Arithmetic Sequential Graceful Labeling on Complete Bipartite Graph

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Abstract: Consider the graph G, which has p vertices and q edges and is simple, finite, connected, undirected, and non-trivial. $G's$ vertex set is V(G), while its edge set is $E(G)$. Let f be an injective function from $V(G)\ to\ \{a, a+d, a+2d, a+3d, \ldots, 2(a+qd)\}$ and for each edge $uv \in E(G), f^*: E(G) \to \{d, 2d, 3d, 4d, \ldots, qd\}$ defined by $f^*(uv) = |f(u) - f(v)|$. Then the f is arithmetic sequential graceful labeling if $f^*(uv)$ is bijective. The graph having arithmetic sequential graceful labeling is called arithmetic sequential graceful graph. This paper examined the arithmetic sequential graceful labeling for a few unique graphs.

Keywords: Graceful labeling, Complete bipartite graph, Open star of graph, Join sum of graph, Path union graph.

1. Introduction

A fascinating area of research in graph theory is labeling. Giving values to edges or vertices is the process of labeling. It was Alexander Rosa [2] who first proposed the idea of graceful labeling. Later, a few labeling techniques were presented. See Gallian's dynamic survey [3] for further details. Golomb [4] proved that the complete bipartite graph is graceful. Kaneria and Makadia [6] proved that star of a cycle is graceful. Kaneria et. al [7] proved that join sum of , path union of and star of are graceful graphs. Barrientos [8] proved that union of complete bipartite graph is graceful. Here, we defined arithmetic sequential graceful and shown that various types of graphs fall under this category in [8]. Here are the some of the definitions which are helpful in this article.

Definition 1.1: A graph is considered bipartite if its vertices V can be split into two separate sets V_1 and V_2, and if each edge of the graph connects a vertex from V_1 to a vertex from V_2. A graph is referred to be a complete bipartite graph if each vertex of V_1 and V_2 are connected to each other. We denote the complete bipartite graph as $K_{r,s}$.

Definition 1.2: Consider n copies of a graph G_i. The graph obtained by adding an edge from the graph G_i to G_{i+1}, $1 \leq i \leq n-1$ and $n > 2$ is called path union of G.

Definition 1.3: Consider τ copies of a graph $G_i, 1 \leq i \leq \tau$. The graph obtained by joining two copies of the graph G_i and $G_{i+1}, 1 \leq i \leq \tau - 1$ by a vertex is called join sum of graphs and it is denoted by $\langle G_1, G_2, G_3, \dots G_\tau \rangle$.

Definition 1.4: "Open star of graph" refers to a graph that is created by substituting the graphs G_1, G_2,…, G_n for each vertex of $K_{1,n}$ other than the apex vertex. Such a graph will be represented as $\mathcal{S}(G_1, G_2, \dots, G_n)$.

Definition 1.5: The open star of a graph that results when all the vertices of $K_{1,n}$ except the apex vertex are replaced by the graph G is denoted by $\mathcal{S}(n.G)$.

Definition 1.6: The star of a graph that results when all the vertices of $K_{1,n}$ are replaced by the graph G is denoted by $\mathcal{S}(G)$.

2. Main Results

Theorem 2.1: The graph $K_{r,s}$ admits Arithmetic Sequential Graceful labeling.

Proof: Let G be a $K_{r,s}$ graph.

$V(G) = \{u_i : 1 \leq i \leq r\} \cup \{v_\zeta : 1 \leq \zeta \leq s\}$ and

$E(G) = \{u_i v_\zeta : 1 \leq i \leq r, 1 \leq \zeta \leq s\}$

$|V| = (r+s), |E| = rs$

We define $f: V(G) \rightarrow \{a, a+d, a+2d, a+3d, \dots, 2(a+qd)\}$

$f(u_i) = a + (i-1)d, 1 \leq i \leq r$

$f(v_\zeta) = a + (rs - (\zeta - 1))d, 1 \leq \zeta \leq s$

Table 1: Edge labels of the graph $K_{r,s}$

$\boldsymbol{f^*(u\,v), \forall\, uv \in E(G)}$	**Edge labels**
$f^*(u_i v_\zeta)$	$[rs - r(\zeta-1) - (i-1)]\,d, 1 \leq i \leq r, 1 \leq \zeta \leq s$

It is clear that the function f is injective and also table 1 shows that

$f^*: E \rightarrow \{d, 2d, 3d, 4d, \dots, qd\}$ is bijective. Hence the graph $K_{r,s}$ admits arithmetic sequential graceful graph.

