Estimate the Faber Polynomial Coefficient for Bi-Univalent Functions Defined by Linear Differential Operator

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Abstract: In this current work, we estimate the Faber polynomial coefficient for Bi-univalent function which is associated with the polylogarithm functions and also we investigate some existing coefficient bounds.

Keywords: Bi-univalent function, Polylogarithm function, Derivative operator, Faber polynomial.

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1. Introduction:

Let " S " be the class of functions $f(\zeta)$ be the form of analytic functions of class " A ", that are univalent in U and normalized by $f(\zeta)=\zeta+\sum_{k=2}^{\infty}a_k\zeta^k$

(1)

which are analytic function in the unit disc $U=\{\zeta:\zeta\in C,|\zeta|<1\}$.

We know that every function $f\in S$ has an inverse $f^{-1}(f(\zeta))=\zeta(\zeta\in U)$ and $f(f^{-1}(\omega))=\omega,\left(|\omega|<r_0(f);r_0(f)\geq\frac{1}{4}\right)$.

If $g=f^{-1}$ is the inverse function of $f\in S$, then g has a Maclaurin series expansion in some disc about the origin,

which is given by $g(\omega)=f^{-1}(\omega)=\omega-a_2\omega^2+(2a_2^2-a_3)\omega^3-(5a_2^3-5a_2a_3+a_4)\omega^4+...$

$$=\omega+\sum_{k=2}^{\infty}A_k\omega^k \; .$$

(2)

A function $f\in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U.

Let Σ denote the class of bi-univalent function in U given by (1). The class of analytic bi-univalent function was first introduced and studied by Lewin [7], where it was proved that $|a_2| < 1.51$. Brannan and Clunie [12] improved Lewin's result to $|a_2| \leq \sqrt{2}$ and later Netanyahu[9] proved that $|a_2| \leq \frac{4}{3}$. Brannan and Taha [11] and Taha [13] also investigated certain subclasses of bi-univalent functions and found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. In recent years Srivastava et al. [14], Frasin and Aouf [10], Xu et al. [15], Hayami and Owa [16] investigated the various subclasses of bi-univalent functions to estimate the first two coefficients. Not much is known about the bounds on general coefficient $|a_n|$ for $n > 3$. In this paper we use the Faber [8] polynomial expansions for a new subclass of analytic bi-univalent functions to estimate the coefficient bounds $|a_n|$.

Let the function $f(\zeta)$ given by (1) and $h(\zeta)$ given by $h(\zeta) = \zeta + \sum_{k=2}^{\infty} b_k \zeta^k$

(3)

then the Hadamard product of $f(\zeta)$ and $h(\zeta)$ are expressed by $(f * h)(\zeta) = \zeta + \sum_{k=2}^{\infty} a_k b_k \zeta^k$

(4)

For $f \in A$, Ruscheweyh [1] established the following differential operator:

$$R^{\lambda} f(\zeta) = \frac{\zeta}{(1-\zeta)^{\lambda+1}} * f(\zeta) = \sum_{k=2}^{\infty} \frac{(\lambda + k - 1)!}{\lambda!(k-1)!} a_k \zeta^k, (\lambda > -1) \quad (5)$$

Now consider the Polylogarithm function $I(n,\delta)$ given by (See Patel [23])

$$I(n,\delta) = \sum_{k=1}^{\infty} \frac{\zeta^k}{[1+(k-1)\delta]^n} \quad (6)$$

Note that $I(-1,1) = \frac{\zeta}{(1-\zeta)^2}$ for k = 1,2,3, ...is Koebe function. For more details about polylogarithms in theory of univalent functions see Ponnusamy and Sabapathy [3], K.Al Shaqsi and M.Daraus [24] , Danyal Soybas, Santosh B. Joshi and Haridas Pawar [4], S. Bulut [17] and Ponnusamy [5]. Now we introduce a function $I^*(n,\delta)$ is given by

$$I(n,\delta) * I^*(n,\delta) = \frac{\zeta}{(1-\zeta)^{\delta+1}}, \delta > -1, n \in \zeta .$$

(7)

On obtaining, the linear operator $R_{\delta,\Sigma}^{n,\lambda} f(\zeta) = I^*(n,\delta) * f(\zeta)$

(8)

