Common Fixed Point of Mappings Satisfying Rational Inequalities in Complete Complex Valued Generalized Banach Spaces

Rakesh Kumar Jaiswal and Arihant Jain
Department of Mathematics,
Dr. A. P. J. Abdul Kalam University, Indore (M.P.) - 452016, India
Corresponding Author Email : rakeshjaiswal210@gmail.com

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Abstract
In the present paper we have defined the concept of complex valued generalized Banach spaces. Some common fixed-point theorems involving rational inequalities have been proved. We also prove work periodic point property of common fixed-point problem for two rational type contractive mappings.

Keywords: Weakly Increasing Map. Common Fixed Point. Complex Valued Generalized Banach Spaces. Partially Ordered Set.

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1. INTRODUCTION AND PRELIMINARIES

The most important Banach contraction principle is proved by Stefan Banach in 1922. His valuable work has been elaborated via generalizing the metric conditions or by imposing conditions on the metric spaces. As a consequence of those generalizations so many metric spaces were introduced namely uniformly convex Banach spaces, strictly convex Banach spaces, cone metric spaces, pseudo metric spaces, B-metric spaces, fuzzy metric spaces etc. introduce the concept of a complex valued generalized Banach Space. Also, we present certain common point result for such contraction. We illustrate some fixed-point theorems in generalized Banach space with rational type contractions.

Consistent with [1] and [2] , following definitions and result will be needed in the sequel.

Let C be the set of complex numbers and let $z_1, z_2 \in \mathbb{C}$. Define a partial order $\leq$ on $\mathbb{C}$ as follows : $z_1 \leq z_2$ if and only if $\mathrm{Re}(z_1) \leq (z_2), \mathrm{Im}(z_1) \leq \mathrm{Im}(z_2)$.

It follows that $z_1 \leq z_2$ if one of the following conditions is satisfied:

(1) $\mathrm{Re}(z_1) = (z_2), \mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,

(2) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2), \mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,

(3) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2), \mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,

(4) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2), \mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

We will write $z_1 \leq z_2$ if one of (1), (2) and (3) is satisfied and we will write $z_1 < z_2$ if only (3) is satisfied.

Some elementary properties of the partial order $\leq$ on $\mathbb{C}$ are the following:

(i) If $0 \leq z_1 < z_2$, then $|z_1| \leq |z_2|$.

(ii) $z_1 \leq z_2$ is equivalent to $z_1 - z_2 \leq 0$.

(iii) If $z_1 \leq z_2$ and $r \geq 0$ is a real number, then $rz_1 \leq rz_2$.

(iv) If $0 \leq z_1$ and $0 \leq z_2$ with $z_1 + z_2 \neq 0$, then $\frac{z_1^2}{z_1+z_2} \leq z_1$.

(v) $0 \leq z_1$ and $0 \leq z_2$ do not imply $0 \leq z_1 z_2$.

(vi) $0 \leq z_1$ does not imply $0 \leq \frac{1}{z_1}$. Moreover, if $0 \leq z_1$ and $0 \leq \frac{1}{z_1}$, then $\mathrm{Im}(z_1) = 0$.

Definition 1.1[3] : Let $(X, \preceq)$ be a partially ordered set. A pair (f, g) of self –map of X is said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$. If $f = g$, then we have $fx \preceq f^2 x$ for all x in X and in this case, we say that f is a weakly increasing map.

Definition 1.2 [5] : If $K \neq \emptyset$ is a linear space having $s \geq 1$, Let$\|.\|$ denotes a function from linear space K into R that satisfy the following axioms:

(a)$\forall\, x \in K$, $\|x\| \geq 0$, $\|x\| = 0$ $if and\ only\ if\ x = 0$.

(b)$\forall\, x, y \in K$, $\|x + y\| \leq s\{\|x\| + \|y\|\}$

(c)$\forall\, x \in K , \alpha \in R$, , $\|\alpha x\| = |\alpha|\ \|x\|$.

$\|x\|$ is called norm of x and $(K, \|.\|)$ is called generalized normed space. If for $s = 1$, it reduces to standard normed linear space.

Definition 1.3[5]: A Banach space (K , ‖. ‖) if a normed vector space such that is complete under the metric induced by the ‖. ‖ .

Definition 1.4 [5] : A linear generalized normed space in which every sequence is convergent is called generalized Banach space.

