Solving Delay Differential Equations Using Least Square Method Based on Successive Integration Technique

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Article Info
Page Number: 1104-1115
Publication Issue:
Vol. 72 No. 1 (2023)

Article History
Article Received: 15 October 2022
Revised: 24 November 2022
Accepted: 18 December 2022

Abstract: In this paper, least square method (LSM) based on successive integration technique is proposed for solving delay differential equations (DDEs). Continuous LSM and Discrete LSM have been presented. In this study we adopted four different orthogonal polynomials for weighted basis function. Numerical examples are considered for testing the efficiency of the proposed method. The proposed method gives results with very good accuracy. It demonstrates the reliability and efficiency of this technique for solving DDEs.

Keywords: Continuous Least Square method, Discrete Least Square method, Polynomials, Hermite, Bernoulli, Chebyshev, Fibonacci, Successive integration technique, DDEs.

1. Introduction

Differential equations with delay terms arise frequently in the fields of science and technology. Some notable applications of DDEs are in chemical kinetics [1], climate model [2] and biological network motif [3].

Many authors have been investigated and developed various analytical and numerical methods to solve DDEs. Evans and Raslan [4] presented Adomain decomposition method for solving DDEs. Mustafa and Mehmet [5] used perturbation-iteration algorithms to solve pantograph type DDEs. Ebimene and Ignatius [6] implemented the Elzaki transform method for solving DDEs. Dhinesh and Emimal [7] proposed RK method with higher order derivatives to solve DDEs. Vinci and Emimal [8] presented fourth order composite RK method to solve DDEs.

The Least Square Method (LSM) is a finite element method to solve differential equations. Daniele [9] has applied least square method to initial and boundary value problems of differential equations (DEs). Siti Farhana et al. [10] have solved DEs by using LSM with an implementation of gradient method. Salisu [11] has investigated LSM for finding approximate solutions to ODEs. Parth et al. [12] have examined the performance of LSM on solving first order ODEs. Salisu and Abdulnasir [13] have used continuous LSM in order to solve second order DEs.

In this study, we propose Continuous LSM (CLSM) and Discrete LSM (DLSM) based on successive integration technique for solving DDEs. We adopted four different orthogonal polynomials for weighted basis function. Numerical examples are considered for testing the efficiency of the proposed method. In section 2, basic definition of polynomials is given. The description of discrete and continuous LSM for solving DDEs are provided in section 3. In section 4, illustrative examples are provided.

2. Basic definition of polynomials

In this study, we consider the most widely used classical orthogonal polynomials, namely, the Hermite polynomials, the Bernoulli polynomials, the Chebyshev polynomials and the Fibonacci polynomials.

Hermite Polynomial

The Hermite polynomial $H_n(t)$ of order n is defined on the interval $(-\infty, \infty)$. There are different ways to define for Hermite polynomial, one of them is the so-called Rodrigues' formula

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2} \tag{1}$$

From Eqn. (1), the recurrence relation for the polynomials can be derived as

$$H_n(t) = 2t\, H_{n-1}(t) - H'_{n-1}(t) \tag{2}$$

$H_0(t)$ can be obtained from Eqn. (1) and the remaining terms are determined by using the recursion relation Eqn. (2). Thus, we have the following sequence of polynomials:

$$H_0(t) = 1$$

$$H_1(t) = 2t$$

$$H_2(t) = 4t^2 - 2$$

$$H_3(t) = 8t^3 - 12t$$

$$H_4(t) = 16t^4 - 48t^2 + 12$$

and so on. The n^{th} order Hermite polynomial $H_n(t)$ has a leading coefficient 2^n.

Bernoulli Polynomial

The Bernoulli polynomial is named after Jacob Bernoulli which combines the Bernoulli numbers and binomial coefficients. The generating function of n^{th} order Bernoulli polynomial is defined by

$$\sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!} = \frac{x e^{xt}}{e^x - 1} \tag{3}$$

The Bernoulli polynomial is explicitly written as:

$$B_n(t) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(t^k) \tag{4}$$

for $n \geq 0$.

