A Computational Derivative Operational Matrix Technique for Solving Second-Order Lane-Emden Type Differential Equations via Modified Lucas Wavelets Basis

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Abstract
In this study, we describe a computational derivative operational matrix technique for generating best approximation solution of second-order Lane-Emden type differential equations using modified Lucas wavelets basis. Modified Lucas wavelets basis expansion together with this computational derivative operational matrix technique, by selecting appropriate collocation points, converts the given second-order Lane-Emden type differential equations into a well-known equivalent set of algebraic equations.Some examples have been solved in order to evaluate the accuracy and stability of the suggested technique. Based on the results obtained for the stated problems, the suggested technique provides the best approximate solution to second-order Lane-Emden type differential equations when compared to other current techniques.

Keywords: Wavelet, Modified Lucas wavelets, Collocation points, second-order Lane-Emden differential equations, Derivative operational matrix.

1. Introduction

The analysis of star composition was a significant concern in the early stages of stellar astrophysics. There have been numerous attempts to deduce the spiral form of density, tension, and mass of stars; the major outcome of these attempts is the second- order Lane-Emden type differential equations (SOLETDEs); this type of differential equations also occurs frequently in the field of applied mathematics , astrophysics and mathematical physics. See references [1-8]. J. Lane and R. Emden, was the first astrophysicists, presented Lane-Emden type differential equations(LETDEs) in 1870 [9].

In recent years, the study of SOLETDEs has captured the attention of both mathematicians and physicists. The purpose of this research is to give a simple implementation of the proposed computational derivative operational matrix technique for getting optimal numerical solutions to SOLETDEs. The general second-order Lane-Emden type differential equation (SOLETDE) is written as:

$$v''(x) + \frac{\chi}{x} v'(x) + f(x, v) = g(x), \; x \geq 0, \; \chi \geq 0, \quad (1)$$

with, condition:

$$v(0) = \aleph, \quad v'(0) = \Im, \quad (2)$$

here, $f(x, v) \in R$ and continuousfunction, $g(x) \in C[0,1]$ is an analytical function and $\aleph, \Im$ are constants . Equation (1) is used to represent a variety of phenomena encountered in the study of star structures, astrophysics, cooling, the principal theory of thermionic currents, and the isothermal gas sphere, among other things [7-17].Wavelets and their techniques are now used to improve performance in all areas of science. The benefit of using this computational derivative operational matrix technique is that it directly converts SOLETDEs into the equivalent set of algebraic equations.

There are several uses of LETDEs in the disciplines of science. As a result, many scholars dedicated their efforts to provide a better approximate solution for LETDEs, and many literatures on LETDEs have lately been published [18-22]. For solving the SOLETDEs, a new computational derivative operational matrix technique linked with modified Lucas wavelets (MLWs) is provided in this paper. Using a modified Lucas wavelets basis, we introduced MLWs and their first and second-order derivatives operational matrix. The proposed technique has the advantage of requiring less computational effort and yielding the best approximate solution of SOLETDEs. In the proposed technique, the derivatives term in a class of SOLETDEs is expressed as a MLWs basis, and the solution is expanded by a MLWs basis with unknown coefficients.

The following is the structure of the current article: Section 2 contains basic definition of MLWs and the derivation of first and second order modified Lucas wavelets derivative operational matrices. Section 3 describes the solution procedure for SOLETDEs. In section 4, error estimation and convergence analysis is discussed .Section 5 presents numerical results. Finally, in the final section, the proposed work's conclusions are discussed.

2. Wavelets

The mother wavelet, also known as a wavelet, is a scaled (dilated) and sifted (translated) family of functions descended from a single function ψ .When the scaled parameter y and sifted parameter z vary continuouslythen,we get the resulting family of continuous wavelets $\psi_{y,z}(x)$ is as follows [23-28].

$$\psi_{y,z}(x) = |y|^{-1/2} \psi\left(\frac{x - z}{y}\right); \qquad y, z \in R; \; y \neq 0. \quad (3)$$

If, we take the parameters (y, z) values in discrete form as: $y = y_0^{-j}, \; z = n z_0 y_0^{-j}, \; y_0 > 1, z_0 > 0$ and $n, j \in N$ then, we getdiscrete form of the family of continuous waveletsas

$$\psi_{j,n}(x) = |y_0|^{j/2} \psi(y_0^j x - n z_0), \quad (4)$$

Where, $\psi_{j,n}(x)$ makes asa wavelet basis for $L^2(R)$.

