Tests for Multivariate Normality based on the Empirical Moment Generating Function and other Criteria

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Abstract

In this article, we present test statistics for assessing the compatibility of a multivariate random sample with the multivariate normal distribution. Three criteria are used to develop the proposed tests: the empirical multivariate moment generating function, mixed partial functional moments, and the empirical distribution function (EDF). The suggested tests are weighted integrals of the deviance or square deviance of the EMGF from the MGF, weighted integrals of the deviance or square deviance between the EMPM and the MPM, and EDF-type tests based on the stochastic ordering of the d-dimensional sample points. We derive computational forms of the proposed tests and conduct simulations to find their approximate critical values at a nominal level of 0.05. In addition, simulations approximate the powers of the proposed tests for various sample sizes when testing the bivariate normal distribution against a set of alternatives. The results show that the proposed tests compete well with some existing ones.

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1. **Introduction**

Many test procedures have been used in the literature to test data compatibility to a bivariate or multivariate normal distribution. These procedures are based on some distributional attributes, as in the univariate case. Based on attributes, there are several approaches to constructing test statistics, such as graphical, moments, characteristic function, moment generating function, and empirical distribution function approach. Analogous to the univariate case, graphical methods have been developed to test for MV normality. For example, Mardia (1970) and Ozturk (1992) suggested various Q-Q plot tests based on Mahalanobist distance. Other graphical tests using T_3-plot, which is a plot based on an estimate of the third cumulant, were suggested by Ghosh (1996).

Regarding moments-based tests, Mardia (1970, 1975), Malkovich and Afifi (1973), Small (1980), Mardia and Foster (1983), BozdoganRamirezand (1986), and Zhou and Y. Shao (2014) developed tests using skewness and kurtosis.

If $X_1, \dots, X_n$ is a random sample from a k-variate normal distribution with a mean μ and variance-covariance matrix Σ, then $\Psi(X_i, \mu, \Sigma) = (X_i - \mu)'\Sigma^{-1}(X_i - \mu)$ are independent χ^2_k variates. Utilizing this property, univariate empirical distribution function tests, including Kolmogorov-Smirnov, Cramer-von-Mises, and Anderson-Darling tests, can be applied to test multivariate normality by testing that $\Psi(X_i, \mu, \Sigma)$ are independent and identically distributed χ^2_k random variables. Koziol (1982, 1983) developed testprocedures utilizing this property. Sze'kelya and Rizzo (2005) and McAssey (2013) proposed tests based on the Mahalanobist distance, which is defined as the square root of $(X_i - \mu)'\Sigma^{-1}(X_j - \mu)$.

Extending the Kolmogorov-Smirnov goodness of fit test and other empirical distribution function tests to the multivariate case was considered by many authors such as Peacock (1983), Fasano and Franceschin (1987), Cabaña and Cabaña (1997), Justel et al. (1997), Lopes et al. (2007), Chiu and. Liu (2009), Kesemen et al. (2021).

Epps and Pully (1983) proposed a test for the univariate normal distribution based on the empirical characteristic function; Baringhaus and Henze (1988), Henze and Wagner (1997), and Fan (1997) extended this approach to the multivariate case. In addition, there are other tests based on criteria different from the ones mentioned above, such as those of Royston (1992), Mudholkar et al. (1992), Sürücü (2006), and Zhang et al. (2012), to name a few. Finally, the Mardia (1980) article reviews univariate and multivariate goodness of fit tests before 1980.

Based on a random sample $X_1, \dots, X_n$, Zghoul (2010) proposed the following univariate normality test

$$T_{n,\beta} = n\int_{-\infty}^{\infty} (M_n(t) - M_0(t))^2 \exp(-\beta t^2)dt$$

where $M_0(t) = \exp(t^2/2), t \in \mathbb{R}$, is the moment generating function of the standard normal distribution, β is a fixed positive parameter, and

$$M_n(t) = \frac{1}{n}\sum_{j=1}^{n} \exp(tY_{n,j}), \quad t \in \mathbb{R}$$

is the empirical moment generating function of the standardized variables,

$$Y_{n,j} = \frac{X_j - \bar{X}_n}{S_n}, \quad j = 1, \dots, n.$$

The null hypothesis X_1 is $N(\mu, \sigma^2)$ is rejected for large values of $T_{n,\beta}$.