From equation (8), we define $R_{\delta,\Sigma}^{n,\lambda} f(\zeta) = \zeta + \sum_{k=2}^{\infty} [1+(k-1)\delta]^n \frac{(\lambda+k-1)!}{\lambda!(k-1)!} a_k \zeta^k$

(9)

$R^{n,0}_{\delta,\Sigma} = D^n_\delta$, $R^{0,\lambda}_{\delta,\Sigma} = R^\lambda$, $R^{n,0}_{1,\Sigma} = D^n_\delta$ which give the Al-Oboudi [2], Ruscheweyh [1] and Salagean [6] operators respectively. Also note that $R^{0,0}_{1,\Sigma} = f(\zeta)$ and $R^{0,1}_{1,\Sigma} = R^{1,0}_{1,\Sigma} = \zeta f'(\zeta)$.

It is obvious that the operator $R^{n,\lambda}_{\delta,\Sigma}$ included convolution of two well known operators. Now we define a general class of analytic bi-univalent associated with polylogorithmic functions as follows.

1.1 Definition

For $0 \le \mu \le \gamma \le 1$ and $0 \le \alpha < 1$, a function $f(\zeta) \in \Sigma$ given by (1) is said to be in the class $R^{n,\lambda}_{\delta,\Sigma}$ if the following conditions are satisfied:

$$\mathrm{Re}\left((1-\gamma+\mu)\frac{R^{n,\lambda}_{\delta,\Sigma}}{\zeta} + (\gamma-\mu)\left(R^{n,\lambda}_{\delta,\Sigma}\right)' + \gamma\mu\zeta\left(R^{n,\lambda}_{\delta,\Sigma}\right)'' \right) > \alpha \text{ and}$$

$$\mathrm{Re}\left((1-\gamma+\mu)\frac{R^{n,\lambda}_{\delta,\Sigma}}{\omega} + (\gamma-\mu)\left(R^{n,\lambda}_{\delta,\Sigma}\right)' + \gamma\mu\omega\left(R^{n,\lambda}_{\delta,\Sigma}\right)'' \right) > \alpha$$

where $\zeta, \omega \in U$ and $g = f^{-1}$ is defined by (10).

Using the Faber polynomial expansion of functions $f \in A$ of the form (1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as, [20]:

$$g(\omega) = f^{-1}(\omega)$$

(10)

$$= \omega + \sum_{k=2}^{\infty} \frac{1}{k} \mathrm{K}^{-k}_{k-1}(a_2, a_3, ...)\omega^k$$

(11)

where $$\mathrm{K}^{-k}_{k-1} = \frac{(-k)!}{(-2k+1)!(k-1)!}a_2^{k-1} + \frac{(-k)!}{(2(-k+1))!(k-3)!}a_2^{k-3}a_3 + \frac{(-k)!}{(-2k+3)!(k-4)!}a_2^{k-4}a_4$$

$$+ \frac{(-k)!}{(2(-k+2))!(k-5)!}a_2^{k-5}\left[a_5 + (-k+2)a_3^2\right] + \frac{(-k)!}{(-2k+5)!(k-6)!}a_2^{k-6}\left[a_6 + (-2k+5)a_3a_4\right] + \sum_{j\ge 7}^{\infty} a_2^{k-j}\mathrm{V}j.$$

(12)

Such that V_j with $7 \le j \le k$ is a homogeneous polynomial in the variables $a_2, a_3, ..., a_k$, [21]. In particular, the first
three terms of K^{-k}_{k-1} are $\mathrm{K}^{-2}_1 = -2a_2$,

$$\mathrm{K}^{-3}_2 = 3(2a_2^2 - a_3),$$

$$\mathrm{K}^{-4}_3 = -4(5a_2^3 - 5a_2a_3 + a_4).$$

(13)

In general, for any $p \in \mathrm{N} = 1,2,3,...$ an expansion of K^p_k is as, [20],

$$\mathrm{K}^p_k = p_k + \frac{p(p-1)}{2}D_k^2 + \frac{p!}{(p-3)!3!}D_k^3 + ... + \frac{p!}{(p-k)!k!}D_k^k$$

(14)

where $D_k^p = D_k^p(a_2, a_3, ...)$ and by [22], $D_k^m(a_2, a_3, ...a_k) = \sum_{k=1}^{\infty} \frac{m!}{i_1! i_2! ... . i_k!} a_1^{i_1} ... a_k^{i_k}$. While $a_1 = 1$, and the sum is taken over all non-negative integers $i_1, i_2, ..., i_k$ satisfying

$i_1 + i_2 + ... + i_k = m$

$$i_1 + 2i_2 + ... + ki_k = k \ .$$

It is clear that $D_k^k(a_1, a_{2,...,} a_k) = a_1^k$.

