Solution of Spherical Fuzzy Linear Equations Using Spherical Fuzzy (α, β, γ) - Cut

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Abstract: Fuzzy Set theory with its various branches is a developing area of mathematics in nowadays. Among this relevance of Spherical Fuzzy Set is its three dimensional approach to real-life situations. Fuzzy equation is a hardly discussed area in fuzzy logic. This paper aims to develop a solving method of spherical fuzzy linear equations by using spherical fuzzy (α, β, γ)- cut. An algorithm also developed to handle the situation.

Key words: Spherical fuzzy set, Spherical Fuzzy Number, Triangular Spherical Fuzzy Number, Spherical Fuzzy Linear Equations, Spherical fuzzy (α, β, γ)-cut

Notations: SFS – Spherical Fuzzy Set, SFN – Spherical Fuzzy Number, TSFN – Triangular Spherical Fuzzy Number, SFLE –Spherical Fuzzy Linear Equation

1. Introduction

Probability theory is the base for critical thinking and evaluation. Also logical reasoning is based on probability theory. Aristotle [12] initiated a formal systematic study on logic. Later George Boole [16] introduced modern symbolic logic and its algebra. Hard set theory's failure to handle real–life situations made researchers to make a search for a new tool. In 1965, Lotfi Aliasker Zadeh [21] succeeded in this by introducing fuzzy sets. In some real life situations, human interpretations are possible, but machine interpretations may become impossible. Fuzzy logic concept will become significant in these kind of situations. So far it's various extensions introduced by several researchers and nowadays fuzzy set theory is considered to be as a fast moving research area in all rational directions. Among these Pythagorean and picture concepts of fuzzy sets are recently developed ideas. In 2019, this theory reached out to a new mile stone named as Spherical Fuzzy Set (SFS). This new idea enabled researchers and aspirants to visualize fuzzy set in a three dimensional outlook. After 58 years of development, still fuzzy environment is facing with so many limitations. A few are: failure of FS to define neutral state, non-acceptance of results due to accuracy loss, Needs of constant updation with FLC system etc. Without addressing these limitations properly, its development is meaningless. This made hindrances and inaccuracies in ongoing works. Thus rebuilding of these sets happened in various forms of fuzzy. In this category, Spherical Fuzzy Set introduced by Kutlu Gündoğdu and Kahraman [13] in 2019 deserves a special interest. They discussed on spherical fuzzy TOPSIS method too! It gives a 3D outlook to fuzzy. Sometimes summation of membership degrees crosses 1 in some newly developed fuzzy forms and in fuzzy, it is not allowed too! Using some algebraic manipulations, SFS overcome this difficulty and got well established now.

In 1990, Buckley et.al [6] evaluated fuzzy equations by using α – Cuts. In 1992, Buckley [7] discussed on a solution of fuzzy equations. In 2009, Yanjun Pang et.al [20] discussed on normal fuzzy set. In 2013, Solution to fuzzy system of linear equations with crisp coefficients was discussed by Behera. D & Chakraverty. S [5]. In 2014, Mohammad Keyanpour, Tahereh Akbarian [15] gave a way for solving intutionistic fuzzy nonlinear equations. In 2015, John Harding, Carol Walker et.al [11] studied on equations in Type-2 Fuzzy Sets. In 2015, Shu – Ping Wan, Jiu-Ying Dong [18] gave power geometric operators of trapezoidal intuitionistic fuzzy numbers and an application to multi – attribute group decision making. In 2016, Makwana Vijay. C, Soni Vijay. P et.al [14] gave solution of fuzzy algebraic equations for new fuzzy Number. In 2016, Pramila. K et.al, [17], mentioned an applicational approach to triangular intuitionistic fuzzy number in fuzzy MCDM. In 2017, theory of triangular fuzzy numbers are well mentioned by Anand et.al [3]. In 2017, Clement Joe Anand. M and Janani Bharatraj [8] gave the theory of triangular fuzzy number. In 2018, Babakordi. F, Taghi-Nezhad. N. A [4] gave the definition of α-cut in a compact manner. In 2020, Tipu Sultan Haque et.al [19] gave an approach to solve multi-criteria group decision-making problems by exponential operational law in generalised spherical fuzzy environment. In 2020, Ali Aydoğdu and Sait Gül [2] developed a novel entropy proposition for spherical fuzzy sets and its application in multiple attribute decision – making. In 2021, Alaa F. Momena et.al [1] discussed on a novel aggregation method for generating pythagorean fuzzy numbers in multiple criteria group decision making. In 2021, Dhivya. J , K. Meena. K et.al [9] gave a new technique for solving picture fuzzy differential equation. In 2021, Fatma Kutlu Gündoğdu and Cengiz Kahraman [10] provided properties and arithmetic operations for spherical fuzzy sets.

Nowadays, fuzzy set theory has developed well but still area of fuzzy equation is not developed fully! Works and thoughts of fuzzy equations can be found of from 1984 onwards. Several kinds of trials came out to solve real-life situations using fuzzy equation. Among which solution of fuzzy linear equations using fuzzy α- cuts and super-imposition principle are commendable. One could find out several ways to solve fuzzy equations even in this premature period of its development! This paper concentrates to develop fuzzy linear equation in Spherical atmosphere via Spherical Fuzzy (α, β, γ) – cut'. A real –life application is also discussed in section 3.2.