Illustration 2.1.1: $K_{3,5}$ and its graceful labeling shown in figure-1.

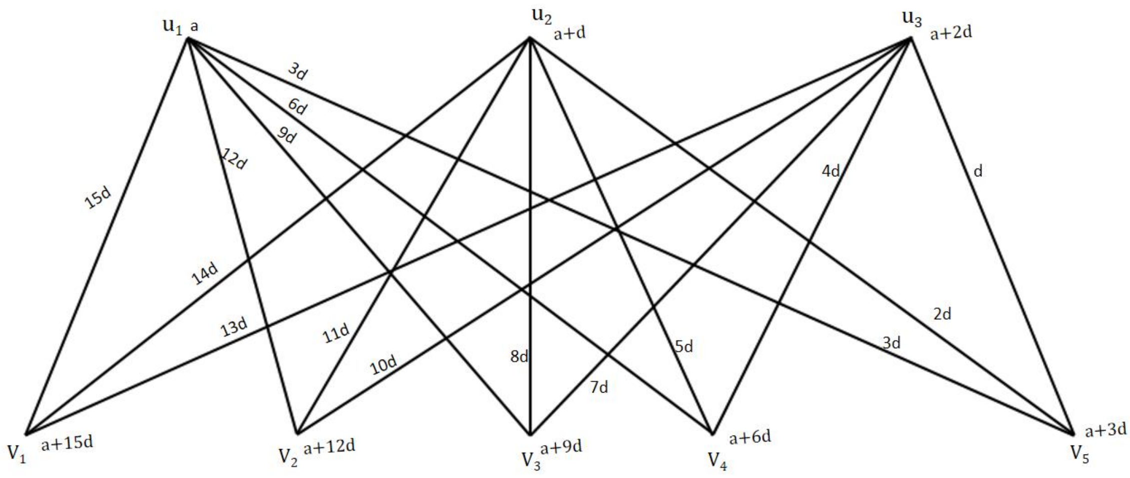

Figure-1: $K_{3,5}$ and its graceful labeling.

Theorem 2.2: The path union of complete bipartite graph is Arithmetic Sequential Graceful labeling.

Proof: Let G be a path union of complete bipartite graph.

$V(G) = \{u_{i,\zeta} : 1 \le i \le \tau, 1 \le \zeta \le r_i\} \cup \{v_{i,\kappa} : 1 \le i \le \tau, 1 \le \kappa \le s_i\}$ and $E(G) = \{ u_{i,\zeta}\, v_{i,\kappa} : 1 \le i \le \tau, 1 \le \zeta \le r_i, 1 \le \kappa \le s_i \}$

$\cup \{u_{i,ri}, v_{i+1,1} : 1 \le i \le \tau - 1\}$

$$|V| = \sum_{i=1}^{\tau}(r_i + s_i)\ , |E| = (\tau - 1) + \sum_{i=1}^{\tau}(r_i s_i)$$

We define a function $f: V(G) \to \{a, a+d, a+2d, a+3d, \dots, 2(a+qd)\}$

$f(u_{1,\zeta}) = a + (\zeta - 1)d\ , 1 \le \zeta \le r_1$

$$f(v_{1,\kappa}) = a + \left[\left[(\tau - 1) + \sum_{i=1}^{\tau}(r_i s_i)\right] - r_1(\kappa - 1)\right] d\ , 1 \le \kappa \le s_1$$

$$f(u_{i,\zeta}) = a + \left[\sum_{l=1}^{i-1} r_l + (\zeta - 1)\right] d, 2 \le i \le \tau\ , 1 \le \zeta \le r_i$$

$$f(v_{i,\kappa}) = a + \left[\left[(\tau - 1) + \sum_{i=1}^{\tau}(r_i s_i)\right] - \sum_{l=1}^{i-1} r_l(s_l - 1) - (i-1) - r_i\,(\kappa - 1)\right] d,$$

$$2 \le i \le \tau\ , 1 \le \kappa \le s_i$$

Table 2: Edge labels of the graph $< K_{r_1,s_1}, K_{r_2,s_2}, \ldots, K_{r_\tau,s_\tau} >, \tau \geq 2$