Definition 1.5 [5] : Let $(K, \|.\|)$ be a generalized normed space then the sequence $\{x_n\}$ in K is called

(a) Cauchy sequence iff for each $\varepsilon > 0$, $\exists\, n(\varepsilon) \in N$ s.t. $\forall\, m, n \geq n(\varepsilon)$ we have

$\|x_m - x_n\| < \varepsilon$.

(b)Convergent sequence iff there exists $x \in K$ s.t. $\in > 0$, $\exists\, n(\varepsilon) \in N$for every

$n \geq n(\varepsilon)$ we have $\|x_n - x\| < \varepsilon$

Definition 1.6 [5] : The generalized Banach space is complete if every Cauchy sequence converges .

2. MAIN RESULTS

Theorem. Let$(X, \leq)$be a partially ordered set such that there exists a complete complex valued generalized Banach Space on X and (S,T) a pair of weakly increasing self – maps on X. Suppose that for every comparable $x, y \in X$we have either

$$\|(Sx,Ty)\| \leq \mu \frac{[\|(y,Sx)\|\|(x,Ty)\|^2 + \|(x,Ty)\|\|(y,Sx)\|^2]}{\|(x,Ty)\|^2 + \|(y,Sx)\|^2}$$

$$+\gamma \frac{\|(x,Ty)\|\|(y,Sx)\|}{\|(x,Ty)\|) + \|(y,Sx)\|} + \delta \frac{\|(x,Ty)\|^2 + \|(y,Sx)\|^2}{\|(x,Ty)\| + \|(y,Sx)\|}$$

$$+\rho\{max\|(x,Sx)\|.\|y,Ty\|.\|x,y\|\} \tag{1}$$

In case $\|(x,Ty)\| + \|(y,Sx)\| \neq 0$, $\mu, \gamma, \delta, \rho \geq 0, \|(Sx,Ty)\| = 0$

if $\|(x,Ty)\| + \|(y,Sx)\| = 0$. (2)

If S or T is continuous or for any non-decreasing sequence $x_n$with $x_n \to z$ in X we necessarily have $x_n \leq z$for all$n \in N$. Then S and T have a common fixed point. Moreover, the set of common fixed points of S and T is totally ordered iff S and T have one and only one common fixed point.

Proof: First we shall show that if S or T has a fixed point, then it is a common fixed point of S and T. Let u be a fixed point of S. Then from (1) with $x = y = u$ we have for $u \neq Tu$.

$$\|(u,Tu)\| = \|(Su,Tu)\|$$

$$\leq \mu \frac{[\|(u,Su)\|\{\|(u,Tu)\|\}^2 + \|(u,Tu)\|(\|(u,Su)\|)^2]}{\{\|(u,Tu)\|\}^2 + \{\|(u,Su)\|\}^2}$$

$$+\gamma \frac{\|(u,Tu)\|\|(u,Su)\|}{\|(u,Tu)\| + \|(u,Su)\|} + \delta \frac{(\|(u,Tu)\|)^2 + \{\|(u,Su)\|\}^2}{\|(u,Tu)\| + \|(u,Su)\|}$$

$$+\rho\{\, max\|\,(u,Su)\|.\|(u,Tu)\|.\|(u,u)\|$$

$$\leq \frac{\mu[\|(u,u)\|\}^2\{\|(u,Tu)\|\}^2 + \|(u,Tu)\|(\|(u,u)\|)^2]}{\{\|(u,Tu)\|\}^2 + \{\|(u,Su)\|\}^2}$$

$$+\gamma \frac{\|(u,Tu)\|\|(u,u)\|}{\|(u,Tu)\| + \|(u,u)\|} + \delta \frac{(\|(u,Tu)\|)^2 + \{\|(u,u)\|\}^2}{\|(u,Tu)\| + \|(u,u)\|}$$

$$+\rho max\{\|\,(u,u)\|.\|(u,u)\|.0\}$$

$$\leq \mu.0 + \gamma.0 + \delta\|(u,Tu)\| + \rho.0$$

$$\|\,(u,Tu)\| \leq \delta(\|(u,Tu)\|)$$

which implies that $\| (u,Tu)\| \leq \delta(\|(u,Tu)\|)$.