2. Preliminaries

In this section some basic definitions and properties are provided

Definition 2.1 (Fuzzy set) [21] : Let X be a space of points (Universe of Discourse) with generic element denoted by x such that X = {x}. A fuzzy set A in X is characterized by a membership value function $f_A(x)$ which associates with each point in X, a real number in the interval [0,1] with the value of $f_A(x)$ represents the "grade of membership" of x in A.

Definition 2.2 (Normal Fuzzy Set) [20]: A fuzzy set A is said be normal if its membership function A(x) is unity for at least one $x \in E$.

Definition 2.3 (Convex Fuzzy Set) [20]: A fuzzy set is convex if,

$\mu_A(tx_1 + (1-t)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}$, $x_1, x_2 \in E$ and $t \in [0,1]$.

Definition 2.4 (α-cut of a fuzzy set) [4, 20]:

(a) α-cut of a fuzzy set A denoted by ${}^{\alpha}A$ is defined by ${}^{\alpha}A = \{x\,;\, \mu(x) \geq \alpha\}$

(b) Let A be a fuzzy set in R: $(A = \{x, \mu_A(x)/x \in R\})$. Then

(i) A is normal if there exists an $x \in R$ such that $\mu_{\tilde{A}}(x) = 1$. Otherwise, A is subnormal.

(ii) Support of A, denoted supp(A), is the subset of R whose elements all have nonzero membership grades in A. In other words, supp (A) = $\{x \in R \,/\, \mu_A(x) > 0\}$

(iii) An α – level set (or α −cut) of a fuzzy set A in R is a non-fuzzy set denoted by A_α and defined by $A_\alpha = \begin{cases} \{x \in R/\mu_{\tilde{A}}(x) > 0\}; & \alpha > 0 \\ cl(supp(A)) & ; \ \alpha = 0 \end{cases}$

Remark 2.1 (Properties of α-cut of a fuzzy set)[20]:

(i) ${}^{\alpha}(A + B) = {}^{\alpha}A + {}^{\alpha}B$

(ii) A fuzzy number is a convex normal fuzzy set A defined on the real line such that A(x) is piecewise continuous.

(iii) Support of a fuzzy set A is denoted by sup ρ(A) and is defined as the set of elements with membership nonzero i.e., sup $\rho(A) = \{x \in E, A(x) > 0\}$

Definition 2.5 (Fuzzy number) [8] : A fuzzy set A on R is called a fuzzy number if it satisfies at least the following three properties

- A must be a normal fuzzy set.
- ${}^{\alpha}A$ is must be a closed interval $\forall \alpha \in (0,1]$.
- The support of A, ${}^{0+}A$ must be bounded.

Remark 2.2 [20]: A fuzzy number is a generalization of a regular, real number. It refers to a connected set of possible values, where each possible value has its own height between 0 and 1. A fuzzy number is thus a special case of a convex, normalized fuzzy set of the real line.

Definition 2.6 (Triangular Fuzzy Number) [1, 8]: A Fuzzy number [a, b, c] is a triangular fuzzy number with membership function, $\mu(x) = \begin{cases} \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ \frac{c-x}{c-b} & \text{if } b \leq x \leq c \\ o & \text{elsewhere} \end{cases}$

Definition 2.7 (Properties of Fuzzy set) [21]:

(i) Two fuzzy sets A and B are equal written by $A = B$, If $f_A(x) = f_B(x)$ for all x in X

(ii) Complement of a set denoted by A' is defined by $f'_A = 1 - f_A$

(iii) A is contained in B or A is a subset of B if and only if $f_A \le f_B$ denoted by $A \subset B$

(iv) Union of two fuzzy sets A and B with membership functions $f_A(x)$ and $f_B(x)$ respectively is a fuzzy set C denoted by $C = A \cup B$ and membership function related to those of A and B by $f_c(x) = \max[f_A(x), f_B(x)]$ for all x in X.

(v) Intersection of two fuzzy sets A and B with membership functions $f_A(x)$ and $f_B(x)$ respectively is a fuzzy set C denoted by $C = A \cap B$ whose membership functions related to those of A and B is $f_C(x) = \min[f_A(x), f_B(x)]$.

Definition 2.8 (Fuzzy equations)[5, 20]: Equations of the form $A + X = B$ and $A.X = B$ are fuzzy equations where A, B and X are fuzzy numbers.

Remark 2.3: (Condition for the existence of the Solution X in equation $A + X = B$) [6,7,8]

(i) $X = B - A$ is not the solution for the equation $A + X = B$. In this case, solution exists if and only if $b_1 - a_1 < b_2 - a_2$

(ii) $\forall \alpha \in (0, 1]$ ${}^{\alpha}A + {}^{\alpha}X = {}^{\alpha}B$, $[{}^{\alpha}a_1, {}^{\alpha}a_2] + [{}^{\alpha}x, {}^{\alpha}x_2] = [{}^{\alpha}b_1, {}^{\alpha}b_2]$. Solving this gives ${}^{\alpha}X = [{}^{\alpha}b_1 - {}^{\alpha}a_1, {}^{\alpha}b_2 - {}^{\alpha}a_2]$. Here the solution exists if and only if ${}^{\alpha}b_1 - {}^{\alpha}a_1 < {}^{\alpha}b_2 - {}^{\alpha}a_2$

