_F}(\mathbf{x}^{(k)}) + 6\mathbf{J_F}(\mathbf{y}^{(k)})\right]^{-1}\left[\mathbf{J_F}(\mathbf{x}^{(k)}) + 3\mathbf{J_F}(\mathbf{y}^{(k)})\right]\mathbf{J_F}(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),\\
\mathbf{x}^{(k+1)} &= \mathbf{z}^{(k)} - 2\left[-\mathbf{J_F}(\mathbf{x}^{(k)}) + 3\mathbf{J_F}(\mathbf{y}^{(k)})\right]^{-1}\mathbf{F}(\mathbf{z}^{(k)}).
\end{aligned}$$

In the instance of TMS1 3 and TMS2 3, we set the value = 0.1. In MJS6, we set the a parameter to the value of 1/2. The problems are used for numerical demonstration from

$$\begin{cases} x_1 + e^{x_2} - \cos(x_2) = 0, \\ 3x_1 - x_2 - \sin(x_2) = 0. \end{cases}$$

$$\begin{cases} x_1^2 - 2x_1 - x_2 + 0.5 = 0, \\ x_1^2 + 4x_2^2 - 4 = 0. \end{cases}$$

$$\begin{cases} x_1^2 + x_2^2 - 1 = 0, \\ x_1^2 - x_2^2 + 0.5 = 0. \end{cases}$$

From the Hammerstein equation for m = 8, we derive the following system of nonlinear equations: (see problem IV, section 1.1)

$$x_i - 1 - \frac{1}{5}\sum_{j=1}^{8} a_{ij}x_j^3 = 0, \ \ i = 1, 2, \ldots, 8.$$

Where

$$a_{ij} = \begin{cases} \omega_j t_j(1-t_i) & \text{if} \quad j \le i, \\ \omega_j t_i(1-t_j) & \text{if} \quad i \le j, \end{cases} \qquad i = 1, 2, \ldots 8,$$

For m = 8, the abscissae tj and the weights j are listed in Table 6.1.

Table 1: Abscissas and weights of the Gauss-Legendre formula for m = 8

j	t_j	ω_j
1	0.019855071751231884158219 57	0.050614268145188129576265 67
2	0.10166676129318663020422303	0.11119051722668723527217800
3	0.23723379504183550709113047	0.15685332293894364366898110
4	0.40828267875217509753026193	0.18134189168918099148257522
5	0.59171732124782490246973807	0.18134189168918099148257522
6	0.76276620495816449290886952	0.15685332293894364366898110
7	0.89833323870681336979577696	0.11119051722668723527217800
8	0.98014492824876811584178043	0.05061426814518812957626567

5. CONCLUSION

By integrating these approximations, we get matching quadrature formulae, such as Newton-Cotes formulas. Taylor series, decomposition, quadrature formulae, homotopy, etc. are only a few of the iterative approaches among many. We propose and critically examine a novel iterative approach for solving nonlinear equations, and we are inspired and motivated to do so by the continuing research activity in this field. We have adapted several approaches for scalar equation to systems of nonlinear equations due to the growing importance of numerical methods in the solution of scientific and engineering issues made possible by the advent of powerful, cost-effective digital computers. The numerical findings here provide strong confirmation of the theoretical conclusions we've drawn. The efficiency indices are graphically shown, revealing that the fourth, fifth, and sixth order approaches outperform the Newton method and the other higher order methods. As a result, there are two families where the only parameter is the starting point.

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