The test of Zghoul showed strong compatibility with prominent tests such as the Anderson–Darling test, the Shapiro–Wilk test, the Epps–Pulley test, and the D'Agostino test. This test was studied further by Henze and Kotch (2017), who proved that the test has attractive properties. They showed that the test has a non-degenerate asymptotic null distribution and is consistent against general alternatives. They also showed that an affine transformation of the test statistic has a non-generic asymptotic distribution. Moreover, they proved that as the parameter associated with the weight function tends to infinity, an affine transformation of the test statistic approaches squared sample skewness.

A test based on an empirical truncated mean was introduced and studied by Zghoul and Awad (2010).

The test is a weighted L^2-type statistic and is given by

$$T_n = n\int_{-\infty}^{\infty}\left(n^{-1}\sum_{i=1}^{n} Z_j I_{Z_j>t} - \phi(t)\right)^2 \phi(t)dt.$$

Z_j here is the standardized variable $Z_j = (X_j - \mu)/\sigma$.

The idea of the test is based on the fact that if f is a differentiable density function on $\mathbb{R}$, and t is a nonzero real number, then

$$\int_t^{\infty} xf(x)dx = f(t) \text{ if and only if } f(x) = (2\pi)^{-1}e^{-\frac{x^2}{2}}.$$

Under the null hypothesis $X_j \sim N(\mu, \sigma^2)$, and $\phi(t)$ is the expected value of the truncated variable $Z_j I_{Z_j>t}$, where I is the usual indicator function. That is,

$$\phi(t) = E\left[Z_j I_{Z_j>t}\right] = \frac{1}{\sqrt{2\pi}}\int_t^{\infty} z\, e^{-z^2/2}\, dz.$$

$\phi(t)$ is also used as a weight function in (2). By the laws of large numbers, $n^{-1}\sum_{i=1}^{n} Z_j I_{Z_j>t}$ converges to its mean $\phi(t)$. Thus, if the sample is from a distribution other than the normal distribution, the value of the test T is expected to be significantly large, in which case the null hypothesis will be rejected. This test also competes well with classical goodness of fit tests.

As we mentioned at the beginning of this section, many articles addressed goodness of fit for multivariate distributions, including the multivariate normal. Nevertheless, there is still room to add to the field because, although some tests perform better than others, none of the suggested tests is best for all alternatives. In this article, we extend the tests of Zghoul (2010) and Zghoul and Awad (2010) to higher dimensions. First, test statistics will be introduced, and their computational forms will be derived. Next, the properties of the suggested tests will be studied, and then simulations will be conducted to compare the introduced tests with other multivariate tests.

2. Test Statistics

Let $X_1, X_2, \ldots, X_n$ be d-dimensional independent and identically distributed (iid) random vectors sampled from an absolutely continuous distribution function with mean μ and nonsingular covariance matrix Σ. Our purpose is to test the null hypothesis H_0: X_j is distributed $N_d(\mu, \Sigma)$ against the alternative that it is not, where $N_d(\mu, \Sigma)$ denotes a d-dimensional normally distributed random vector with mean μ and covariance matrix Σ. We will first introduce two tests based on the empirical MGF. It is easy to verify that if $X \sim N_d(\mu, \Sigma)$, then $Z=\Sigma^{-\frac{1}{2}}(X - \mu)$ is distributed $N_d(0, I_d)$, where I_d is $d \times d$identity matrix. The sample mean and sample covariance matrix are calculated as $\overline{X}_n = n^{-1}\sum_j^n X_j$ and $S_n = n^{-1}\sum_{j=1}^{n}(X_j - \bar{X}_n)(X_j - \bar{X}_n)'$.