$\boldsymbol{f^*(u\,v)}$, $\forall\, \boldsymbol{uv} \in \boldsymbol{E(G)}$	**Edge labels**
$f^*(u_{1,\zeta} v_{1,\kappa})$	$[(\zeta-1)-[(\tau-1)+\sum_{i=1}^{\tau}(r_i s_i)]-r(\kappa-1)]d$, $1 \leq \kappa \leq s_1, 1 \leq \zeta \leq r_1$
$f^*(u_{i,\zeta} v_{i,\kappa})$	$\left[\sum_{l=1}^{i-1} r_l + (\zeta-1) - \left[\begin{matrix}[(\tau-1)+\sum_{i=1}^{\tau}(r_i s_i)] \\ -\sum_{l=1}^{i-1} r_l(s_l-1)-(i-1)-r_i(\kappa-1)\end{matrix}\right]\right]d$, $2 \leq i \leq \tau, 1 \leq \zeta \leq r_i, 1 \leq \kappa \leq s_i$

It is clear that the function f is injective and also table 2 shows that

$f^*: E \rightarrow \{d, 2d, 3d, 4d, \ldots, qd\}$ is bijective. Hence f is arithmetic sequential graceful labeling and the graph$< K_{r_1,s_1}, K_{r_2,s_2}, \ldots, K_{r_\tau,s_\tau} >, \tau \geq 2$ is arithmetic sequential graceful graph.

Example 2.2.1: Path union of $G = < K_{3,4}, K_{3,3}, K_{2,3} >$ and its graceful labeling shown in figure-2.

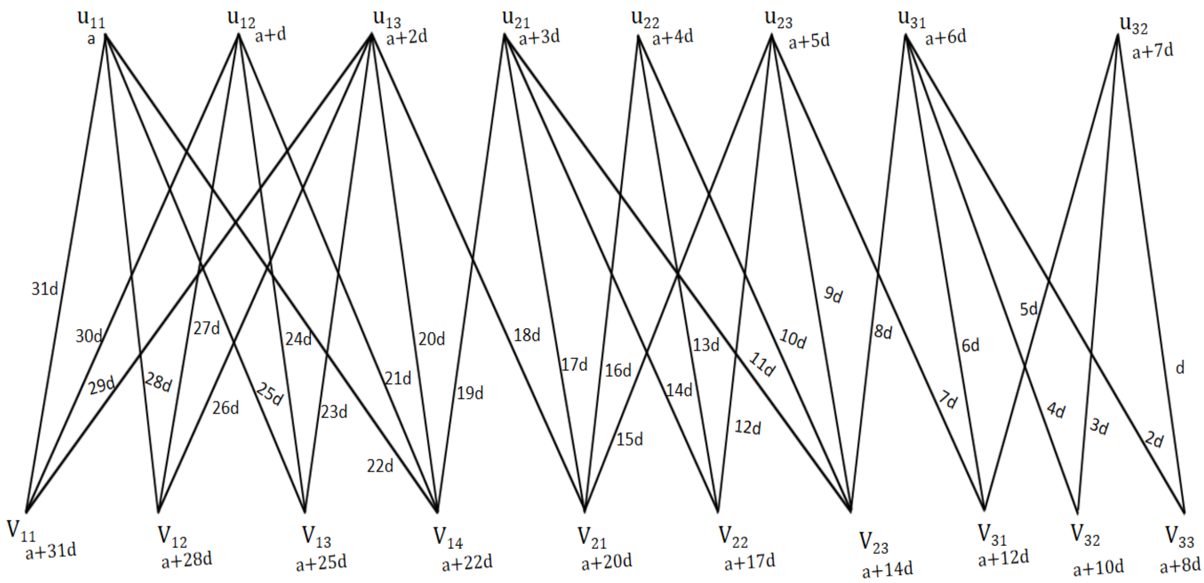

Figure-2: Path union of $G = < K_{3,4}, K_{3,3}, K_{2,3} >$ and its graceful labeling.

Theorem 2.3: The join sum of complete bipartite graph is Arithmetic Sequential Graceful labeling.

Proof: Let G be a $\langle K_{r_1,s_1}, K_{r_2,s_2}, \ldots, K_{r_\tau,s_\tau} \rangle$ graph, $\tau \geq 2$.