As $\delta < 1$ so we have $\| (u,Tu)\| = 0$ and u is a common fixed point of S and T. Similarly, if u is a fixed point of T, then it is also fixed point of S.

Now let $x_0$be an arbitrary point on X. If $Sx_0 = x_0$then the proof is finished. Assume that $Sx_0 \neq x_0$. Construct a sequence $\{x_n\}\ in\ X$ as follows:

$x_1 = Sx_0 \leq TSx_0 = Tx_1 = x_2$ and

$x_2 = Tx_1 \leq STx_1 = Sx_2 = x_3$.

Continuing this way we have $x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$. Assume that $d(x_{2n}, x_{2n+1}) > 0$ for every $n \in N$. If not, then $x_{2n} = x_{2n+1}$ for some n. For all those $n, x_{2n} = x_{2n+1} = Sx_{2n}$ and the proof is finished. Assume that $d(x_{2n,}x_{2n+1}) > 0\ forn = 0,1,2,3 \ldots. Asx_{2n} andx_{2n+1}$ are comparable, so we have

$$\|(x_{2n+1,}x_{2n+2})\| = \|(Sx_{2n}, Tx_{2n+1})\|$$

$$\leq \mu \frac{[\|(x_{2n+1}, Sx_{2n})\|\{\|(x_{2n}, Tx_{2n+1})\|\}^2 + \|(x_{2n}, Tx_{2n+1})\|\|(x_{2n+1}, Sx_{2n})\|]}{\{\|(x_{2n}, Tx_{2n+1})\|\}^2 + \{\|(x_{2n+1}, Sx_{2n})\|\}^2}$$

$$+\gamma \frac{\|(x_{2n}, Tx_{2n+1})\|\|(x_{2n+1}, Sx_{2n})\|}{\|(x_{2n}, Tx_{2n+1})\| + \|(x_{2n+1}, Sx_{2n})\|}$$

$$+\delta \frac{[\{\|(x_{2n}, Tx_{2n+1})\|\}^2 + \{\|(x_{2n+1}, Sx_{2n})\|\}^2]}{d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n})}$$

$$+\rho \max\{\|(x_{2n}, Sx_{2n})\|. \|(x_{2n+1}, Tx_{2n+1})\|. \|(x_{2n}, x_{2n+1})\|\}$$

$$= \mu \frac{[\|(x_{2n+1}, x_{2n+1})\|\{\|(x_{2n}, x_{2n+2})\|\}^2 + d\|(x_{2n}, x_{2n+2})\|\{\|(x_{2n+1}, x_{2n+1})\|\}^2]}{\{\|(x_{2n}, x_{2n+2})\|\}^2 + \|(x_{2n+1}, x_{2n+1})\|\}}$$

$$+ \gamma \frac{\|(x_{2n}, x_{2n+2})\|\|(x_{2n+1}, x_{2n+1})\|}{\|(x_{2n}, x_{2n+2})\| + \|(x_{2n+1}, x_{2n+1})\|}$$

$$+ \delta \frac{[\{\|(x_{2n}, x_{2n+2})\|\}^2 + \{\|(x_{2n+1}, x_{2n+1})\|\}^2]}{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})}$$

$$+ \rho\max\{\|(x_{2n}, x_{2n+1})\|. \|(x_{2n+1}, x_{2n+2})\|. \|(x_{2n}, x_{2n+1})\|\}$$

$$= \delta\{\|(x_{2n+1}, x_{2n+2})\| + \|(x_{2n}, x_{2n+1})\|\}$$

$$+ \rho\ max\{\|(x_{2n}, x_{2n+1}\|)\|(x_{2n+1}, x_{2n+2})\|\|(x_{2n}, x_{2n+1})\|\}$$

$$\|(x_{2n+1}, x_{2n+2})\|(1 - \rho - \delta) = (\delta)\|(x_{2n}, x_{2n+1})\|$$

$$\|(x_{2n+1}, x_{2n+2})\| \leq \frac{(\delta)}{(1 - \rho - \delta)}\|(x_{2n}, x_{2n+1})\|$$