In this section, we propose two tests based on the empirical moment generating function and two other tests based on the partial sample mean.

2.1 Tests based on the empirical moment generating function

Under H_0, the MGF of Z is $M_Z(t) = \exp(-\frac{1}{2}t't)$, where $t \in \mathbb{R}^d$, and t' is the transpose of t. The empirical MGF based on a random sample of size n from a d-dimensional distribution is

$$M_n(t) = \frac{1}{n}\sum_{j=1}^{n} \exp(t'Z_j) \quad .$$

We will consider the following two-sided test based on the deviance of the empirical moment generating function from its mean.

$$MT1 = \sqrt{n}\int_{-\infty}^{\infty}\left(\frac{1}{n}\sum_{j=1}^{n} \exp(t'Z_j) - \exp\left(-\frac{1}{2}t't\right)\right)\exp(-\beta t't)dt.$$

Applying the integral on both terms of the integrand yields the following computational form for MT1:

$$MT1 = \left(\frac{\pi}{\beta}\right)^{\frac{d}{2}}\frac{1}{\sqrt{n}}\sum_{j=1}^{n} \exp\left(\sum_{i=1}^{d}\frac{Z_{ij}^2}{4\beta}\right) - \sqrt{n}\left(\frac{2\pi}{1+2\beta}\right)^{\frac{d}{2}}.$$

A weighted L^2-type test statistic usually has the form

$$n\int_{-\infty}^{\infty}\left(\Psi_n(X,t) - \Psi_0(t)\right)^2\omega(t)dt,$$

where for each t, Ψ_n is a function of the sample, $\Psi_0(t) = E\left(\Psi_n(X,t)\right)$, and $\omega(t)$is a conveniently chosen weight function. Setting $\Psi_n(Z,t) = M_n(t)$, $\Psi_0(t) = M_Z(t)$, and $\omega(t) = \exp(-\beta t't)$, we propose the following test for multivariate normality.

$$MT2 = n\int_{-\infty}^{\infty}\left(\frac{1}{n}\sum_{j=1}^{n} \exp(t'Z_j) - \exp\left(-\frac{1}{2}t't\right)\right)^2 \exp(-\beta t't)dt.$$

We may notice that the MT1 test is just a weighted integral of the bias of $M_n(t)$ as an estimator for $M_Z(t)$, and the MT2 test is a weighted integral of the mean square error of $M_n(t)$ as an estimator for $M_Z(t)$. If the null hypothesis is true, the laws of large numbers assure the convergence of $M_n(t)$ to its expectation $M_Z(t)$, hence MT1 and MT2 are expected to be small, especially for large samples. Thus, it is convenient to propose rejecting H_0 for large values of T_n.

When μ and Σ are unknown, which is usually the case, they will be replaced by their maximum likelihood estimates, $\bar{X}_n = \frac{1}{n}\sum_{j=1}^{n} X_j$ and $S_n = \frac{1}{n}\sum_{j=1}^{n}\left(X_j - \bar{X}_n\right)\left(X_j - \bar{X}_n\right)'$, respectively. In this case, we set $Z_j = S_n^{-1/2}\left(X_j - \bar{X}_n\right), \quad j = 1, \dots, n.$