$V(G) = \{u_{i,\zeta} : 1 \leq i \leq \tau, 1 \leq \zeta \leq r_i\} \cup \{v_{i,\kappa} : 1 \leq i \leq \tau, 1 \leq \kappa \leq s_i\}$

$\cup \{w_i : 1 \leq i \leq \tau - 1\}$ and

$E(G) = \{u_{i,\zeta}\, v_{i,\kappa} : 1 \le i \le \tau, 1 \le \zeta \le r_i, 1 \le \kappa \le s_i\} \cup \{u_{i,r_i}, w_i\} : 1 \le i \le \tau - 1$

$\cup\ \{w_i, u_{i+1}, 1 \le i \le \tau - 1\}$

$$|V| = (\tau - 1) + \sum_{i=1}^{\tau}(r_i + s_i)\ , |E| = 2(\tau - 1) + \sum_{i=1}^{\tau}(r_i\, s_i)$$

We define a function $f: V(G) \to \{a, a + d, a + 2d, a + 3d, \ldots, 2(a + qd)\}$

$f(u_{1,\zeta}) = a + (\zeta - 1)d\ , 1 \le \zeta \le r_1$

$$f(v_{1,\kappa}) = a + \left[\left[2(\tau - 1) + \sum_{i=1}^{\tau}(r_i s_i)\right] - r_1(\kappa - 1)\right] d\ , 1 \le \kappa \le s_1$$

$$f(w_{\zeta-1}) = a + \left[2(\tau - 1) + \sum_{i=1}^{\tau}(r_i s_i) - \left[\sum_{l=1}^{\zeta-1} r_l(s_l - 1)\right] - (2\zeta + 3)\right] d, 2 \le \zeta \le \tau$$

$$f(u_{i,\zeta}) = a + \left[\sum_{l=1}^{i-1} r_l + (\zeta - 1)\right] d, 2 \le i \le \tau\ , 1 \le \zeta \le s_i$$

$$f(v_{i,\kappa}) = a + \left[\left[2(\tau - 1) + \sum_{i=1}^{\tau}(r_i s_i)\right] - \sum_{l=1}^{i-1} r_l(s_l - 1) - 2(i - 1) - r_i(\kappa - 1)\right] d\ ,$$

$2 \le i \le \tau\ , 1 \le \kappa \le s_i$

Table 3: Edge labels of the graph $< K_{r_1,s_1}, K_{r_2,s_2}, \ldots, K_{r_\tau,s_\tau} >, \tau \ge 2$

$\boldsymbol{f^*(u\, v), \forall\, uv \in E(G)}$	**Edge labels**
$f^*(u_{1,\zeta} v_{1,\kappa})$	$[(\zeta - 1) - [2(\tau - 1) + \sum_{i=1}^{\tau}(r_i s_i)] - r_1(\kappa - 1)]d$, $1 \le \kappa \le s_1, 1 \le \zeta \le r_1$.
$f^*(u_{1,\zeta} w_{\zeta-1})$	$\left[(\zeta - 1) - [2(\tau - 1) + \sum_{i=1}^{\tau}(r_i s_i)] - \left[\left[\sum_{l=1}^{\zeta-1} r_l(s_l - 1)\right] - (2\zeta + 3)\right]\right] d$, $2 \le i \le \tau\ , 1 \le \zeta \le r_1$,
$f^*(u_{i,\zeta} v_{i,\kappa})$	$\left[\sum_{l=1}^{i-1} r_l + (\zeta - 1) - \left[[2(\tau - 1) + \sum_{i=1}^{\tau}(r_i s_i)] - \sum_{l=1}^{i-1} r_l(s_l - 1) - 2(i - 1) - r_i(\kappa - 1)\right]\right] d$, $2 \le i \le \tau\ , 1 \le \zeta \le r_i, 1 \le \kappa \le s_i$

It is clear that the function f is injective and also table 3 shows that

$f^*: E \to \{d, 2d, 3d, 4d, \dots, qd\}$ is bijective. Hence f is arithmetic sequential graceful labeling and the graph $< K_{r_1,s_1}, K_{r_2,s_2}, \dots, K_{r_\tau,s_\tau} >, \tau \geq 2$ is arithmetic sequential graceful graph.

Illustration 2.3.1: Join sum of $G =< K_{3,4}, K_{3,3}, K_{2,3} >$ and its graceful labeling shown in figure-3.