2.1. Modified Lucas Wavelets

Modified Lucas wavelets (MLWs) $\Psi_{\eta,k}(x) = \Psi(q, \hat{\eta}, k, x)$ have four arguments:

$\hat{\eta} = \eta - 1,\ \eta = 1,2,\ldots,2^{q-1}, k = 0, 1, 2, \ldots, K-1,\ q \in Z^+$, k is the order and x is the normalized time for modified Lucas polynomials (MLPs)they are defined on the interval $[0, 1]$ as:

$$\Psi_{\eta,k}(x) = \begin{cases} 2^{\frac{(q-1)}{2}} \alpha_k \tilde{L}_k(2^{q-1}x - \eta + 1), & if\ \frac{\eta - 1}{2^{q-1}} \le x < \frac{\eta}{2^{q-1}}, \\ 0, & otherwise \end{cases} \quad (5)$$

where

$$\alpha_k = \begin{cases} \frac{1}{\sqrt{\pi}}, & both\ k = 0 \\ \frac{-\sqrt{2}\, i}{\sqrt{\pi}}, & both\ k = odd \\ \frac{\sqrt{2}}{\sqrt{\pi}}, & both\ k = even \\ 0, & otherwise \end{cases}, \quad (6)$$

and $\tilde{L}_k(2^{q-1}x - \eta + 1)$ are orthonormal MLPs of degree k,taking into account the weight function $w_\eta(x) = w(2^{q-1}x - \eta + 1) = \frac{1}{\sqrt{16x - 16x^2}}$ on the interval $[0, 1]$. The following recursive formula yields these MLPs:

$$\tilde{L}_k(2^{q-1}x - \eta + 1) = \frac{1}{2^k}\left[\left(i(4x-2) + \sqrt{16x - 16x^2}\right)^k + \left(i(4x-2) - \sqrt{16x - 16x^2}\right)^k\right]. \quad (7)$$

The MLWs $\Psi_{\eta,k}(x)$ presented in equation (5) are orthonormal taking into account the weight function $w_\eta(x) = w(2^{q-1}x - \eta + 1) = \frac{1}{\sqrt{16x - 16x^2}}$ in $L^2[0,1)$,

i.e.,

$$\int_0^1 \Psi_{\eta,k}(x)\Psi_{\eta',k'}(x) w_\eta(x) dx = \begin{cases} 1, & (\eta, k) = (\eta', k') \\ 0, & (\eta, k) \ne (\eta', k') \end{cases}. \quad (8)$$

By inserting, $q = 1$, $\eta = 1$ and $K = 8$ into equation (5) and using the relationships indicated in equations (6) and (7), we obtain the eight MLWs shown below, and these resulting eight MLWs are visually depicted in figure-1.

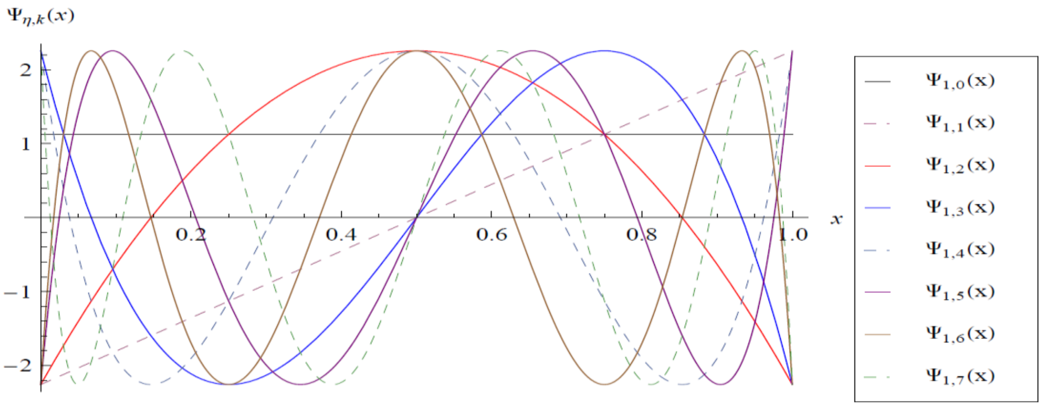

Fig.1. EightMLWs for $\eta = 1$ **,** $q = 1$ **and** $K = 8$

2.2. Function approximation

A square integrable function $g(x)$ on $[0, 1]$ can be expressed as linear combination of MLWs series as

$$g(x) \cong \sum_{\eta=1}^{\infty} \sum_{k=0}^{\infty} \zeta_{\eta,k} \Psi_{\eta,k}(x), \qquad (9)$$

where $\zeta_{\eta,k} = \langle g(x), \Psi_{\eta,k}(x) \rangle$ areMLWscoefficients and the $\langle ., . \rangle$ symbolizes the inner product in $L_2[0, 1]$. Now, we will truncate the above series provided in equation (9) as follows:

$$g(x) \cong \sum_{\eta=1}^{2^{q-1}} \sum_{k=0}^{K-1} \zeta_{\eta,k} \Psi_{\eta,k}(x) = \zeta^T \Psi(x), \qquad (10)$$

where ζ and $\Psi(x)$ are vectorof order $2^{q-1}K \times 1$, written as

$$\zeta = \left[\zeta_{1,0}, \ldots, \zeta_{1,K-1}, \zeta_{2,0}, \ldots, \zeta_{2,K-1}, \ldots, \zeta_{2^{q-1},0}, \ldots, \zeta_{2^{q-1},K-1}\right]^T \quad (11)$$

$$\Psi(x) = \begin{bmatrix} \Psi_{1,0}(x), \ldots, \Psi_{1,K-1}(x), \Psi_{2,0}(x), \ldots, \Psi_{2,K-1}(x), \ldots, \\ \Psi_{2^{q-1},0}(x), \ldots, \Psi_{2^{q-1},K-1}(x) \end{bmatrix}^T \quad (12)$$