Remark 2.4 [6,7, 8]: Let [a, b] and [c, d] be two intervals then

- $[a, b] + [c, d] = [a + c, b + d]$
- $[a, b] - [c, d] = [a - d, b - c]$
- If k is a scalar and [a, b] is an interval then k [a, b] = [ak, bk]

Definition 2.9 (Spherical Fuzzy Set) [10, 12, 19] : Let U be a universe. Let A be a Spherical Fuzzy Set. Then A is defined by, $A = \{\langle(x, (\mu_A, \vartheta_A, \pi_A))\rangle / x \in U\}$. The triplet $(\mu_A, \vartheta_A, \pi_A)$ such that $(\mu_A^2 + \vartheta_A^2 + \pi_A^2) \le 1$ is known as Spherical Fuzzy Number, where μ, ϑ, π in A are membership, non-membership and hesitancy degrees of x. Values of all these three will be in [0, 1].

Definition 2.10 (Spherical Fuzzy Number) [10, 13, 19] : Let X be the universe of discourse and A be a Spherical Fuzzy Set (SFS) defined by, $A = \left\{\langle\left(x, \left(\mu_A(x), \vartheta_A(x), \pi_A(x)\right)\right)\rangle / x \in X\right\}$. The triple component $\langle\mu_A(x), \vartheta_A(x), \pi_A(x)\rangle$ is called a Spherical Fuzzy Number represented by $a = \langle\mu_A(x), \vartheta_A(x), \pi_A(x)\rangle$ where $\mu_A(x)$, $\vartheta_A(x)$ and $\pi_A(x) \in [0, 1]$ along with restrictions $0 \le \mu_A^2(x) + \vartheta_A^2(x) + \pi_A^2(x) \le 1$

Definition 2.11 (Basic Operations of spherical fuzzy sets) [10, 13, 19]:

(i) $(A_S \cup B_S) = \begin{Bmatrix} \max\{\mu_A, \mu_B\} \\ \min\{\vartheta_A, \vartheta_B\} \\ \min\left\{\left(1-((\max\{\mu_A,\mu_B\})^2+(\min\{\vartheta_A,\vartheta_B\})^2)\right)^{\frac{1}{2}}, \max\{\pi_A,\pi_B\}\right\} \end{Bmatrix}$

(ii) $(A_S \cap B_S) = \begin{Bmatrix} \min\{\mu_A, \mu_B\} \\ \max\{\vartheta_A, \vartheta_B\} \\ \max\left\{\left(1-((\max\{\mu_A,\mu_B\})^2+(\min\{\vartheta_A,\vartheta_B\})^2)\right)^{\frac{1}{2}}, \min\{\pi_A,\pi_B\}\right\} \end{Bmatrix}$

(iii) Multiplication by scalar: $\lambda.A_S = \begin{Bmatrix} \left(1-(1-\mu_A)^\lambda\right)^{\frac{1}{2}} \\ \vartheta_A^\lambda \\ \left((1-\mu_A)^\lambda-(1-\mu_A^2-\pi_A^2)^\lambda\right)^{\frac{1}{2}} \end{Bmatrix}$; where $\lambda > 0$

3. Main Results

Fuzzy equations are not well established yet and researches are ongoing in this area. In this section, a new concept, Spherical Fuzzy Linear Equation (abbreviated as SFLE) is introduced with example. For that, some preliminary terms are introducing first.

3.1. Spherical Fuzzy Linear Equations and Algorithm to find its solution

In this subsection some basic definitions are provided.

Definition 3.1(Spherical Fuzzy $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$-cut) : Let U be the universe of discourse and A_s be a spherical fuzzy set with membership degree $\mu_{A_s}(x)$, non-membership degree $\vartheta_{A_s}(x)$ and hesitance degree $\pi_{A_s}(x)$ with $\mu_{A_s}(x) : U \longrightarrow [0,1]$, $\vartheta_{A_s}(x) : U \longrightarrow [0,1]$ and $\pi_{A_s}(x) : U \longrightarrow [0,1]$. Suppose $\alpha \in [0, 1]$ in $\mu_{A_s}(x)$, $\beta \in [0, 1]$ in $\vartheta_{A_s}(x)$ and $\gamma \in [0, 1]$ in $\pi_{A_s}(x)$.

Then (α, β, γ)-cut of a spherical fuzzy set can be defined by,

$(\delta,\beta,\delta)_{A_S} = \left\{x \,/\, \mu_{A_s}(x) \geq \alpha,\ \vartheta_{A_s}(x) \leq \beta,\ \pi_{A_s}(x) \leq \gamma\right\}$ and $0 \leq \mu_{A_s}^2(x) + \vartheta_{A_s}^2(x) + \pi_{A_s}^2(x) \leq 1$