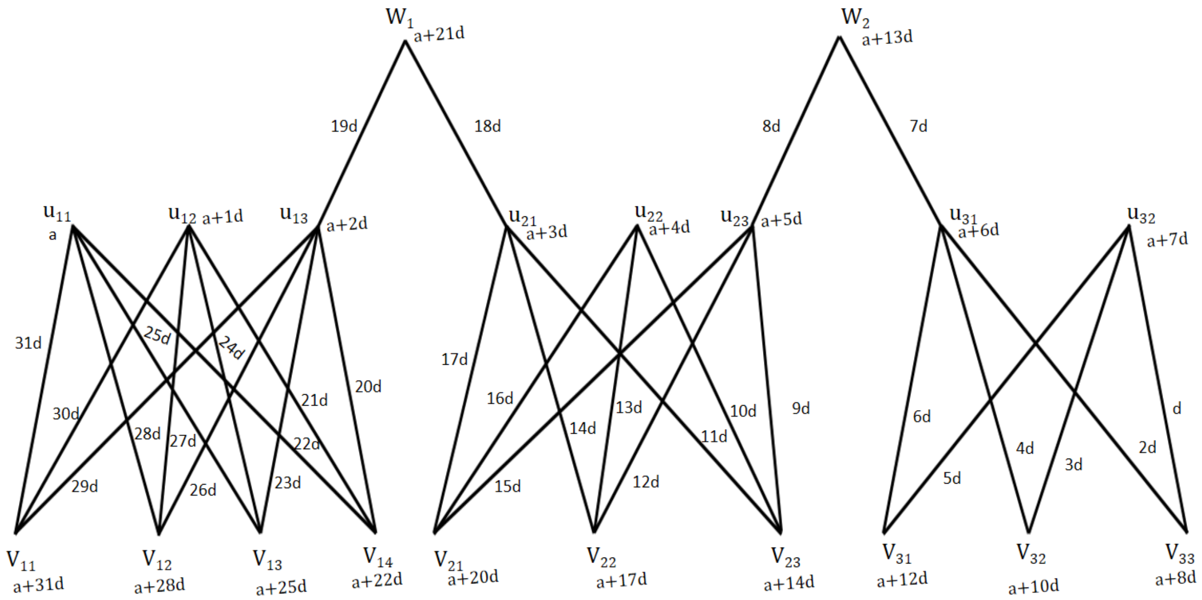

Figure-3: Join sum of $G =< K_{3,4}, K_{3,3}, K_{2,3} >$ and its graceful labeling.

Theorem 2.4: The graph $\mathcal{S}(\tau. K_{r,s})$ is Arithmetic sequential graceful labeling.

Proof: Let G be a $\mathcal{S}(\tau. K_{r,s})$ graph.

$V(G) = \{u_{li} : 1 \leq l \leq \tau, 1 \leq i \leq r\} \cup \{v_{l\zeta} : 1 \leq l \leq \tau, 1 \leq \zeta \leq s\} \cup \{u_0\}$ and

$E(G) = \{v_{l\zeta} u_{l\zeta} : 1 \leq l \leq \tau, 1 \leq \zeta \leq s, 1 \leq i \leq r\} \cup \{u_0 u_{l,1} : 1 \leq l \leq \tau\}$

$|V| = \tau(r+s) + 1, \qquad |E| = \tau(rs+1)$

We define a function $f: V(G) \to \{a, a+d, a+2d, a+3d, \dots, 2(a+qd)\}$

$f(u_0) = a$

$f(v_{1,\zeta}) = a + \zeta d, \; 1 \leq \zeta \leq s$

$f(u_{1,i}) = a + [\tau(rs+1) - (i-1)s]d, 1 \leq i \leq r$

$f(v_{2,\zeta}) = f(v_{1,\zeta}) + [\tau(rs+1) - (rs+1)]d, 1 \leq \zeta \leq s$

$f\ (u_{2,i})\ =\ f\ (u_{1,i}) - [\tau\ (rs+1) - (rs+1)]\ d\ , 1\ \leq\ i\ \leq\ r$

$f\ (v_{l,\zeta})\ =\ f(v_{l-2,\zeta}) - [\ (-1)^{\ l}(rs+1)\ ]\ d\ , 1\ \leq\ \zeta\ \leq\ s\ , 3\ \leq\ l\ \leq\ \tau$