The approximation of vector $\Psi(x)$ differentiationcan be obtainedby

$$\frac{d\Psi(x)}{dx} = \Theta \Psi(x). \quad (13)$$

2.3. Derivative operational matrix of modified Lucas wavelets

In the subsection 2.3, we determine thederivative operational matrix (DOM) of MLWs. To explain the working procedure, we first derive derivatives of MLWs.
So,

$$\frac{d\,\Psi_{1,0}(x)}{dx} = 0,$$

$$\frac{d\,\Psi_{1,1}(x)}{dx} = \frac{4\sqrt{2}}{\sqrt{\pi}},$$

$$\frac{d\,\Psi_{1,2}(x)}{dx} = \frac{\sqrt{2}}{\pi}(16 - 32x),$$

$$\frac{d\,\Psi_{1,3}(x)}{dx} = \frac{\sqrt{2}}{\sqrt{\pi}}\left(-36 + 192x - 192x^2\right),$$

$$\frac{d\,\Psi_{1,4}(x)}{dx} = \frac{\sqrt{2}}{\sqrt{\pi}}\left(-64 + 640x - 1536x^2 + 1024x^3\right),$$

$$\frac{d\,\Psi_{1,5}(x)}{dx} = \frac{\sqrt{2}}{\sqrt{\pi}}\left(100 - 1600x + 6720x^2 - 10240x^3 + 5120x^4\right),$$

$$\frac{d\,\Psi_{1,6}(x)}{dx} = \frac{\sqrt{2}}{\sqrt{\pi}}\left(144 - 3360x + 21504x^2 - 55296x^3 + 61440x^4 - 24576x^5\right),$$

$$\frac{d\,\Psi_{1,7}(x)}{dx} = \frac{\sqrt{2}}{\sqrt{\pi}}\begin{pmatrix} -196 + 6272x - 56448x^2 + 215040x^3 - 394240x^4 \\ +344064x^5 - 114688x^6 \end{pmatrix}.$$

By using above procedure, the DOM of MLWs of first and second orderfor $K = 8$ and $q = 1$ are given by respectively

$$\Theta_{8\times8}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6\sqrt{2} & 0 & 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 16 & 0 & -16 & 0 & 0 & 0 & 0 \\ 10\sqrt{2} & 0 & -20 & 0 & 20 & 0 & 0 & 0 \\ 0 & -24 & 0 & 24 & 0 & -24 & 0 & 0 \\ 14\sqrt{2} & 0 & 28 & 0 & -28 & 0 & 28 & 0 \end{bmatrix},$$

$$\Theta^2{}_{8\times8}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -16\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -96 & 0 & 0 & 0 & 0 & 0 & 0 \\ 128\sqrt{2} & 0 & -192 & 0 & 0 & 0 & 0 & 0 \\ 0 & 480 & 0 & -320 & 0 & 0 & 0 & 0 \\ -432\sqrt{2} & 0 & 768 & 0 & -480 & 0 & 0 & 0 \\ 0 & -1344 & 0 & 1120 & 0 & -672 & 0 & 0 \end{bmatrix}.$$

By using above procedure, the ηth DOM of MLWs is defined as

$$\frac{d^\eta \Psi(x)}{dx^\eta}=\Theta^\eta \Psi(x) \qquad (14)$$

3. Method of solution

In this section, we will apply the computational DOM of MLWs technique with collocation points to solve the SOLETDE given in equations (1) and (2). By using MLWsapproximation, let

$$v(x)=\zeta^T \Psi(x) \qquad (15)$$

Using eq. (14), we can write

$$v'(x)=\zeta^T \Theta \Psi(x) \text{ and } v''(x)=\zeta^T \Theta^2 \Psi(x) \qquad (16)$$

Using equations (15) and (16), equation (1) becomes

$$\zeta^T \Theta^2 \Psi(x)+\frac{\chi}{x}\zeta^T \Theta \Psi(x)+ f(x,\zeta^T \Psi(x))=g(x) \qquad (17)$$

Furthermore, the initial and boundary conditions from equation (2), produces

$$\zeta^T \Psi(0)=\aleph, \quad \zeta^T \Theta \Psi(0)=\Im \qquad (18)$$

To obtain the unknown coefficients $\zeta_{\eta,k}$, we observe that there should be $(2^{q-1}K)-\Re$, ($\Re$ is the number of specified boundary conditions) extra conditions.These conditions can be derived by solving equation (18) at the appropriate collocation points

$$x_i=\frac{i}{(2^{q-1}K)-\Re}, \qquad i=1,2,3..., \text{ where } (2^{q-1}K)-\Re\neq 0.$$

Equation (17) and (18) produce $2^{q-1}K$ systems of algebraic equations.We got unknown coefficients $\zeta_{\eta,k}$ after solving this system.