$B_0(t)$ can be obtained from Eqn. (3) and the remaining terms are determined by using the recursion relation. Thus, we have few terms of the Bernoulli polynomials as:

$B_0(t) = 1$

$B_1(t) = t - 1/2$

$B_2(t) = t^2 - t + 1/6$

$B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t$

$B_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30}$

Chebyshev Polynomial

The Chebyshev polynomial related to cosine functions on the interval $[-1, 1]$ of order n is defined as

$$T_n(\cos t) = \cos(nt) \quad (5)$$

The recursion relation of Chebyshev polynomial is:

$$T_{n+1}(t) = 2t\, T_n(t) - T_{n-1}(t) \quad (6)$$

$T_0(t)$ and $T_1(t)$ can be obtained from Eqn. (5). Then the remaining terms are determined by from Eqn. (6). Thus, we have the following sequence of polynomials:

$T_0(t) = 1$

$T_1(t) = t$

$T_2(t) = 2t^2 - 1$

$T_3(t) = 4t^3 - 3t$

$T_4(t) = 8t^4 - 8t^2 + 1$

Fibonacci Polynomial

The Fibonacci polynomials are generated by Fibonacci numbers. The recurrence relation of Fibonacci polynomial is:

$$F_n(t) = \begin{cases} 0, & if\ n = 0 \\ 1, & if\ n = 1 \\ tF_{n-1}(t) + F_{n-2}(t), & if\ n \geq 2. \end{cases}$$

Using this relation, we have the following sequence of polynomials:

$F_0(t) = 0$

$F_1(t) = 1$

$F_2(t) = t$

$F_3(t) = t^2 + 1$

$F_4(t) = t^3 + 2t$

3. Solving DDEs using Least Square method based on successive integration technique

Consider the n[th] order DDE

$$y^{(n)}(t) = f(t, y(t), y(t-\tau), y'(t), y'(t-\tau),), \dots, y^{(n-1)}(t), y^{(n-1)}(t-\tau)\big), \quad t > t_0 \quad (7)$$

with initial conditions

$$y^{(i)}(t_0) = \emptyset(t),\ i = 1, 2, 3, \dots \quad t \le t_0 \quad (8)$$

Here τ is the delay term and $\emptyset(t)$ is the history function.

Let P(t) represent any orthogonal polynomials. For the proposed method, we assume that

$$y^{(n)}(t) \approx B^T P(t)^T = \sum_{j=0}^{N} c_j P_j(t) \quad (9)$$

where N being any positive integer,

$B^T = (c_0, c_1, \dots c_N)$

$P(t) = \big(P_0(t), P_1(t) \dots P_N(t)\big)$

Our aim is to determine the polynomial coefficients $c_j's$. For this, we integrate Eqn. (9) with respect to t from $t_0\ to\ t$,

$$\left.\begin{aligned} y^{(n-1)}(t) &= y(t_0) + \int_{t_0}^{t} B^T\, P_j(t)\, dt \\ y^{(n-2)}(t) &= y(t_0) + y'(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} B^T\, P_j(t)\, dt \\ &\ \ \vdots \\ y'(t) &= \sum_{i=0}^{N-1} y^{(i)}(t_0) + \int_{t_0}^{t}\int_{t_0}^{t}\dots\int_{t_0}^{t} B^T\, P_j(t)\, dt \\ y(t) &= \sum_{i=0}^{N} y^{(i)}(t_0) + \int_{t_0}^{t}\int_{t_0}^{t}\dots\int_{t_0}^{t} B^T\, P_j(t)\, dt \end{aligned}\right\} \quad (10)$$

Now, for delay terms

$$\left.\begin{aligned} y^{(n-1)}(t-\tau) &= y(t_0) + \int_{t_0}^{t} B^T P_j(t-\tau)\, dt \\ y^{(n-2)}(t-\tau) &= y(t_0) + y'(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} B^T P_j(t-\tau)\, dt \\ &\ \ \cdots \\ y'(t-\tau) &= \sum_{i=0}^{N-1} y^{(i)}(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} \dots \int_{t_0}^{t} B^T P_j(t-\tau)\, dt \\ y(t-\tau) &= \sum_{i=0}^{N} y^{(i)}(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} \dots \int_{t_0}^{t} B^T P_j(t-\tau)\, dt \end{aligned}\right\} \qquad (11)$$

By substituting (10) and (11) in (7), we get the residue function R(t). The coefficients $c_j's$ can be obtained using the LSM which is based on weighted residuals minimization. In this study, we introduce Continuous Least Square Method (CLSM) and Discrete Least Square Method (DLSM).