$f\ (u_{l,i})\ =\ f(u_{l-2,i}) - [\ (-1)^{\ l}\ (rs+1)\ ]\ d\ , 1\ \leq\ i\ \leq\ r\ , 3\ \leq\ l\ \leq\ \tau$

From the function $f^*: E(G) \to \{d, 2d, 3d, 4d, \ldots, qd\}$ we get the edge labels of the graph $S(\tau. K_{r,s}), s \geq 1$ as follows

Table:4 Edge labels of the graph $S(\tau. K_{r,s}), s \geq 1$

$\boldsymbol{f^*(u\, v)}, \forall\, \boldsymbol{uv} \in \boldsymbol{E(G)}$	**Edge labels**
$f^*(u_0, u_{1,1})$	$[\tau(rs+1)]d$
$f^*(u_0, u_{2,1})$	$(rs+1)d$
$f^*(u_0, u_{l,i})$	$f(u_{l-2,\mathrm{i}}) + [\ (-1)^l\ (rs+1)]d,$ $1\ \leq\ \mathrm{i}\ \leq\ r, 1\ \leq\ \zeta\ \leq\ s$
$f^*(v_{1\zeta}, u_{1,\mathrm{i}})$	$[\tau(rs+1) - (\mathrm{i}-1)s - \zeta]d,$ $1\ \leq\ \mathrm{i}\ \leq\ r, 1\ \leq\ \zeta\ \leq\ s$
$f^*(v_{2,\zeta}, u_{2,\mathrm{i}})$	$f\ (u_{1\mathrm{i}}) - f(v_{1\zeta}) + 2[\tau(rs+1) - (rs+1)]d,$ $1\ \leq\ \mathrm{i}\ \leq\ r, 1\ \leq\ \zeta\ \leq\ s$
$f^*(v_{l,\zeta}, u_{l,i})$	$f(u_{l-2,i}) -\ f(v_{l-2,\zeta}) + 2[\ (-1)^l\ (rs+1)]d,$ $3\ \leq\ l\ \leq\ \tau, 1\ \leq\ i\ \leq\ r, 1\ \leq\ \zeta\ \leq\ s$

It is clear that the function f is injective and also table 4 shows that

$f^*: E \to \{d, 2d, 3d, 4d, \ldots, qd\}$ is bijective. Hence f is arithmetic sequential graceful labeling and the graph $S(\tau. K_{r,s})$ is arithmetic sequential graceful graph.

Illustration 2.4.1: $\mathcal{S}(3. K_{r,s}\ )$ and its graceful labeling shown in figure-4.