which implies that $\|(x_{2n+1}, x_{2n+2})\| \leq k\|(x_{2n}, x_{2n+1})\|$for all $n \geq 0$, $where\ 0 \leq k = \frac{(\delta)}{(1-\rho-\delta)} < 1$. Similarly $\|(x_{2n}, x_{2n+1})\| \leq k\|(x_{2n-1}, x_{2n})\|$ for all $n \geq 0$. Hence for all $n \geq 0$, we have $\|(x_{n+1}, x_{n+2})\| \leq k\|(x_n, x_{n+1})\|$. Consequently,
$\|(x_{n+1}, x_{n+2})\| \leq k\|(x_n, x_{n+1})\| \leq \cdots \leq k^{n+1}\|(x_0, x_1\ )\|$for all $n \geq 0$. Now for $m > n$, we have

$$\begin{aligned}\|(x_n, x_m)\| &\leq \|(x_n, x_{n+1})\| + \|(x_{n+1}, x_{n+2})\| + \cdots + \|(x_{n+1}, x_m)\| \\ &\leq k^n\|(x_0, x_1)\| + k^{n+1}\|(x_0, x_1)\| + \cdots + k^{m-1}\|(x_0, x_1)\| \\ &\leq \frac{k^n}{1-k} d\|(x_{0,} x_1)\|.\end{aligned}$$

Therefore $\|(x_n, x_m)\| \leq \frac{k^n}{1-k}\|(x_{0\ ,} x_1)\|. So\|(x_n, x_m)\| \to 0$ as $n, m \to \infty$ gives that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete the sequence $\{x_n\}$ converges to a point u in X.

If S or T is continuous, then it is clear that *Su=u=Tu*.

If neither S nor T is continuous, then by given assumption $x_n \leq u$ for all $n \in N$. We claim that u is a fixed point of S. If not then $\|(u, Su)\| = z > 0$ from (1), we obtain

$$\|z\| \leq \|(u, x_{n+1})\| + \|(x_{n+1}, x_{n+2})\| + \|(x_{n+2}, Su)\|$$

$$= \|(u, x_{n+1})\| + \|(x_{n+1}, x_{n+2})\| + \|(Su, Tx_{n+1})\|$$

$$\begin{aligned}&\leq \|(u, x_{n+1})\| + \|(x_{n+1}, x_{n+2})\| \\ &\quad + \frac{\mu\|(x_{n+1}, Su)\|\|(u, Tx_{n+1})\|\}^2 + \|(u, Tx_{n+1})\|\{\|(x_{n+1}, Su)\|)\}^2}{\{\|(u, Tx_{n+1})\|\}^2 + \{d(x_{n+1}, Su)\}^2} \\ &\quad + \frac{\gamma\|(u, Tx_{n+1})\|\|(x_{n+1}, Su)\|}{\|(u\ , Tx_{n+1})\| + \|(\ x_{n+1}\ , Su)\|} + \delta\frac{\{\|(u, Tx_{n+1})\|\}^2 + \{\|(x_{n+1}, Su)\|\}^2}{\|(u, Tx_{n+1})\| + \|(x_{n+1}, Su)\|} \\ &\quad + \rho\max\{\|(u, Su)\|. (x_{n+1}, Tx_{n+1}). (u, x_{n+1})\}\end{aligned}$$

and so

$$\begin{aligned}\|z\| &\leq \|(u, x_{n+1})\| + \|(x_{n+1}, x_{n+2})|\| + \mu\frac{\|\|(d(x_{n+1,}Su)\|\|\{\|(u,Tx_{n+1})\|^2\}+\|(u,Tx_{n+1})\|\|\|\{(x_{n+1},Su)\|\}^2}{\|(u,Tx_{n+1}\|)^2+\{\|(x_{n+1},Su)\|^2\}} \\ &\quad +\gamma\frac{\|(u, Tx_{n+1})\|\|(x_{n+1,}Su)|\|}{\|(u, Tx_{n+1})\| + \|(x_{n+1}, Su)\|} + \delta\frac{\{\|(u, Tx_{n+1})\|\}^2 + \{\|(x_{n+1}, Su)\|\}^2}{\|(u, Tx_{n+1})\| + \|(x_{n+1}, Su)\|} \\ &\quad + \rho\max\{\|d(u, Su)\|. \|(x_{n+1}, Tx_{n+1})\|\ . \|(u, x_{n+1})\|\}\end{aligned}$$

which on taking limit as $n \to \infty$ given $\|z\| < \rho\ \|z\|$ a contradiction and So, $u = Su$. Therefore $Su = u = Su$.