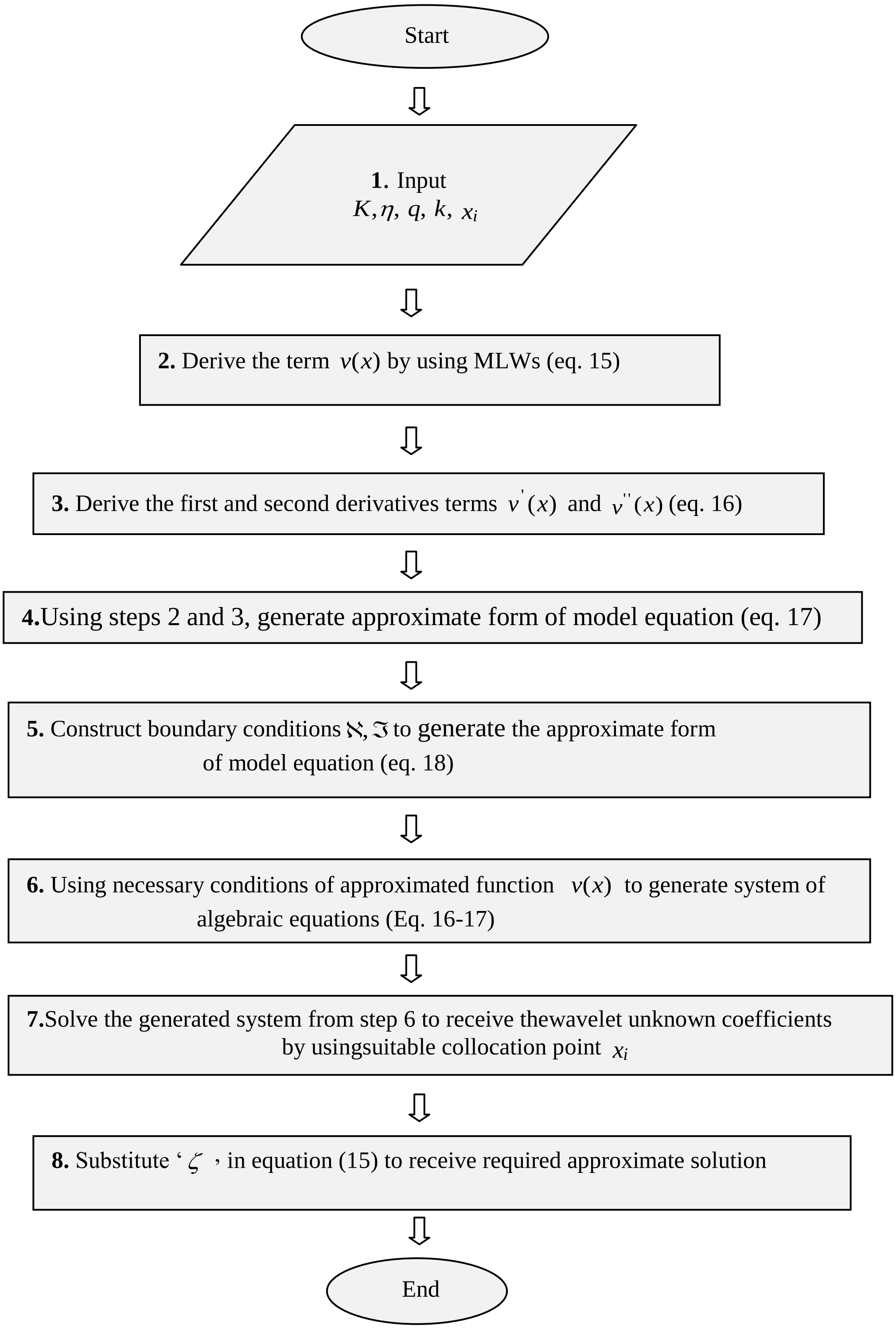

Fig.2. Flow chart of the imposition of proposed technique

Algorithm

Input: $k, n, m, M, \alpha \geq 0, z \geq 0, z_i, i = 1,2,3,\ldots(2^{k-1}M) - C$ **.**

Step 1. Define the MLWs $\Psi(x)$ through eq. 12.

Step 2. Introduced unknown MLWs coefficient vector 'ζ' by using eq. 11.

Step 3. In this stepApproximate $v(x)$ in terms of MLWs series from eq. 15.

Step 4. Approximate: $v'(x)$ and $v''(x)$ using nth derivative operational matrix from eq. (16).

Step 5. Substituting the approximated form of $v(x)$ in eq. (1) according to given problem by using step 3, 4.

Step 6. Composeboundary conditions from eq. (18) according to given Problem.

Step 7. Introduce $x_i = \frac{i}{(2^{q-1}K)-\Re}$, $i = 1,2,3...$, where $(2^{q-1}K)-\Re \neq 0$.

Step 8. In step 7 ‘$\Re$’ represents total number of specified boundary conditions.

Step 9. Take set of $(2^{q-1}K)-\Re$ algebraic equations from eq. (17) for given problem using step 7.

Step 10. Take rest set of $\Re$ algebraic equations by using eq. 18for given problem.

Step 11. Solve this combined system which is received in step 9 and 10 and find ζ.

Step 12. Increases: K or q to get moreaccurateapproximatesolution $v(x)$.