OR

$$(\delta,\beta,\delta)_{A_S} = \left\{ \begin{Bmatrix} x \in R/\mu_{\tilde{A}_s}(x) > 0\ ;\ \alpha > 0 \\ \mathrm{cl}(\mathrm{supp}(A_s))\ ;\ \alpha = 0 \end{Bmatrix}, \begin{Bmatrix} \vartheta_{\tilde{A}_s}(x) < \beta\ ;\ \beta < 0 \\ \mathrm{cl}(\mathrm{supp}(A_s))\ ;\ \beta = 0 \end{Bmatrix}, \begin{Bmatrix} \pi_{\tilde{A}_s}(x) < \gamma\ ;\ \gamma < 0 \\ \mathrm{cl}(\mathrm{supp}(A_s))\ ;\gamma = 0 \end{Bmatrix} \right\}$$

where $\left(\mu_{\tilde{A}_s}(x),\ \vartheta_{\tilde{A}_s}(x), \pi_{\tilde{A}_s}(x)\right) = (1,0,0)$ when A_s normal &

supp $(A_s) = \{x \in R/\mu_{A_s}(x) > 1, \vartheta_{A_s}(x) < 0, \pi_{A_s}(x) < 0\}$ and $0 \le \mu^2_{A_s}(x) + \vartheta^2_{A_s}(x) + \pi^2_{A_s}(x) \le 1$

Remark 3.1 : $(\delta,\beta,\gamma)_{(A_S+B_S)} = (\delta,\beta,\gamma)_{A_S} + (\delta,\beta,\gamma)_{B_S}$

Definition 3.2 (Triangular Spherical Fuzzy Number)

Type I : (Triangular Spherical Fuzzy Number (TSFN) using $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$-cut):

A spherical fuzzy number S is said to be triangular if

(i) There exists exactly one $x_0 \in R$ with
$\left(\mu_{A_S}(x_0), \vartheta_{A_S}(x_0), \pi_{A_S}(x_0)\right) = (1,0,0)$; or ; $\left(\mu_{A_S}(x_0), \vartheta_{A_S}(x_0), \pi_{A_S}(x_0)\right) = (0,1,0)$

or ; $\left(\mu_s(x_0), \vartheta_{A_S}(x_0), \pi_{A_S}(x_0)\right) = (0,0,1)$

where x_0 is the mean value of A_S where μ_{A_S}, ϑ_{A_S}, π_{A_S} represents the membership, non-membership and hesitancy functions of the spherical fuzzy set.

(ii) $\mu_{A_S}(x_0), \vartheta_{A_S}(x_0), \pi_{A_S}(x_0)$ are piecewise continuous.
Membership, non-membership and hesitancy of an arbitrary triangular spherical fuzzy number S = (a,b,c) may be defined as follows:

$$\mu_{A_S}(x) = \begin{cases} \frac{x-a}{b-a}\ if\ a \le x \le b \\ \frac{c-x}{c-b}\ if\ b < x \le c \\ 0\ elsewhere \end{cases} ;\quad \vartheta_{A_S}(x) = \begin{cases} \frac{b-x}{b-a}\ if\ a \le x \le b \\ \frac{x-b}{c-b}\ if\ b < x \le c \\ 1\ elsewhere \end{cases} ;\quad \pi_{A_S}(x) = \begin{cases} \frac{b-x}{b-a}\ if\ a \le x \le b \\ \frac{x-b}{c-b}\ if\ b < x \le c \\ 1\ elsewhere \end{cases}$$

Any arbitrary triangular spherical fuzzy number can be represented by an ordered pair of functions through spherical fuzzy (α,β,γ)-cut approach as

$\left[\underline{u}(\alpha), \overline{u}(\alpha)\right] = [(b-a)\alpha + a, -(c-b)\alpha + c]$ where $\alpha \in [0,1]$.

$\left[\underline{v}(\beta), \overline{v}(\beta)\right] = [-(b-a)\beta + b, (c-b)\beta + b]$ where $\beta \in [0,1]$.

$\left[\underline{w}(\gamma), \overline{w}(\gamma)\right] = [-(b-a)\gamma + b, (c-b)\gamma + b]$ where $\gamma \in [0,1]$.

This satisfy the following requirements:

(1) $\underline{u}(\alpha)$, $\underline{v}(\beta)$, $\underline{w}(\gamma)$ are bounded left continuous non-decreasing functions over $[0,1]$
(2) $\overline{u}(\alpha)$, $\overline{v}(\beta)$, $\overline{w}(\gamma)$ is a bounded right continuous non-increasing functions over $[0,1]$
(3) $\underline{u}(\alpha) \le \overline{u}(\alpha)$; $\alpha \in [0,1]$; $\underline{v}(\beta) \ge \overline{v}(\beta)$, $\beta \in [0,1]$; $\underline{w}(\gamma) \ge \overline{w}(\gamma)$, $\gamma \in [0,1]$

Type II : Triangular Spherical Fuzzy Number using $(1-\vartheta_{A_S}^2 - \pi_{A_S}^2)$:

$$\left(\mu_{A_S}(x), \vartheta_{A_S}(x), \vartheta_{A_S}(x)\right)$$

=

$$\left(\begin{cases} \frac{x-a}{b-a} \; if \; a \le x \le b \\ \frac{c-x}{c-b} \; if \; b < x \le c \\ 0 \; elsewhere \end{cases} ; \begin{cases} \frac{b-x}{b-a} \; if \; a \le x \le b \\ \frac{x-b}{c-b} \; if \; b < x \le c \\ 1 \; elsewhere \end{cases} ; \begin{cases} (1-\mu_{A_S}^2(x) - \vartheta_{A_S}^2(x))^{0.5} \; ; if \; a \le x \le b \\ (1-\mu_{A_S}^2(x) - \vartheta_{A_S}^2(x))^{0.5} \; ; if \; b < x \le c \\ 1 \quad elsewhere \end{cases} \right)$$