A computational form of $T_{n,1}$can be obtained by expanding the integrand and applying term-by-term integration:

$$MT2 = \frac{1}{n}\sum_{j,k=1}^{n}\int_{-\infty}^{\infty} \exp\left[t'\left(Z_j + Z_k - \beta t\right)\right]dt$$

$$-2\sum_{j=1}^{n}\int_{-\infty}^{\infty} \exp\left[t'Z_j - \left(\frac{1}{2}+\beta\right)t't\right]dt + n\int_{-\infty}^{\infty} \exp[-(1+\beta)t't]\,dt$$

$$= \left(\frac{\pi}{\beta}\right)^{\frac{d}{2}} \frac{1}{n} \sum_{j,k=1}^{n} \exp\left[\sum_{i=1}^{d} \frac{\left(Z_{ij} + Z_{ik}\right)^2}{4\beta}\right] - 2 \left(\frac{2\pi}{1+2\beta}\right)^{\frac{d}{2}} \sum_{j=1}^{n} \exp\left(\sum_{i=1}^{d} \frac{Z_{ij}^2}{2+4\beta}\right) + n\left(\frac{\pi}{1+\beta}\right)^{\frac{d}{2}}.$$

2.2 Tests Based on Partial Functional Mean Characterization

Let X be a random variable with an absolutely continuous distribution function F and probability density function f. We define the partial functional mean of $\psi(X)$ on the interval $[-\infty, t]$ as

$$\mu(t) = \int_{-\infty}^{t} \psi(z)f(z)\, dz,$$

where ψ is chosen in such a way that the integral exists. We extend this definition to the multivariate case. See Mazouz and Zghoul (2022).

Definition 4.1. Let $Z = (Z_1, Z_2, \dots, Z_d)$ be a d-dimensional random vector with a continuous density function f defined on $\mathbb{R}^d$ and $\psi(Z)$ is a function of Z, then for any $t = (t_1, \dots, t_d) \in \mathbb{R}^d$,

$$\mu(t_1, \dots, t_d) = \int_{-\infty}^{t_1} \dots \int_{-\infty}^{t_d} \psi(z)f(z)\, dz \tag{5.1}$$

is called the partial functional mean of $\psi(Z)$, provided the integral exists.

In the following theorem, we prove a characterization of the MVN based on a partial functional mean.

Theorem 2.1: Given a d-dimensional vector of random variables X with joint probability density function $f(x)\colon \mathbb{R}^d \to \mathbb{R}$ whose all first partial derivatives exist, then

$$\int_{A(t)}^{\infty} \left(\prod_{j=1}^{d} x_j\right) f(x)dx = f(t) \text{ iff } f(t) = \prod_{j=1}^{d} \phi(t_j)$$

where $A(t) = \prod_{j=1}^{d} [t_j, \infty)$, $\phi(t) = (2\pi)^{-1} e^{-t^2/2}$, and t is a nonzero vector of real numbers.

Proof: Assume $f(t) = \prod_{j=1}^{d} \phi(t_j)$, then

$$\int_{A(t)}^{\infty} \left(\prod_{j=1}^{d} x_j\right) f(x)dx = \int_{A(t)}^{\infty} \prod_{j=1}^{d} x_j \prod_{j=1}^{d} \phi(x_j)\, dx$$

$$= \prod_{j=1}^{d} \int_{t_j}^{\infty} x_j\, \phi(x_j) dx_j$$

$$= \prod_{j=1}^{d} \int_{t_j}^{\infty} -\, \phi'(x_j) dx_j$$

$$= \prod_{j=1}^{d} \phi(t_j).$$

Conversely, let $\int_{A(t)}^{\infty} \left(\prod_{j=1}^{d} x_j\right) f(x)dx = f(t)$, then stepwise partial differentiation of both sides of () with respect to t_j, $j = 1, \dots, d$, yields

$$(-1)^d\left(\prod_{j=1}^{d} t_j\right) f(t) = \frac{\partial}{\partial t} f(t).$$

A rearrangement of the above equation gives

$$\frac{\frac{\partial}{\partial t} f(t)}{f(t)} = (-1)^d\left(\prod_{j=1}^{d} t_j\right)$$

The solution of this equation, after a proper normalization, is $f(t) = \prod_{j=1}^{d} \phi(t_j)$. □

Now, let

$$\mu_X(t) = \int_{A(t)}^{\infty}\left(\prod_{j=1}^{d} x_j\right) f(x)dx = \prod_{j=1}^{d} \phi(t_j)$$