Now suppose that set of common fixed points of S and T is totally ordered. We prove that common fixed point of S and T. By supposition, we can replace x by p and y by q in (1) to obtain.

$\|p, q\| = \|(Sp, Tq)\|$

$$\leq \mu \frac{[\{\|(q,Sp)\|\}\{\|(p,Tq)\|\}^2 + \|(p,Tq)\|\{\|(q,Sp)\|\}^2]}{\{\|(p,Tq)\|\}^2 + \{\|(q,Sp)\|\}^2} + \gamma \frac{\|(p,Tq)\|\|(q,Sp)\|}{\|(p,Tq)\| + \|(q,Sp)\|}$$

$$+ \delta \frac{\{\|(p,Tq)\|\}^2 + \|(q,Sp)\|\}^2}{\|(p,Tq)\| + \|(q,Sp)\|} + \rho \max\{\|(p,Sp)\|.\|(q,Tq)\|.\|(p,q)\|\}$$

$$= \mu \frac{[\{\|(q,p)\|\}\{\|(p,q)\|\}^2 + \|(p,q)\|\{\|(q,p)\|\}^2]}{\{\|(p,q)\|\}^2 + \{\|(p,q)\|\}^2} + \gamma \frac{\|(p,q)\|\|(q,p)\|}{\|(p,q)\| + \|(q,p)\|}$$

$$+ \delta \frac{\|(p,q)\|^2 + \|(q,p)\|^2}{\|(p,q)\| + \|(p,q)\|} + \rho\max\{\|(p,p)\|.\|(q,q)\|.\|(p,q)\|\}$$

$$= \mu \left[\frac{2\|(p,q)\|^3}{2\|(p,q)\|^2}\right] + \gamma \left[\frac{\|(p,q)\| * \|(p,q)\|}{2\|(p,q)\|}\right] + \delta \left[\frac{2\|(p,q)\|^2}{2\|(p,q)\|}\right] + 0$$

$$= \mu\|(p,q)\| + \frac{\gamma}{2}\|(p,q)\| + \delta\|(p,q)\|$$

$$= \left(\mu + \frac{\gamma}{2} + \delta\right)\|(p,q)\|$$

which implies that $\|(p,q)\| \leq \left(\mu + \frac{\gamma}{2} + \delta\right)\|(p,q)\|$ a contradiction. Hence $p = q$.

Conversely, if S and T have only one common fixed point then the set of common fixed point of S and T being singleton is totally ordered.

Although we studied a common fixed-point problem for two mapping to consider a more general result, we could use even one and yet the result would have been new. In theorem (1) take S=T, to obtain the following corollary.

Corollary: Let $(X, \leq)$ be a partially ordered set such that there exists a complete complex valued generalized Banach Space on X and let T be a weakly increasing self – map on X. Suppose that for every comparable $x, y \in X$, either

$$d\|Tx,Ty\| \leq \mu \frac{[\|(y,Tx)\|\,\{\|(x,Ty)\|\}^2 + \|(x,Ty)\|\,\{\|(y,Tx)\|\}^2]}{\{\|(x,Ty)\|\}^2 + \{\|(y,Tx)\|\}^2}$$

$$+\gamma \frac{\|(x,Ty)\|\|(y,Tx)\|}{\|(x,Ty)\| + \|(y,Tx)\|} + \delta \frac{\{\|(x,Ty\|\}^2 + \{\|(y,Tx)\|\}^2}{\|(x,Ty)\| + \|(y,Tx)\|}$$

$$+\rho \max\{\|(x,Tx)\|.\|(y,Ty)\|.\|(x,y)\|\}$$

If $\|(x,Ty)\| + \|(x,Tx)\| \neq 0,\ \mu, \gamma, \delta, \rho \geq 0$.

$\|(Tx,Ty)\| = 0$ if $\|(x,Ty)\| + \|(y,Tx)\| = 0$ if T is continuous or for a non decreasing sequence $\{x_n\}$ with $x_n \to Z$ in X we necessarily have $x_n \leq Z$ for all $n \in N$ then T has a fixed point. Moreover, the set of fixed point of T is totally ordered if and only if T has one and only one fixed point.

3. CONCLUSION

The concept of a complex valued generalized Banach spaces is defined and prove the fixed-point theorem involving rational inequalities,work periodic point property of common fixed-point problem for two rational type contractive mappings were also discussed.

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