Output: The approximated MLWs solution:

$$v(x) = \zeta^T \Psi(x).$$

4. Error estimation

Let us suppose $E_q(x)$ is error function between exact solution $v(x)$ and approximate solution $\hat{v}(x)$ of LETDEs given in eq. (1) and (2), then there corresponding error at q^{th} level can be explain as

$$E_q(x) = \max |v(x) - \hat{v}(x)|$$
$$= \max \left| v(x) - \sum_{i=0}^{2^{q-1}K} \zeta_i \, \Psi_{i,j}(x) \right|.$$

Hence, if we have exact solution of discussed problem then we can estimate the error for the discussed problem.

4.1. Convergence analysis

Theorem 4.1.1 If $v(x)$ is square integrablecontinuous function as well as bounded function on $[0,1)$, such that $v(x) \leq l$, then MLWs series of $v(x)$ converges uniformly.

Proof: According to the given hypothesis $v(x)$ is bounded function on the interval $[0,1)$ then MLWs coefficients of $v(x)$ is defined as

$$\zeta_{\eta,k} = \int_0^1 v(x) \Psi_{\eta,k}(x)\, dx$$
$$= \int_{\frac{\eta-1}{2^{q-1}}}^{\frac{\eta}{2^{q-1}}} v(x) 2^{\frac{(q-1)}{2}} \alpha_k \tilde{L}_k (2^{q-1}x - \eta + 1) dx, \qquad (19)$$

now, putting $2^{q-1}x - \eta + 1 = \ell$ in equation (19).

Then,

$$\zeta_{\eta,k} = \int_{\frac{\eta-1}{2^{q-1}}}^{\frac{\eta}{2^{q-1}}} v\left(\frac{\ell - 1 + \eta}{2^{q-1}}\right) 2^{\frac{(q-1)}{2}} \alpha_k \tilde{L}_k(\ell) 2^{-q+1} d\ell \qquad (20)$$
$$= \sqrt{2^{-q+1}}\, \alpha_k \int_0^1 v\left(\frac{\ell - 1 + \eta}{2^{q-1}}\right) \tilde{L}_k(\ell) d\ell,$$

by mean-value theorem (generalized) for integrals,

$$=\sqrt{2^{-q+1}}\,\alpha_k\, v(\frac{\varepsilon-1+\eta}{2^{q-1}})\int_0^1 \widetilde{L}_k(\ell)\,d\ell\,, \quad (21)$$

for any $\varepsilon\in(0,1)$

$$=\sqrt{2^{-q+1}}\,\alpha_k\; v(\frac{\varepsilon-1+\eta}{2^{q-1}})\aleph\,, \quad (22)$$

where $\aleph=\int_0^1 \widetilde{L}_k(\ell)d\ell,$ (23)

then,

$$\left|\zeta_{\eta,k}\right|=\left|\alpha_k\right|\left|\sqrt{2^{-q+1}}\right|\left|v(\frac{\varepsilon-1+\eta}{2^{q-1}})\aleph\right|. \quad (24)$$

Since $v(x)$is bounded, so series $\sum_{\eta,k=0}^{\infty}\zeta_{\eta,k}$ is absolutely convergent.

Therefore, MLWs series of $v(x)$converges uniformly.

5. Illustrative Examples

In this part, we compare the effectiveness of the proposed techniques to that of certain previously known techniques or methods. In this section, we apply DOM techniques based on MLWs basis for some SOLETDEs.

Example 5.1 For $f(x,\,v)=v,\;\; g(x)=x^3+x^2+12x+6$ and $\chi=2,$ equation (1) gives one of the LETDE:

$$v^{''}(x)+\frac{2}{x}v^{'}(x)+v(x)=6+12x+x^2+x^3\,,\quad x\ge 0\,, \quad (25)$$

with conditions

$$v(0)=0,\quad v^{'}(0)=0\,. \quad (26)$$

This problem has exact solution $v(x)=x^2+x^3$

We solve equation (25) and (26) by the proposed technique for fix value of $q=1,\;K=4,$ We obtain the appropriate system of algebraic equations, and after solving the system by Mathematica 7.0, we obtain the values for the unknown coefficients as follows:

$\zeta_{10}=0.6092810112487715,\ \zeta_{11}=0.6070740352621954,\ \zeta_{12}=-\,0.1958303339555469,$
$\zeta_{13}=-\,0.01958303339555469$

and the solution $v(x)$ of equation (25) and (26) is approximated by

$$\begin{aligned} v(x)=\zeta^T\,\Psi(z) &= \zeta_{10}\Psi_{10}(x)+\zeta_{11}\Psi_{11}(x)+\zeta_{12}\Psi_{12}(x)+\zeta_{13}\Psi_{13}(x)\\ &=0.6092810112487715\,\Psi_{10}(x)+0.6070740352621954\,\Psi_{11}(x)\\ &\quad -0.1958303339555469\,\Psi_{12}(x)-0.01958303339555469\,\Psi_{13}(x)\\ &\approx x^2+x^3 \end{aligned}$$

To check the efficiency of the proposed technique we compared the results obtained from the proposed technique to the exact solution and other current methods through absolute erroris given in table-1, and in figure-3, the graph between numerical solution and exact solution of equation (25) are compared and in figure-4,absolute error graph of example 5.1 is discussed.