If $X_1, \ldots, X_n$ is a random sample from a d-variate normal distribution with a mean μ and variance-covariance matrix Σ, then

$$Z_j = \Sigma^{-\frac{1}{2}}\left(X_j - \mu\right) \sim N(0, I_d).$$

Therefore, the empirical analog of $\mu_X(t)$ is

$$\psi_n(t) = \frac{1}{n}\sum_{i=1}^{n}\left(\prod_{j=1}^{d} z_{ij}\, I_{Z_{ij} > t_i}\right).$$

Utilizing the characterization of Theorem 1, we introduce two test statistics:

- A weighted integral of the deviance of the empirical from the theoretical partial functional mean,

$$PM1 = \int_{-\infty}^{\infty}\left(\psi_n(t) - \mu_X(t)\right)\omega(t)dt.$$

- A weighted integral of the squared deviance of the empirical from the theoretical partial functional mean,

$$PM2 = \int_{-\infty}^{\infty}\left(\psi_n(t) - \mu_X(t)\right)^2\omega(t)dt,$$

where the weight function $\omega(t)$ is chosen so that the integrals converge. A convenient choice for $\omega(t)$ is

$$\omega(t) = (2\pi\beta^2)^{-d/2} e^{-\sum_{j=1}^{n}\left(t_j^2/2\beta^2\right)}.$$

By the laws of large numbers, for fixed t, $\psi_n(t)$ converges to $\mu_X(t)$ as n approaches ∞, PM1 and PM2 are expected to be small, especially for a reasonably large sample size. Therefore, the null hypothesis must be rejected for large absolute values of the tests.

To derive a computational form for tests $T_n^1(z)$, and $T_n^2(z)$ we have

$$T_n^1(z) = \int_{-\infty}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\prod_{i=1}^{d} z_{ij}\, I_{Z_{ij}>t_i} - \prod_{i=1}^{d}\phi(t_i)\right)(2\pi\beta^2)^{-d/2}e^{-\sum_{i=1}^{n}(t_i^2/2\beta^2)}dt$$

$$= \frac{1}{n}\sum_{j=1}^{n}\left(\prod_{i=1}^{d} z_{ij}\right)\int_{-\infty}^{z_{ij}}(2\pi\beta^2)^{-d/2}e^{-\sum_{i=1}^{n}(t_i^2/2\beta^2)}\,dt_i$$

$$+\int_{-\infty}^{\infty}\left(\prod_{j=1}^{d}\phi(t_j)\right)(2\pi\beta^2)^{-d/2}e^{-\sum_{j=1}^{n}(t_j^2/2\beta^2)}d\boldsymbol{t}$$

$$= \frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{d} z_{ij}\ \Phi(z_{ij}/\beta) - [2\pi\beta^2(\beta^2+1)]^{-d/2}.$$

And

$$T_n^2(\boldsymbol{z}) = \int_{-\infty}^{\infty}\left(\frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{d} z_{ij}\, I_{Z_{ij}>t_i} - \prod_{i=1}^{d}\phi(t_i)\right)^2(2\pi\beta^2)^{-d/2}e^{-\sum_{i=1}^{n}(t_i^2/2\beta^2)}d\boldsymbol{t}$$

$$= \frac{1}{n^2}\sum_{j,k=1}^{n}\prod_{i=1}^{d} z_{ij}\, z_{ik}\int_{-\infty}^{(z_{ij}\wedge z_{ik})}(2\pi\beta^2)^{-d/2}e^{-\sum_{j=1}^{n}(t_j^2/2\beta^2)}\,dt_j$$

$$-\frac{2}{n}\sum_{j=1}^{n}\prod_{i=1}^{d} z_{ij}\int_{-\infty}^{z_{ij}}\prod_{i=1}^{d}\phi(t_i)\,(2\pi\beta^2)^{-\frac{d}{2}}e^{-\sum_{i=1}^{n}\left(\frac{t_i^2}{2\beta^2}\right)}dt_i$$