Continuous Least Square Method (CLSM)

In CLSM, we make the residue function R tend to zero by minimizing the error function

$$E = \int_{\Omega} R^2(t)dt \qquad (12)$$

for $t \epsilon \Omega$.

To obtain an optimum solution with minimal error E, we differentiate the Eqn. (12) with respect to c_j and then equate to zero. Thus, we have

$$\frac{\partial E}{\partial c_j} = \frac{\partial}{\partial c_j} \int R^2(t)\ dt = 0,\ \text{for } j = 1, 2, \dots, N,$$

which implies

$$\int_{t=0}^{t=1} R(t)\ \frac{\partial R(t)}{\partial c_j}\ dt\ \ = 0, \text{for } j = 1, 2, \dots, N \qquad (13)$$

This yields an algebraic system of linear and nonlinear equations subject to the linear and nonlinear terms involving in the Eqn. (7). By solving this system of equations, we get the respective polynomial co-efficient c_j's from which the solution of the DDE (7) can be obtained.

Discrete Least Square Method (DLSM)

In DLSM, we consider the residuals at the points $t_i, 1 \leq i \leq N$. Let

$$E = \sum_{i=1}^{N} R^2(t) \qquad (14)$$

To obtain an optimum solution with minimal error E, we differentiate the Eqn. (14) with respect to c_j and then equate to zero. Thus, we have

$$\frac{\partial E}{\partial c_j} = 0,\ \text{for } j = 1, 2, \dots, N,$$

This yields an algebraic system of linear and nonlinear equations. By solving this system of equations, we get the respective polynomial coefficients c_j's from which the solution of the DDE (7) can be obtained.

4. Numerical examples

Here, linear and nonlinear DDEs are considered for testing the efficiency of the proposed method. We solved these DDEs by using CLSM and DLSM based on successive integration technique by using four orthogonal polynomials, namely Hermite, Bernoulli, Chebyshev, and Fibonacci. Here, for convenience, in the case of CLSM, we denote them as H-CLSM, B-CLSM, C-CLSM and F-CLSM respectively. Similarly, in the case of DLSM, we denote them as H-DLSM, B-DLSM, C-DLSM and F-DLSM respectively.

Problem 1

$$y''(t) = \frac{3}{4}\, y(t) + y\left(\frac{t}{2}\right) - t^2 + 2$$

with $y'(0) = 0$ and $y(0) = 0$.

The analytical solution is $y(t) = t^2$.

The numerical results obtained by the proposed methods CLSM and DLSM using the four polynomials with different values of N have been compared with the exact solution. The absolute error results at t = 1 are given in Table 1 and Table 2. The solution graphs by CLSM and DLSM with N = 7 are presented in Fig. 1 and Fig. 2.

Table 1 Error Results in CLSM for Problem 1

Methods	**N = 3**	**N = 5**	**N = 7**
H-CLSM	6.85 e-09	3.92 e-10	7.42 e-10
B-CLSM	1.25 e-09	1.40 e-09	1.23 e-09
C-CLSM	1.07 e-10	1.02 e-08	3.04 e-09
F-CLSM	1.37 e-08	2.70 e-09	7.16 e-09

Table 2 Error Results in DLSM for Problem 1

Methods	**N = 3**	**N = 5**	**N = 7**
H-DLSM	1.34 e-09	4.22 e-16	2.68 e-17
B-DLSM	6.18 e-10	1.75 e-10	5.73 e-10
C-DLSM	5.80 e-09	1.78 e-09	4.14 e-0.9
F-DLSM	1.58 e-09	5.60 e-15	7.96 e-13

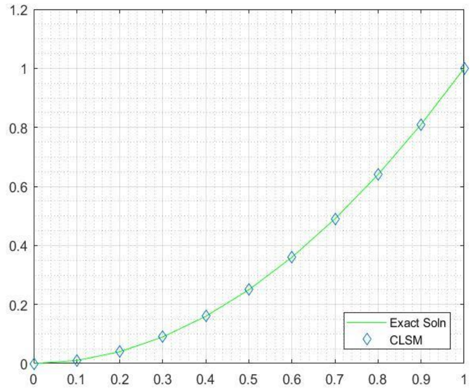

Fig. 1 CLSM - Solution (Problem 1)