Definition 3.3 (Spherical Fuzzy Linear Equation) : Equations of the form $A_S + X_S = B_S$ and $A_S.X_S = B_S$ are called as Spherical Fuzzy Linear Equation where A_s, X_s and B_s are TSFN's

where

$$\mu_{A_S}(x) = \begin{cases} \frac{x-a}{b-a} \; if \; a \le x \le b \\ \frac{c-x}{c-b} \; if \; b < x \le c \\ 0 \; elsewhere \end{cases} ; \quad \vartheta_{A_S}(x) = \begin{cases} \frac{b-x}{b-a} \; if \; a \le x \le b \\ \frac{x-b}{c-b} \; if \; b < x \le c \\ 1 \; elsewhere \end{cases} ; \quad \pi_{A_S}(x) = \begin{cases} \frac{b-x}{b-a} \; if \; a \le x \le b \\ \frac{x-b}{c-b} \; if \; b < x \le c \\ 1 \; elsewhere \end{cases}$$

$$\mu_{X_S}(x) = \begin{cases} \frac{x-a}{b-a} \; if \; a \le x \le b \\ \frac{c-x}{c-b} \; if \; b < x \le c \\ 0 \; elsewhere \end{cases} ; \quad \vartheta_{X_S}(x) = \begin{cases} \frac{b-x}{b-a} \; if \; a \le x \le b \\ \frac{x-b}{c-b} \; if \; b < x \le c \\ 1 \; elsewhere \end{cases} ; \quad \pi_{X_S}(x) = \begin{cases} \frac{b-x}{b-a} \; if \; a \le x \le b \\ \frac{x-b}{c-b} \; if \; b < x \le c \\ 1 \; elsewhere \end{cases}$$

$$\mu_{B_S}(x) = \begin{cases} \frac{x-a}{b-a} \; if \; a \le x \le b \\ \frac{c-x}{c-b} \; if \; b < x \le c \\ 0 \; elsewhere \end{cases} ; \quad \vartheta_{B_S}(x) = \begin{cases} \frac{b-x}{b-a} \; if \; a \le x \le b \\ \frac{x-b}{c-b} \; if \; b < x \le c \\ 1 \; elsewhere \end{cases} ; \quad \pi_{B_S}(x) = \begin{cases} \frac{b-x}{b-a} \; if \; a \le x \le b \\ \frac{x-b}{c-b} \; if \; b < x \le c \\ 1 \; elsewhere \end{cases}$$

OR

$\mu_{A_S}(x)=\begin{cases}\frac{x-a}{b-a} \; if \; a \le x \le b \\ \frac{c-x}{c-b} \; if \;\; b < x \le c \\ 0 \; elsewhere\end{cases}$; $\vartheta_{A_S}(x) = \begin{cases}\frac{b-x}{b-a} \; if \; a \;\le x \le b \\ \frac{x-b}{c-b} \; if \; b < x \le c \\ 1 \; elsewhere\end{cases}$; $\pi_{A_S}(x) = \begin{cases}(1-\mu_{A_s}^2(x)-\vartheta_{A_s}^2(x))^{0.5}\,; \; if \; a \;\le x \le b \\ (1-\mu_{A_s}^2(x)-\vartheta_{A_s}^2(x))^{0.5}\,; if \; b < x \le c \\ 1 \qquad\qquad elsewhere\end{cases}$

$\mu_{X_S}(x)=\begin{cases}\frac{x-a}{b-a} \; if \; a \le x \le b \\ \frac{c-x}{c-b} \; if \;\; b < x \le c \\ 0 \; elsewhere\end{cases}$; $\vartheta_{X_S}(x) = \begin{cases}\frac{b-x}{b-a} \; if \; a \;\le x \le b \\ \frac{x-b}{c-b} \; if \; b < x \le c \\ 1 \; elsewhere\end{cases}$; $\pi_{X_S}(x) = \begin{cases}(1-\mu_{X_s}^2(x)-\vartheta_{X_s}^2(x))^{0.5}\,; \; if \; a \;\le x \le b \\ (1-\mu_{X_s}^2(x)-\vartheta_{X_s}^2(x))^{0.5}\,; if \; b \;\le x \le c \\ 1 \qquad\qquad elsewhere\end{cases}$

$\mu_{B_S}(x)=\begin{cases}\frac{x-a}{b-a} \; if \; a \le x \le b \\ \frac{c-x}{c-b} \; if \;\; b < x \le c \\ 0 \; elsewhere\end{cases}$; $\vartheta_{B_S}(x) = \begin{cases}\frac{b-x}{b-a} \; if \; a \;\le x \le b \\ \frac{x-b}{c-b} \; if \; b < x \le c \\ 1 \; elsewhere\end{cases}$; $\pi_{B_S}(x) = \begin{cases}(1-\mu_{A_s}^2(x)-\vartheta_{A_s}^2(x))^{0.5}\,; \; if \; a \;\le x \le b \\ (1-\mu_{A_s}^2(x)-\vartheta_{A_s}^2(x))^{0.5}\,; if \; b < x \le c \\ 1 \qquad\qquad elsewhere\end{cases}$