$$+\int_{-\infty}^{\infty}\left(\prod_{i=1}^{d}\phi(t_i)\right)^2(2\pi\beta^2)^{-\frac{d}{2}}e^{-\sum_{i=1}^{n}\left(\frac{t_i^2}{2\beta^2}\right)}dt_i$$

$$= \frac{1}{n^2}\sum_{j,k=1}^{n}\prod_{i=1}^{d} z_{ij} z_{ik}\ \Phi\left(\sqrt{(\beta^2+1)/\beta^2}(z_{ij}\wedge z_{ik})\right)$$

$$-\frac{2[2\pi(\beta^2+1)]^{-d/2}}{n}\sum_{j=1}^{n}\left(\prod_{i=1}^{d} z_{ij}\ \Phi(\sqrt{(\beta^2+1)/\beta^2}z_{ij}\right)+[2\pi(\beta^2+1)]^{-d/2}.$$

2.3 Empirical Distribution Function based tests

Following Koziol (1982), three tests analogous to the Kolmogorov-Smirnov, Cramer-von Mises, and Anderson-Darling tests are considered. To conduct these tests, we first compute the value of the null distribution at each of the d-dimension sample points, then sort these values, and then evaluate the value of the tests as in the univariate case. To illustrate, let $z_{ij}, i = 1 \dots, d; j = 1, \dots, n$, be the sample points from some d-dimensional distribution, and $F_0(z_{1j}, z_{2j}, \dots, z_{dj})$ are the sorted values of the assumed distribution under the null hypothesis, then the computation formulas for KS, CvM, and AD tests:

a) $KS = \max_{1 \le j \le n}\left(F_0(z_{1j}, z_{2j}, \dots, z_{dj}) - \frac{i-1}{n}, \frac{i}{n} - F_0(z_{1j}, z_{2j}, \dots, z_{dj})\right)$

b) $CvM = \frac{1}{12n} + \sum_{j=1}^{n}\left[\frac{2j-1}{2n} - F_0(z_{1j}, z_{2j}, \dots, z_{dj})\right]^2$

c) $AD = -n - \sum_{j=1}^{n}\frac{2i-1}{n}\left[\ln\left(F_0\left(z_{1j}, z_{2j}, \dots, z_{dj}\right)\right) + \ln\left(1 - F_0\left(z_{1(n+1-j)}, z_{2(n+1-j)}, \dots, z_{d(n+1-j)}\right)\right)\right]$.

3. Simulated Percentage Points

In this section, we simulate specific percentiles for the proposed tests and other tests to which the performance of the proposed tests will be compared. The percentiles required to calculate the powers of tests for significance levels $\alpha = 0.01, 0.05,$ and $0.1,$ and sample sizes $n = 20, 30, 50,$ and 100 will be simulated. In each case, 10,000 replications will be performed. In addition to the suggested tests, the following tests will be considered for comparison purposes:

1) Mardia (1970) tests

a) Skewness-based test:

$$MA = \frac{1}{6n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[(x_i - \overline{X}_n)'\hat{\Sigma}^{-1}\left(x_j - \overline{X}_n\right)\right]^3$$

where

$$\hat{\Sigma} = \frac{1}{n}\sum_{j=1}^{n}\left(X_j - \overline{X}_n\right)\left(X_j - \overline{X}_n\right)'$$

b) Kurtosis-based test:

$$MB = \sqrt{\frac{n}{8d(d+2)}}\left\{\frac{1}{n}\sum_{i=1}^{n}\left[(X_i - \overline{X}_n)'\hat{\Sigma}^{-1}(X_i - \overline{X}_n)\right]^2 - d(d+2)\right\}.$$

2) Characteristic function-based test:

Baringhaus and Henze (1988) developed this test as a generalization of the univariate normality test by Epps and Pully (1983).