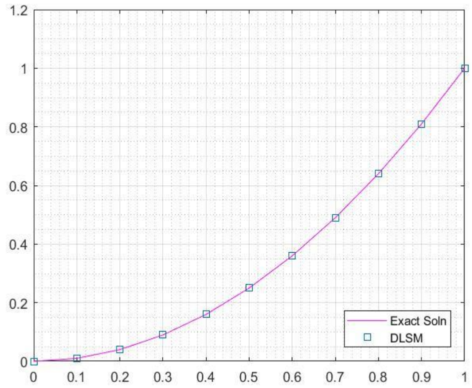

Fig. 2 DLSM - Solution (Problem 1)

Problem 2

$y'(t) + ty(t - t^2) + ty^2(t) = 1 + t^2$

with $y(0) = 0$.

The analytical solution is $y(t) = t$.

The numerical results obtained by the proposed methods CLSM and DLSM using the four polynomials with N = 5 have been compared with the exact solution. The absolute error results are given in Table 3 and Table 4. The solution graphs by CLSM and DLSM with N = 7 are presented in Fig. 3 and Fig. 4.

Table 3 Error Results in CLSM for Problem 2

t	H-CLSM	B-CLSM	C-CLSM	F-CLSM
0.2	1.39e-13	4.07e-09	2.47e-14	1.54e-08
0.4	1.09e-12	8.70e-09	2.39e-13	1.16e-08

0.6	8.37e-13	1.30e-08	1.52e-13	4.80e-08
0.8	5.39e-13	1.57e-08	9.77e-14	1.42e-08
1.0	2.76e-14	1.54e-08	1.88e-14	5.62e-09

Table 4 Error Results in DLSM for Problem 2

t	H-DLSM	B-DLSM	C-DLSM	F-DLSM
0.2	4.31e-13	1.08e-11	1.74e-15	7.16e-15
0.4	1.70e-13	9.16e-12	3.16e-15	5.09e-15
0.6	7.07e-13	7.85e-12	4.54e-15	1.46e-14
0.8	8.50e-13	1.96e-11	4.86e-15	5.20e-14
1.0	5.57e-14	2.90e-10	3.52e-16	1.49e-13

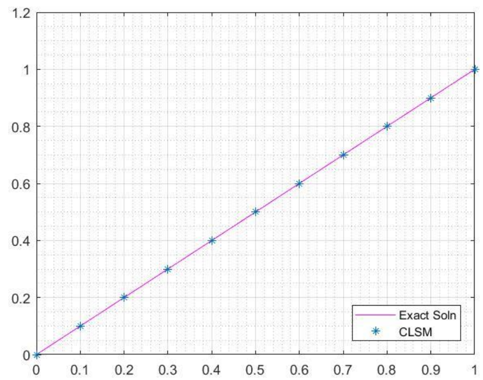

Fig. 3 CLSM - Solution (Problem 2)

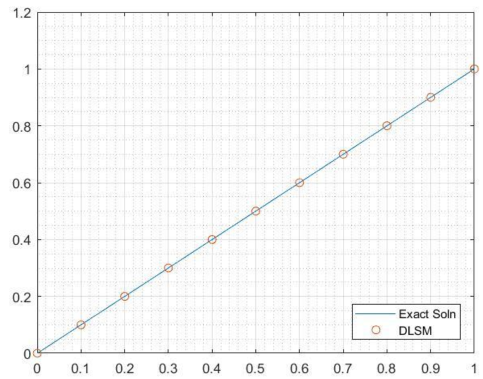

Fig. 4 DLSM - Solution (Problem 2)

Problem 3

$$\frac{d^3y(t)}{dt^3} = -y(t) - y(t-0.3) + e^{-t+0.3}, \ 0 \le t \le 1$$

with $y(0) = 1,\ y'(0) = -1$ and $y''(0) = 1$.

The analytical solution is $y(t) = e^{-t}$.

The numerical results obtained by the proposed methods CLSM and DLSM using the four polynomials with N = 7 have been compared with the exact solution. The absolute error results are given in Table 5 and Table 6. The solution graphs by CLSM and DLSM with N = 7 are presented in Fig. 5 and Fig. 6.