Generally,

$\mu(x) = \begin{cases}\frac{x-a}{b-a} \; if \; a \le x \le b \\ \frac{c-x}{c-b} \; if \;\; b < x \le c \\ 0 \; elsewhere\end{cases}$; $\vartheta(x) = \begin{cases}\frac{b-x}{b-a} \; if \; a \;\le x \le b \\ \frac{x-b}{c-b} \; if \; b < x \le c \\ 1 \; elsewhere\end{cases}$; $\pi(x) \; == \begin{cases}\frac{b-x}{b-a} \; if \; a \;\le x \le b \\ \frac{x-b}{c-b} \; if \; b < x \le c \\ 1 \; elsewhere\end{cases}$

OR

$\mu(x)=\begin{cases}\frac{x-a}{b-a} \; if \; a \le x \le b \\ \frac{c-x}{c-b} \; if \;\; b < x \le c \\ 0 \; elsewhere\end{cases}$; $\vartheta(x) = \begin{cases}\frac{b-x}{b-a} \; if \; a \;\le x \le b \\ \frac{x-b}{c-b} \; if \; b < x \le c \\ 1 \; elsewhere\end{cases}$; $\pi(x) = \begin{cases}(1-\mu_{A_s}^2(x)-\vartheta_{A_s}^2(x))^{0.5}\,; \; if \; a \;\le x \le b \\ (1-\mu_{A_s}^2(x)-\vartheta_{A_s}^2(x))^{0.5}\,; if \; b < x \le c \\ 1 \qquad\qquad elsewhere\end{cases}$

3.2 Algorithm for finding the Solution for SFLE of the form $A_s + X_s = B_s$

where A_s , X_s , B_s are TSFN's

In this section one algorithm is provided to solve SFLE

Steps:

(1) Frame A_s , X_s, B_s

(2) Assign TSFN values for the SFS's: A_s , X_s and B_s as $\tilde{a} = [a_1, b_1, c_1], \tilde{b} = [a_2, b_2, c_2]$, $\tilde{c} = [a_3, b_3, c_3]$ respectively.

(3) Find Spherical Fuzzy (α, β, γ)-cut for A_s, X_s, B_s. Mark each SFLE in appropriate intervals using spherical (α, β, γ)-cut.Find corresponding interval of x to obtain $^{\alpha}$A and $^{\alpha}$B and thereby $^{\alpha}$X.

(4) X_s is the solution set. Consider, ^{k}X = $[pk + q, rk + s]$ where, k = α, β, γ
If all the 3 membership degrees satisfies $(pk + q) \le (rk + s)$ then solution exists.

(5) If step 4 got satisfied then proceed to step 6 . Otherwise, solution does not exist.

(6) Find solution set X_s= $\langle \mu_{A_S}(x), \vartheta_{A_S}(x), \pi_{A_S}(x) \rangle$.

Example 3.2.1: Let A_s = $[1, 2, 4]$ and B_s = [20, 40, 50]. Solve for X_s if $A_s + X_s = B_s$

Solution: Given $A_s = [1, 2, 4]$ and $B_s = [20, 40, 50]$

Step 1:

$\left(\mu_A(x), \vartheta_A(x), \pi_A(x)\right)$

$$= \left(\begin{cases} x-1 \; if \; 1 \le x \le 2 \\ \frac{4-x}{2} \quad if \; 2 < x \le 4 \\ 0 \quad elsewhere \end{cases} ; \begin{cases} \frac{x-2}{-1} \quad if \; 1 \le x \le 2 \\ \frac{x-2}{2} \quad if \; 2 < x \le 4 \\ 1 \quad elsewhere \end{cases} ; \; (1 - \mu_A^2(x) - \vartheta_A^2(x))^{0.5} \right)$$

$\left(\mu_B(x) , \vartheta_B(x), \pi_B(x)\right)$

$$= \left(\begin{cases} \frac{x-20}{20} \; if \; 20 \le x \le 40 \\ \frac{50-x}{10} \; if \; 40 < x \le 50 \\ 0 \quad elsewhere \end{cases} ; \begin{cases} \frac{x-40}{-20} \; if \; 20 \le x \le 40 \\ \frac{x-40}{10} \; if \; 40 < x \le 50 \\ 1 \quad elsewhere \end{cases} ; \; (1 - \mu_B^2(x) - \vartheta_B^2(x))^{0.5} \right)$$

Step 2 : Using algorithm in 3.2,

$^{\alpha}\mu_A = [1 + \alpha, 4 - 2\alpha]$, $^{\alpha}\mu_B = [20 + 20\alpha, 50 - 20\alpha]$

$^{\beta}\vartheta_A = [2 - \beta, 2 + 2\beta]$, $^{\beta}\vartheta_B = [40 - 20\beta, 40 + 10\beta]$