$$\mathrm{BHEP} = n\int_{\mathbb{R}^d} \left|\frac{1}{n}\sum_{j=1}^{n} e^{it'\hat{\Sigma}^{-1/2}(X_j-\bar{X}_n)} - e^{-|t|^2/2}\right|^2 \omega(t)dt$$

$$= \frac{1}{n}\sum_{i,j=1}^{n} e^{-\frac{1}{2}(X_i-X_j)'\hat{\Sigma}^{-1}(X_i-X_j)} - 2^{1-d/2}\sum_{j=1}^{n} e^{-\frac{1}{4}(X_i-X_j)'\hat{\Sigma}^{-1}(X_i-X_j)} + n3^{-d/2}.$$

We will restrict our computations to the two-dimensional case.

The simulated critical values for all tests under consideration at $\alpha = 0.1, 0.05,$ and 0.01 levels for the one-sided and two-sided tests and different sample sizes are displayed in Tables 1 and 2.

Table 1. Simulated critical values at $\alpha = 0.1, 0.05, 0.01$ for the one-sided tests under consideration for sample sizes $n = 10, 20, 30, 50$.

Test	n	α			Test	α		
		0.1	0.05	0.01		0.1	0.05	0.01
		90%	95%	99%		90%	95%	99%
MA	10	4.874	6.303	8.289	CvM	0.724	0.770	0.855
	20	6.328	7.994	11.833		1.313	1.375	1.514
	30	6.840	8.565	12.573		1.888	1.965	2.147
	50	7.163	9.083	13.036		3.028	3.138	3.320
BHEP	10	0.456	0.516	0.661	AD	3.633	3.821	4.173
	20	0.465	0.535	0.689		6.418	6.756	7.353
	30	0.468	0.538	0.706		9.386	9.679	10.367
	50	0.473	0.547	0.719		15.059	15.465	16.136
G	10	7.406	9.422	14.351	MGF2	1.390	1.564	2.048
	20	10.275	13.950	26.307		3.088	3.470	4.822
	30	12.254	16.648	29.938		4.582	5.345	7.422
	50	13.045	17.678	30.984		7.804	8.954	12.244

Table 2. Simulated critical values at $\alpha = 0.1, 0.05, 0.01$ for the two-sided tests under consideration for sample sizes $n = 10, 20, 30, 50$.

Test	N	α					
		0.1		0.05		0.01	
		5%	95%	2.5%	97.5%	0.5%	99.5%
MB	10	-1.160	0.215	-1.234	0.404	-1.347	0.854
	20	-1.250	0.611	-1.339	1.281	-1.421	1.651
	30	-1.343	1.105	-1.473	1.534	-1.697	2.378
	50	-1.419	1.233	-1.578	1.652	-1.867	2.831
MGF1	10	2.458	2.660	2.449	2.660	2.434	2.763
	20	3.505	4.052	3.514	3.948	3.533	3.837
	30	4.350	4.785	4.330	4.880	4.298	5.140

	50	5.663	6.170	5.639	6.277	5.599	6.676
PM1	10 20 30 50	-0.033 -0.032 -0.031 -0.031	-0.025 -0.024 -0.027 -0.027	-0.033 -0.032 -0.032 -0.031	-0.024 -0.026 -0.026 -0.027	-0.034 -0.032 -0.032 -0.032	-0.023 -0.026 -0.025 -0.026

4. Power simulation and Conclusions

Simulated powers of the proposed tests and other tests, which are taken into account, are displayed in Table 3. The powers are calculated at nominal level α and samples of size $n = 10, 20,$ and 50 against the following alternatives:

- Bivariate Student T distributions with 2 and 8 degrees of freedom; BT(2) and BT(8).
- Bivariate chi-square distribution with 4 degrees of freedom; BCHI(4).
- Bivariate Skew Normal distribution with shape parameters 0.5 and 0.9; BSKN(0.5) and BSKN(0.9).
- Bivariate Stable distribution with index parameter $\alpha = 1.2$ and skewness parameter $\beta = -0.8$ and 0.8; STB(1.2,-0.8) and BSTB(1.2,0.8).
- Bivariate Lognormal distribution with shape parameters 0.2, 0.5, and 1; BLN(0.2), BLN(0.5), and BLN(1).