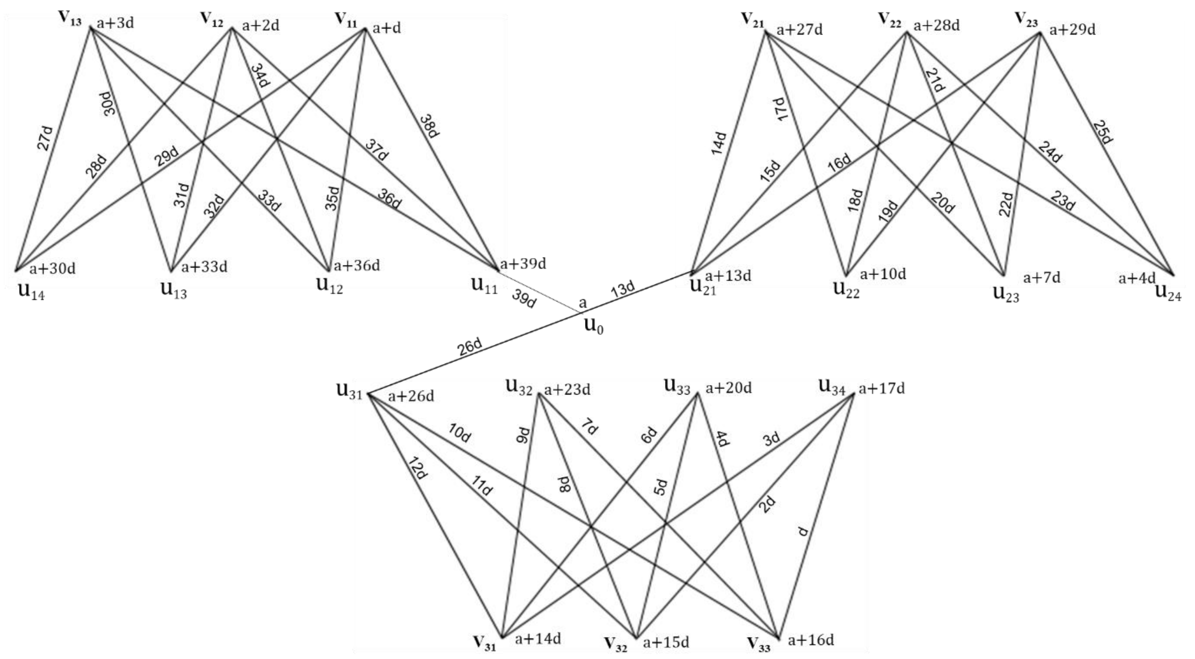

Figure -4: $\mathcal{S}(3.K_{r,s})$ and its graceful labeling.

Theorem 2.5: The graph $\mathcal{S}(K_{r,s})$ admits Arithmetic Sequential Graceful labeling.

Proof: Let $V(G) = \{u_{0,i} : 1 \le i \le r\} \cup \{v_{0,\zeta} : 1 \le \zeta \le s\}$

$\cup \{u_{\kappa,i} : 1 \le \kappa \le r+s, 1 \le i \le r\}$

$\cup \{v_{\kappa,\zeta} : 1 \le \kappa \le r+s, 1 \le \zeta \le s\}$ and

$E(G) = \{u_{0,i}\, v_{0,\zeta} : 1 \le i \le r, 1 \le \zeta \le s\}$

$\cup \{u_{\kappa,i}\, v_{\kappa,\zeta} : 1 \le \kappa \le r+s, 1 \le i \le r, 1 \le \zeta \le s\}$

$\cup \{u_{0,i} u_{\kappa,i} : 1 \le \kappa \le r+s, 1 \le i \le r\}$

$\cup \{v_{0,\zeta}\, v_{\kappa,\zeta} : 1 \le \kappa \le r+s, 1 \le \zeta \le s\}$

$|V| = (r+s)(r+s), |E| = (r+s+1)(rs) + (r+s)$

We define a function $f: V(G) \to \{a, a+d, a+2d, a+3d, \ldots, 2(a+qd)\}$

$f(u_{0,i}) = a + (i-1)d, 1 \le i \le r$

$f(v_{0,\zeta}) = a + [(r+s+1)rs + (r+s) - (\zeta-1)r]d, 1 \le \zeta \le s$

$f(u_{1,i}) = a + [(r+s+1)rs + (r+s) - (rs+1) + i]d, 1 \le i \le r$

$f(v_{1,\zeta}) = a + (\zeta r)d, 1 \le \zeta \le s$

$f(u_{\kappa,i}) = f(u_{(\kappa-2),i}) + (rs+1)\, d, 2 \le \kappa \le r+s, 1 \le i \le r$

$f(v_{\kappa,\zeta}) = f(v_{(\kappa-2),\zeta}) - (rs+1)\,d\,, 2 \leq \kappa \leq r+s\,, 1 \leq \zeta \leq s$

From the function $f^*: E(G) \rightarrow \{d, 2d, 3d, 4d, \ldots, qd\}$ we get the edge labels of the graph as follows