Calculations will give as

$$\langle \mu_{X_S}(x), \vartheta_{X_S}(x), \pi_{X_S}(x)\rangle = \left\{\begin{array}{c} \mu_{X_S}(x) = \begin{cases} \frac{x-19}{19} & if\ 19 \le x \le 38 \\ \frac{46-x}{8} & if\ 38 < x \le 46 \\ 0 & elsewhere \end{cases}; \\ \vartheta_{X_S}(x) = \begin{cases} \frac{x-38}{-19} & if\ 19 \le x \le 38 \\ \frac{x-38}{8} & if\ 38 < x \le 46 \\ 1 & elsewhere \end{cases}; \\ \pi_{X_S}(x) = \begin{cases} (1-\mu_{X_S}^2(x) - \vartheta_{X_S}^2(x))^{0.5}\ ; & if\ 19 \le x \le 38 \\ (1-\mu_{X_S}^2(x) - \vartheta_{X_S}^2(x))^{0.5}\ ; & if\ 38 < x \le 46 \\ 1 & elsewhere \end{cases} \end{array}\right\}$$

where $\pi_{X_S}(x)$ is the value obtained by assigning values to $\mu_{X_S}(x)$ and $\vartheta_{X_S}(x)$

4. Real –Life Application for Spherical Fuzzy Linear Equation

In this section a real-life situation is discussed using Spherical Fuzzy Linear Equation of the form $A_S + X_S = B_S$.

Laboratory investigation on Blood Urea level

A spherical atmosphere is created in blood urea level analysis and explained the abnormal results ($x< 10$ or $x > 40$) in light of spherical conditions. Let $A_S + X_S = B_S$ be a SFLE with membership degree $\mu_{A_s}(x)$, non-membership degree $\vartheta_{A_s}(x)$ and hesitancy degree $\pi_{A_s}(x)$. Let A_S be the amount of substances mixing to get the reaction in the given blood sample X_S . Resulting reaction will get as the result in the form of B_S. Then the result of the test, B_S could be explained in terms of its various membership functions, $\mu_{A_s}(x)$, $\vartheta_{A_s}(x)$ and $\pi_{A_s}(x)$. In urea analysis, 1 ml of Urea reagent and 10 U/L of serum is mixed. After 5 mins at 37^o C add Buffer reagent and finally read the calorimeter values to obtain the result. Any result value between 10 and 40 is within a normal limit. Let X_S be the sample of blood to be studied with membership degrees $\langle \mu_{A_s}(x), \vartheta_{A_s}(x), \pi_{A_s}(x)\rangle$; where A_S be [0.5, 1, 10] and B_S be [10, 40, u]. Solving this gives the idea about the relation between higher urea level and relation between membership degrees. Equations in 3.2 gives

$$\left(\mu_{A_S}(x), \vartheta_{A_S}(x), \pi_{A_S}(x)\right) =$$

$$\left(\begin{cases} \dfrac{x-0.5}{0.5}\ if\ 0.5 \le x \le 1 \\ \dfrac{10-x}{9}\ if\ \ 1\ \le x \le 10 \\ 0\ elsewhere \end{cases};\ \begin{cases} \dfrac{x-1}{-0.5}\ if\ 0.5\ \le x \le 1 \\ \dfrac{x-1}{9}\ if\ 1\ \le x \le 10 \\ 1\ elsewhere \end{cases};\ \begin{cases} (1-\mu_{A_s}^2(x) - \vartheta_{A_s}^2(x))^{0.5}\ ;\ if\ 0.5\ \le x \le 1 \\ (1-\mu_{A_s}^2(x) - \vartheta_{A_s}^2(x))^{0.5}\ ; if\ 1\ \le x \le 10 \\ 1 \quad elsewhere \end{cases}\right)$$

Similarly, $\left(\mu_{B_S}(x), \vartheta_{B_S}(x), \pi_{B_S}(x)\right) =$

$$\left(\begin{cases} \frac{x-10}{30} \; if\ 10 \le x \le 40 \\ \frac{u-x}{u-40} \; if\ 40 \le x \le u \\ 0\ elsewhere \end{cases} ; \begin{cases} \frac{x-40}{-30} \; if\ 10 \le x \le 40 \\ \frac{x-40}{u-40} \; if\ 40 \le x \le u \\ 1\ elsewhere \end{cases} ; \begin{cases} (1-\mu_{B_s}^2(x)-\vartheta_{B_s}^2(x))^{0.5}\ ;\ if\ 10 \le x \le 40 \\ (1-\mu_{B_s}^2(x)-\vartheta_{B_s}^2(x))^{0.5}\ ; if\ 40 \le x \le u \\ 1 \qquad elsewhere \end{cases} \right)$$