Table 1: Comparison of absolute errors for example 5.1

x	Exact	Error in [29]	Error in [30] $k=2,\ M=2$	Error in [30] $k=2,\ M=3$	Error in [22] $k=1,\ M=4$	Error in (DOMMLWT) $q=1,\ K=4$
0.00	0	0	0	0	1.11E-16	4.44E-16
1.00	2	1.25E-06	0	0	2.22E-16	4.44E-16
2.00	12	6.93E-07	3.38E-14	0	3.55E-15	0
3.00	36	7.58E-08	2.37E-14	3.33E-16	0	0
4.00	80	3.07E-07	4.35E-14	5.63E-16	0	0
5.00	150	3.21E-07	2.08E-14	4.33E-16	0	0
6.00	252	9.74E-08	6.22E-14	3.68E-16	0	0
7.00	392	2.05E-07	4.76E-14	4.34E-16	5.68E-14	0
8.00	576	7.36E-07	7.78E-14	2.12E-16	0	0
9.00	810	4.61E-06	2.28E-14	5.38E-16	0	0
10.00	1100	1.24E-05	5.48E-14	2.12E-16	2.27E-13	0

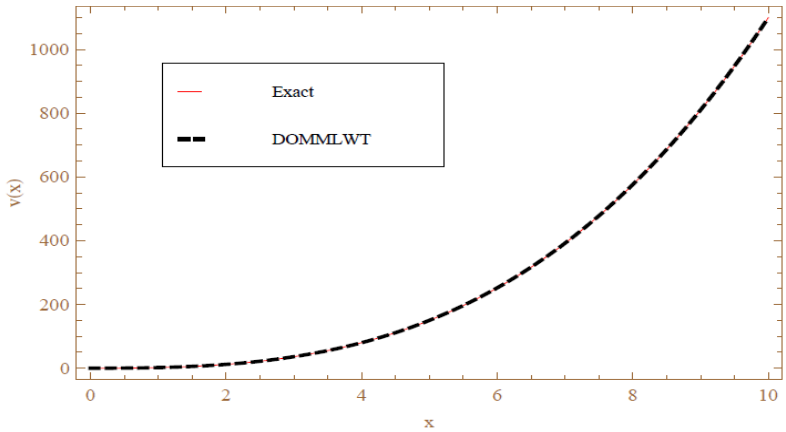

Fig.3. Graph between DOMMLWT (approximate) and Exact solution for example 5.1

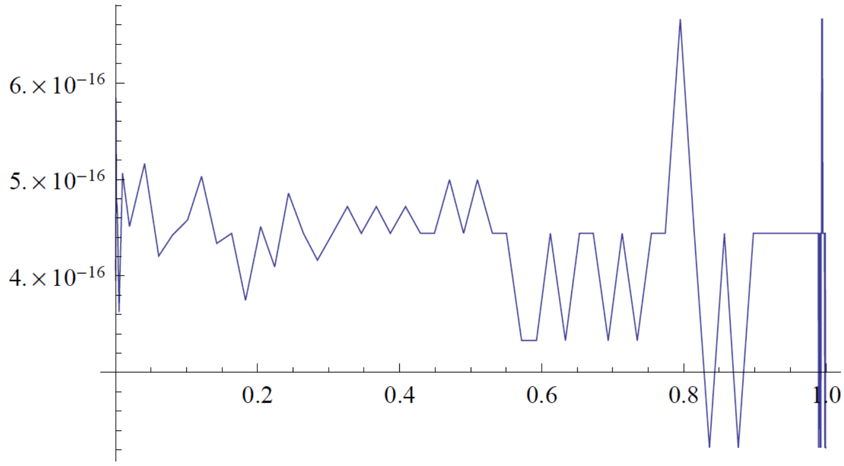

Fig- 4. Absolute error graph of example 5.1

Example 5.2 For $f(x, v) = xv,\ g(x) = x^5 - x^4 + 44x^2 - 30x$ and $\chi = 8,$ equation (1) gives one of the LETDE

$$v''(x) + \frac{8}{x}v'(x) + xv(x) = x^5 - x^4 + 44x^2 - 30x, \quad x \geq 0 \tag{27}$$

With conditions

$$v(0) = 0, \quad v'(0) = 0 \tag{28}$$

This equation has exact solution $v(x) = x^4 - x^3$

We solve equation (27) and (28) by the proposed technique for fix value of $q = 1,\ K = 5,$ We obtain the appropriate system of algebraic equations, and after solving the system by Mathematica 7.0, we obtain the values for the unknown coefficients as follows:

$\zeta_{10} = -0.03461823927549676,\ \zeta_{11} = -0.019583033395554696\ ,\ \zeta_{12} = -0.019583033395554707,$
$\zeta_{13} = -0.01958303339555469,\ \zeta_{14} = 0.0048957583488886715$

and the solution $v(x)$ of equation (27) and (28) is approximated by

$$\begin{aligned} v(x) &= \zeta^T\ \Psi(x) \\ &= \zeta_{10}\ \Psi_{10}(x) + \zeta_{11}\ \Psi_{11}(x) + \zeta_{12}\ \Psi_{12}(x) + \zeta_{13}\ \Psi_{13}(x) + \zeta_{14}\ \Psi_{14}(x) \\ &= -0.0346182392754976\,\Psi_{10}(x) - 0.019583033395554696\,\Psi_{11}(x) \\ &\quad - 0.019583033395554707\,\Psi_{12}(x) - 0.01958303339555469\,\Psi_{13}(x) \\ &\quad + 0.0048957583488886715\,\Psi_{14}(x) \\ &\approx z^4 - z^3 \end{aligned}$$

In table-2, the efficiency of the proposed technique is compared to the exact solution and other current methods through absolute error. The numerical and exact solution of equation (27) is compared in figure 5, and the absolute error graph of example 5.2 is explained in figure 6.