Table 3. Simulated power points (rejection proportions) for the tests under consideration at nominal level $\alpha = 0.05$ when testing BVN against a set of alternatives.

	$\mathbf{n = 10}$										
	Test										
Alternative	MA	MB	BHEP	MGF1	MGF2	PM1	PM2	G	KS	CvM	AD
BT(2)	47	44	47	52	19	23	12	50	17	15	11
BT(8)	10	09	10	14	00	08	06	12	07	07	10
BCH(4)	19	13	25	20	26	14	13	21	22	37	34
BSN(.5)	04	05	05	08	05	06	05	05	05	05	04
BSN(.9)	05	06	05	09	06	06	05	06	05	06	05
BST(1.2,-8)	61	54	65	61	59	04	05	63	05	61	54
BST(1.2,8)	59	52	62	60	58	14	14	61	44	59	51
BLN(2)	07	07	08	11	09	09	08	08	10	13	11
BLN(5)	22	16	28	24	29	14	13	25	23	38	22
BLN(1)	58	44	70	54	60	18	16	58	48	58	44

	$\mathbf{n = 20}$										
	Test										
Alternative	MA	MB	BHEP	MGF1	MGF2	PM1	PM2	G	KS	CvM	AD
BT(2)	71	71	72	71	47	31	12	75	33	21	71
BT(8)	19	15	14	17	14	13	05	21	11	08	07
BCH(4)	51	25	58	29	48	18	11	45	55	75	51
BSK(.5)	05	05	05	05	05	09	04	05	05	05	05
BSK(.9)	05	06	06	06	05	08	03	06	05	06	06
BST(1.2,-.8)	91	84	93	85	89	10	10	91	28	91	84
BST(1.2,.8)	92	85	93	86	90	28	21	92	83	86	82
BLN(.2)	14	09	14	10	17	12	06	14	22	24	14
BLN(.5)	57	32	61	35	55	21	14	50	54	57	32
BLN(1)	95	73	97	75	89	31	23	90	95	73	97

	$\mathbf{n = 50}$										
	Test										
Alternative	MA	MB	BHEP	MGF1	MGF2	PM1	PM2	G	KS	CvM	AD
BT(2)	92	98	98	98	86	46	20	97	50	30	20
BT(8)	31	36	24	33	27	14	08	39	09	08	05
BCH(4)	96	59	97	58	81	22	20	86	80	99	96
BSN(.5)	05	05	06	05	05	07	05	06	06	05	05
BSN(.9)	06	06	06	06	06	07	06	07	07	08	08
BST(1.2,-.8)	100	100	100	100	100	57	65	100	100	60	11
BST(1.2,.8)	100	100	100	100	100	50	42	100	100	99	99
BLN(.2)	37	18	32	18	30	11	09	30	27	45	47
BLN(.5)	96	70	97	69	87	24	23	100	90	81	99
BLN(1)	100	99	100	98	100	45	45	100	100	100	100

We can draw the following conclusions from Table 3:

- For most alternatives, the power increases considerably when n increases from 10 to 50.
- All tests have poor performance when testing against bivariate skew-normal distributions.
- None of the tests outperforms all of the others for all alternatives.
- The power of all tests rises as the scale parameter increases when testing against a lognormal distribution.
- Most of the tests perform well for stable distribution alternatives.
- MB and MGF1 have almost similar performance for most alternatives, and MB and BHEP have almost similar performance for most alternatives.
- Empirical distribution function-based tests, particularly KS and AD, perform best for CH(4) and LN(0.2) alternatives.

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