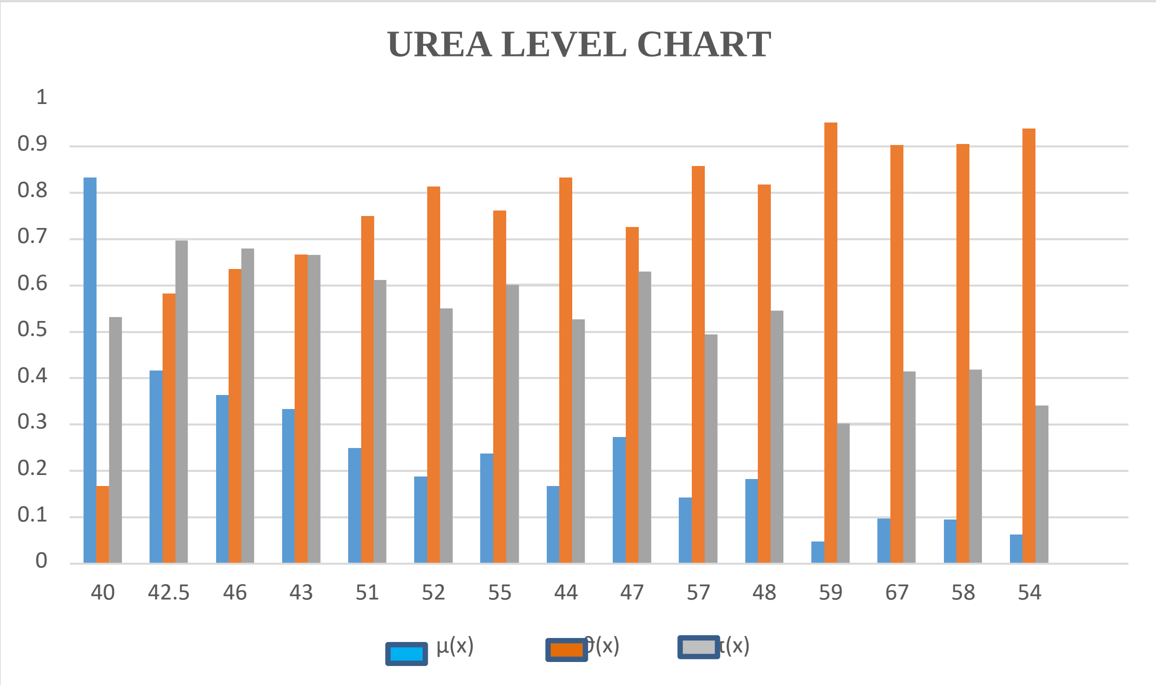
Solving this using algorithm in 3.2: $X_S = \langle \mu_{X_S}(x), \vartheta_{X_S}(x), \pi_{X_S}(x) \rangle$ where

$$\left(\mu_{X_S}(x), \vartheta_{X_S}(x), \pi_{X_S}(x)\right) = \left(\begin{array}{c} \begin{cases} \frac{x-9.5}{29.5} \; if\ 9.5 \le x \le 39 \\ \frac{u-10-x}{u-49} \; if\ 39 \le x \le u-10 \\ 0\ elsewhere \end{cases} ; \\ \begin{cases} \frac{x-39}{-29.5} \quad if\ 9.5 \le x \le 39 \\ \frac{x-39}{u-49} \; if\ 39 \le x \le u-10 \\ 1\ elsewhere \end{cases} ; \\ \begin{cases} (1-\mu_{X_s}^2(x)-\vartheta_{X_s}^2(x))^{0.5}\ ;\ if\ 9.5 \le x \le 39 \\ (1-\mu_{X_s}^2(x)-\vartheta_{X_s}^2(x))^{0.5}\ ; if\ 39 \le x \le u-10 \\ 1 \qquad elsewhere \end{cases} \end{array} \right)$$

Limit values obtained from above, clearly shows u $>$ 49 and u – 10 > x, i.e. u > x+10, where x denotes the urea level. Let urea level be 43 which is an abnormal condition. Choose u > $(43 + 10)$ i.e., u > 53 and let it be 55.Then the 3 membership degrees would be: $\mu_x(x = 43)$ = 0.33333, $\vartheta_x(x = 43)$ = 0.40678, $\pi_x(x = 43)$ = 0.66667

15 Patients urea level table with 3 membership degrees

Patients	**Urea level(x)**	$\mu_{X_S}(x)$	$\vartheta_{X_S}(x)$	$\pi_{X_S}(x)$
P_1	40	0.833	0.167	0.532
P_2	42.5	0.417	0.583	0.697
P_3	46	0.364	0.636	0.680
P_4	43	0.333	0.667	0.666
P_5	51	0.250	0.750	0.612
P_6	52	0.188	0.813	0.551
P_7	55	0.238	0.762	0.602
P_8	44	0.167	0.833	0.527
P_9	47	0.273	0.727	0.630
P_{10}	57	0.143	0.875	0.490
P_{11}	48	0.182	0,818	0.546
P_{12}	59	0.048	0.952	0.302
P_{13}	67	0.097	0.903	0.419
P_{14}	58	0.095	0.905	0.415
P_{15}	54	0.063	0.938	0.341



From these examples, it is clear that if the result is positive with high urea value, the hesitancy or non-membership degree would be the highest among the three membership degrees. Clearly, its contrapositive would be also true, that is if the hesitancy or non-membership degree is the lowest among the three membership functions, the result will be positive with low urea levels.

5. Conclusion

Hard Set theory is inadequate in certain real-life situations where ambiguity or uncertainty arises. Fuzzy set theory is a new tool developed in these situations. Fuzzy equation is considered to be a more complicated area than any other fuzzy zones. Since, while comparing with other fuzzy areas, it has found to get fail in displaying a perfect method to find a solution set! Still by using fuzzy alpha cut-method, this difficulty can be crossed up to a certain level. In this paper, solution set to spherical fuzzy linear equations is developed using spherical fuzzy (α, β, γ)-cut. Elevated urea level in blood may indicates several risk factors like metabolism, kidney dysfunction, dehydration etc. An application, using SFE has also provided, in section 4 in which, level of urea level in blood is well explained in the shades of spherical fuzzy set. As a future scope, this concept can be further extended into Spherical Fuzzy Differential Equations, Spherical Fuzzy Integral Equations, Spherical Volterra Integral Equation, Spherical Fredholm Integral equations etc.

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