Table 2: Comparison of absolute errors for example 5.2

x	Exact	Error in [29]	Error in[31]	Error in [22] $k=1, M=5$	Error in (DOMMLWT) $q=1,\ K=5$
0.00	0	0	0	3.38E-15	1.83E-15
1.00	0	8.28E-07	1.36E-11	3.37E-15	1.83E-15
2.00	8	1.73E-07	1.27E-12	3.55E-15	1.77E-15
3.00	54	2.07E-07	2.46E-12	0	7.10E-15
4.00	192	3.68E-08	2.09E-11	2.84E-14	2.84E-14
5.00	500	1.91E-07	2.05E-11	5.68E-14	1.13E-13
6.00	1080	4.74E-07	6.37E-12	2.27E-13	2.27E-13
7.00	2058	4.14E-07	3.28E-12	0	4.54E-13
8.00	3584	9.36E-06	1.32E-11	0	9.09E-13
9.00	5832	4.40E-05	3.02E-12	0	9.09E-13
10.00	9000	3.39E-04	1.64E-11	1.81E-12	1.81E-12

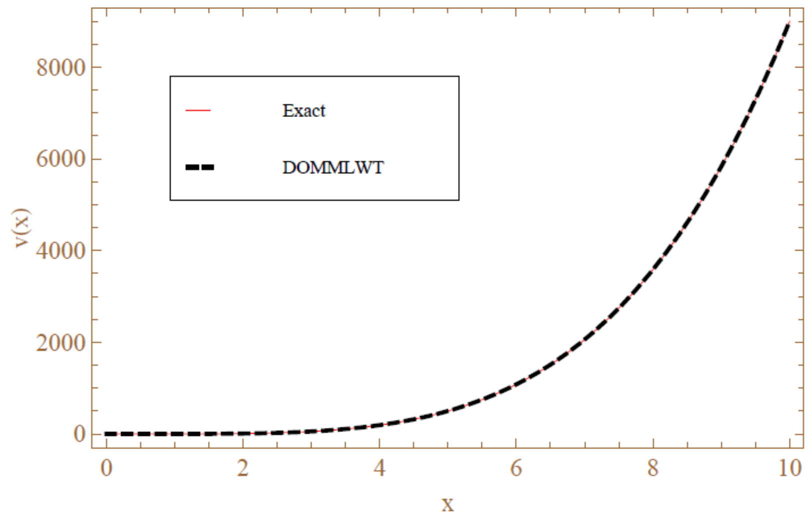

Fig.5. Graph between DOMMLWT (approximate) and Exact solution for example 5.2

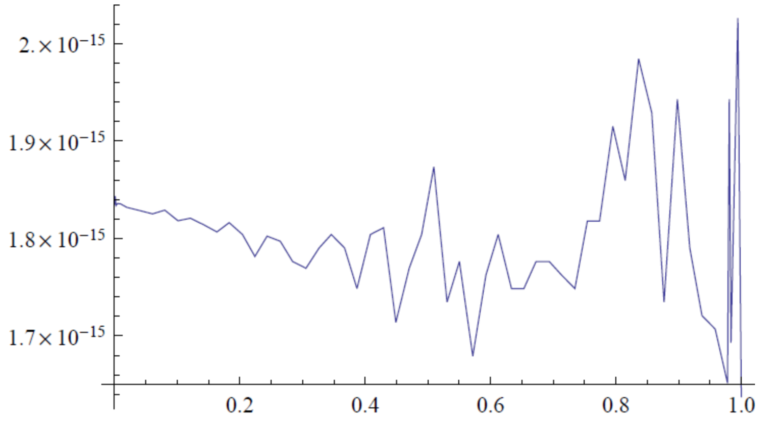

Fig- 6. Absolute error graph of example 5.2

Example 5.3 For $f(x,v)=-2(2x^2+3)v(x)$, $g(x)=0$ and $\chi=2$, equation (1) gives one of the LETDE

$$v''(x)+\frac{2}{x}v'(x)-2(2x^2+3)v(x)=0, \quad x\ge 0 \qquad (29)$$

With conditions

$$v(0)=1, \quad v'(0)=0 \qquad (30)$$

This equation has exact solution $v(x)=e^{x^2}$

We solve equation (29) and (30) by the proposed technique for fix value of $q=1$, $K=8$, We obtain the appropriate system of algebraic equations, and after solving the system by Mathematica 7.0, we obtain the values for the unknown coefficients as follows:

ζ_{10} = 1.3824907669120983, ζ_{11}= 0.5025346420887822 , ζ_{12} = - 0.1801683029901088,
ζ_{13} = - 0.03521701042149476, ζ_{14} = 0.007764252783041935, ζ_{15} = 0.0012613680644173858,
ζ_{16} = - 0.00020787732501664164, ζ_{17} = - 0.000038940077058158394

and the solution $v(x)$ of equation (29) and (30) is approximated by

$$\begin{aligned}v(x)&=\zeta^T\,\Psi(x)\\&=\zeta_{10}\,\Psi_{10}(x)+\zeta_{11}\,\Psi_{11}(x)+\zeta_{12}\,\Psi_{12}(x)+\zeta_{13}\,\Psi_{13}(x)+\zeta_{14}\,\Psi_{14}(x)\\&\quad+\zeta_{15}\,\Psi_{15}(x)+\zeta_{16}\,\Psi_{16}(x)+\zeta_{17}\,\Psi_{17}(x)\end{aligned}$$

$$\begin{aligned}&=1.3824907669120983,\ \Psi_{10}(x)+0.5025346420887822\,\Psi_{11}(x)\\&\quad-0.1801683029901088\,\Psi_{12}(x)-0.03521701042149476\,\Psi_{13}(x)\\&\quad+0.007764252783041935\,\Psi_{14}(x)+\ 0.0012613680644173858\,\Psi_{15}(x)\\&\quad-0.00020787732501664164\,\Psi_{16}(x)-0.000038940077058158394\,\Psi_{17}(x)\\&\approx e^{x^2}\end{aligned}$$

In table-3, the efficiency of the proposed technique is compared to the exact solution through absolute error explanation for different values of q , K . The numerical and exact solution of equation (29) is compared in figure 7, and the absolute error graph of example 5.3 is explained in figure 8.

x	Exact	Error in (DOMMLWT) $q=1,\ K=6$	Error in (DOMMLWT) $q=1,\ K=8$
0.0	1.00000	6.66E-16	0
0.1	1.01005	0.00291951	0.000012912
0.2	1.04081	0.00758725	8.21E-07
0.3	1.09417	0.0108483	0.000138906
0.4	1.17351	0.0123222	0.000317503
0.5	1.28403	0.0130404	0.000499579
0.6	1.43333	0.0142988	0.000709027
0.7	1.63232	0.0168141	0.000964158
0.8	1.89648	0.020324	0.00125742
0.9	2.24791	0.0238558	0.00161279
1	e	0.0270171	0.00212312

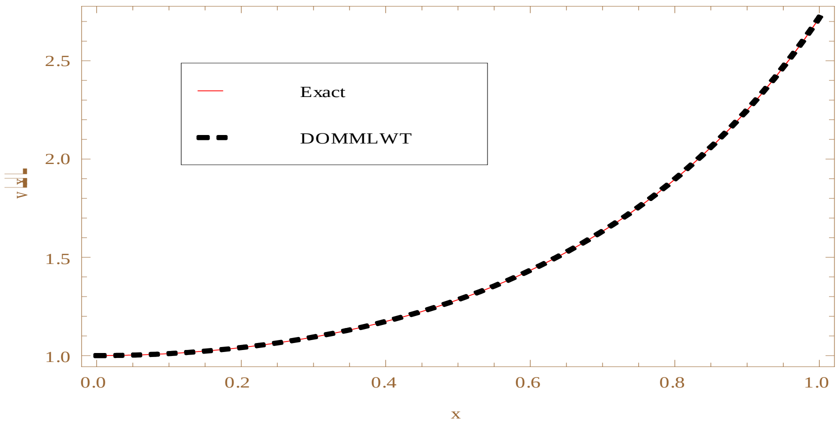

Fig- 7. Graph between DOMMLWT and Exact solution for $q=1,\ K=8$ of example 5.3

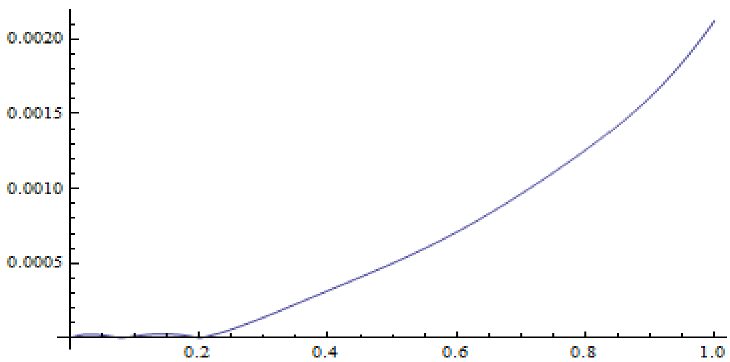

Fig- 8. Absolute error graph for $q=1,\ K=8$ of example 5.3

Conclusions

In this article, a computational DOM technique based on MLWsis derived to find the numeric solution of SOLETDEs. Using this technique, the SOLETDEsand there system has been reduced to solve system of algebraic equations without converting into integral equations. The approximate solution obtained from the proposed DOMMLWT is compared to the solutions obtained from other existing methods using absolute error explanation, which shows that the proposed technique is faster, more efficient, and computationally acceptable than other established methods for solving SOLETDEs. If we increase the value of q , K as well as no. of collocation points then we get more accuracy in results.

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