For this example, the numerical results obtained by using the proposed new approach based on successive integration technique are compared with the results by the conventional operational matrix approach of polynomial collocation method which are available in literature [14]. The comparative results for N = 8 are presented in Table 7 and Table 8.

Table 5 Error Results in CLSM for Problem 3

t	H-CLSM	B-CLSM	C-CLSM	F-CLSM
0.2	1.20e-08	3.07e-11	9.26e-09	6.70e-09
0.4	3.07e-09	3.87e-11	2.77e-09	5.12e-09
0.6	5.77e-08	1.14e-10	3.96e-08	1.98e-09
0.8	8.01e-08	3.02e-10	5.68e-08	1.93e-08
1.0	1.31e-07	5.36e-10	9.23e-08	1.96e-08

Table 6 Error Results in DLSM for Problem 3

t	H-DLSM	B-DLSM	C-DLSM	F-DLSM
0.2	3.76e-11	3.86e-11	4.22e-11	3.76e-11
0.4	2.14e-10	2.15e-10	2.38e-10	2.14e-10
0.6	5.25e-10	5.32e-10	5.49e-10	5.25e-10
0.8	9.66e-10	9.93e-10	9.98e-10	9.66e-10
1.0	1.53e-09	1.58e-09	1.60e-09	1.53e-09

Table 7 Comparative Error Results for Problem 3 (Hermite Polynomial)

t	Polynomial Collocation Method	Proposed Least Square Method	
		H-CLSM	H-DLSM
0.2	6.20e-09	3.38e-10	1.38e-12
0.4	5.76e-08	4.85e-09	7.33e-12
0.6	1.79e-07	1.07e-08	1.77e-11

0.8	3.73e-07	1.91e-08	3.25e-11
1.0	6.36e-07	3.08e-08	5.13e-11

Table 8 Comparative Error Results for Problem 3 (Chebyshev Polynomial)

t	**Polynomial Collocation Method**	**Proposed Least Square Method**	
		C-CLSM	**C-DLSM**
0.2	3.70e-07	3.05e-09	3.53e-12
0.4	2.38e-06	9.42e-09	5.78e-11
0.6	5.97e-06	2.68e-08	1.78e-10
0.8	3.48e-05	4.56e-08	3.65e-10
1.0	2.03e-04	7.30e-08	6.28e-10

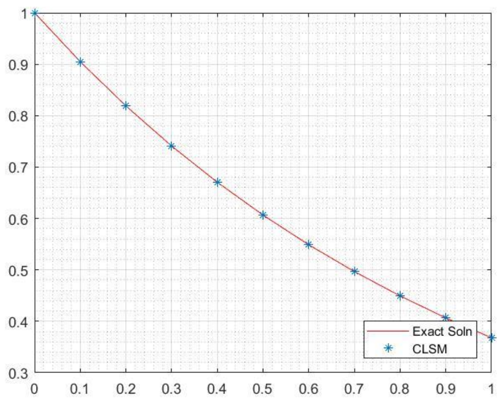

Fig. 5 CLSM - Solution (Problem 3)

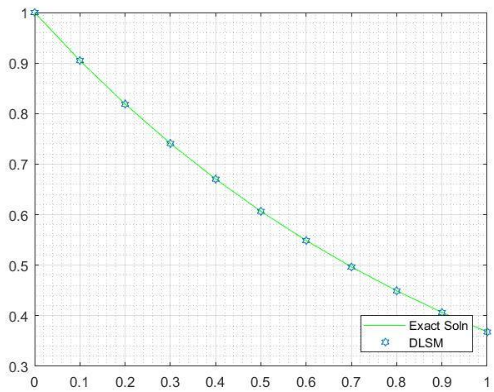

Fig. 6 DLSM - Solution (Problem 3)

5. Conclusion

In this paper, Continuous and Discrete Least Square Methods have been presented to solve differential equations with delay terms. Here four different orthogonal polynomials are utilised for weighted basis function. Numerical examples of linear and nonlinear DDEs are given to test the efficiency of the proposed method.

From the numerical results, it is clear that the results obtained by the continuous and discrete least square methods are reasonably good in accuracy. It is observed that results by discrete square least method gives slightly better results than the continuous least square method. From the comparative Tables 7 and 8, it is evident that the proposed method based on successive integration technique gives better results than the conventional matrix approach. Hence the proposed least square method is very effective and reliable to solve delay differential equations.

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