Table:5 Edge labels of the graph $\mathcal{S}(K_{r,s})\ s \geq 1$

$\boldsymbol{f^*(u\,v), \forall\, uv \in E(G)}$	**Edge labels**
$f^*(v_{0,\zeta}u_{0,i})$	$[(r+s+1)rs + (r+s) - (\zeta-1)r\,(i-1)]d\,,$ $1 \leq i \leq r\,, 1 \leq \zeta \leq s$
$f^*(v_{1,\zeta}\, u_{1,i})$	$[(r+s+1)rs + (r+s) - (rs+1) + i - \zeta r]d,$ $1 \leq i \leq r\,, 1 \leq \zeta \leq s$
$f^*(v_{\kappa,\zeta}\, u_{\kappa,i})$	$[f(u_{(\kappa-2),i}) - f(v_{(\kappa-2),\zeta}) + 2(rs+1)]\,d,$ $2 \leq \kappa \leq r+s\,, 1 \leq i \leq r\,, 1 \leq \zeta \leq s$
$f^*(u_{0,1}u_{1,i})$	$[\,(r+s+1)rs + (r+s) - (rs+1) + i]d\,,$ $1 \leq i \leq r$
$f^*(u_{0,i}\, u_{i,i}\,)$	$\left[f\left(u_{(i-2),i}\right) + (rs+1)d - (a + (i-1)d)\right]$ $2 \leq i \leq r$
$f^*(v_{0,1}\, v_{(r+1),1})$	$[\,(r+s+1)\,rs + r+s - r\,]d$
$f^*(v_{0,\zeta}\, v_{(r+\zeta),\zeta})$	$[f\left(v_{(\zeta-2),\zeta}\right) - (rs+1)d\,] - [a + \begin{bmatrix} (r+s+1)rs \\ + (r+s) - (\zeta-1)r \end{bmatrix} d\,,$ $2 \leq \zeta \leq s$

It is clear that the function f is injective and also table 5 shows that

$f^*: E \rightarrow \{d, 2d, 3d, 4d, \ldots, qd\}$ is bijective. Hence $\mathcal{S}(K_{r,s})$ is arithmetic sequential graceful graph.

Illustration 2.5.1: $\mathcal{S}(K_{3,2})$ and its graceful labeling shown in figure – 5.

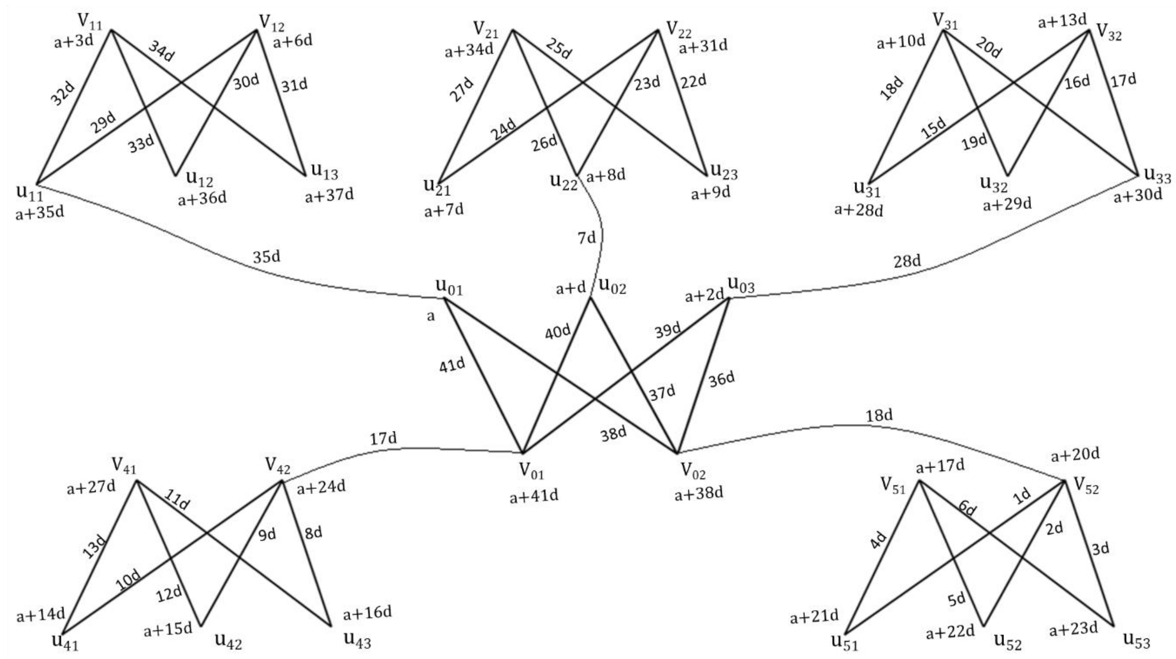

Figure -5: $\mathcal{S}(K_{3,2})$ and its graceful labeling.

3. Conclusion

Here we demonstrated arithmetic sequential graceful labeling of some graphs obtained by complete bipartite graphs. Present work contributes five new results. We discussed gracefulness complete bipartite graph, Open star of graph, Join sum of graph, Path union graph. The labeling patten is showed by means of illustrations which dispense better understanding of attained results.

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