Consequently, for functions $f \in R_{\delta,\Sigma}^{n,\lambda}$ of the form (1), we can write:

$$(1-\gamma+\mu)\frac{R_{\delta,\Sigma}^{n,\lambda} f(\zeta)}{\zeta} + (\gamma-\mu)\left(R_{\delta,\Sigma}^{n,\lambda} f(\zeta)\right)' + \gamma\mu\zeta\left(R_{\delta,\Sigma}^{n,\lambda} f(\zeta)\right)''$$

$$= 1 + \sum_{k=2}^{\infty} [1 + \gamma\mu k(k-1) + (\gamma-\mu)(k-1)]\left([1+(k-1)\delta]^n \frac{(\lambda+k-1)!}{\lambda!(k-1)!}\right) a_k \zeta^{k-1}$$

(15)

2 Coefficient Estimates

2.1 Theorem

For $0 \le \mu \le \gamma \le 1, \delta \ge 0, \lambda > -1$ and $0 \le \alpha < 1$, let the function $f \in R_{\delta,\Sigma}^{n,\lambda}$ be given by (1). If $a_j = 0 (2 \le j \le k-1)$, then

$$|a_k| \le [1 + \gamma\mu k(k-1) + (\gamma-\mu)(k-1)]\left([1+(k-1)\delta]^n \frac{(\lambda+k-1)!}{\lambda!(k-1)!}\right), (k \ge 4).$$

Proof: For the function $f \in R_{\delta,\Sigma}^{n,\lambda}$ of the form (1), we have the expansion (15) and for the inverse map $g = f^{-1}$,

considering (10) we obtain $(1-\gamma+\mu)\frac{R_{\delta,\Sigma}^{n,\lambda} g(\omega)}{\omega} + (\gamma-\mu)\left(R_{\delta,\Sigma}^{n,\lambda} g(\omega)\right)' + \gamma\mu\zeta\left(R_{\delta,\Sigma}^{n,\lambda} g(\omega)\right)''$

$$= 1 + \sum_{k=2}^{\infty} [1 + \gamma\mu k(k-1) + (\gamma-\mu)(k-1)]\left([1+(k-1)\delta]^n \frac{(\lambda+k-1)!}{\lambda!(k-1)!}\right) A_k \omega^{k-1}, \qquad (16)$$

where $A_k = \frac{1}{k} K_{k-1}^{-k}(a_2, a_3, ...)$. On the other hand, since $f \in R_{\delta,\Sigma}^{n,\lambda}$ and $g = f^{-1} \in R_{\delta,\Sigma}^{n,\lambda}$ by definition, there exist two positive real part functions

$p(\zeta) = 1 + \sum_{k=1}^{\infty} c_k \zeta^k \in A$ and $q(\omega) = 1 + \sum_{k=1}^{\infty} d_k \omega^k \in A$

where $\mathrm{Re}(p(\zeta)) > 0$ and $\mathrm{Re}(q(\omega)) > 0$ in U so that

$$(1-\gamma+\mu)\frac{R_{\delta,\Sigma}^{n,\lambda} f(\zeta)}{\zeta} + (\gamma-\mu)\left(R_{\delta,\Sigma}^{n,\lambda} f(\zeta)\right)' + \gamma\mu\zeta\left(R_{\delta,\Sigma}^{n,\lambda} f(\zeta)\right)'' = \alpha + (1-\alpha)p(\zeta) =$$

$$1 + (1-\alpha)\sum_{k=1}^{\infty} K_k^1(c_1, c_2, ..., c_k)\zeta^k$$

(17)

$$(1-\gamma+\mu)\frac{R_{\delta,\Sigma}^{n,\lambda}g(\omega)}{\omega}+(\gamma-\mu)\left(R_{\delta,\Sigma}^{n,\lambda}g(\omega)\right)'+\gamma\mu\zeta\left(R_{\delta,\Sigma}^{n,\lambda}g(\omega)\right)''=\alpha+(1-\alpha)g(\omega)=$$

$$1+(1-\alpha)\sum_{k=1}^{\infty}\mathrm{K}_k^1(d_1,d_2,...,d_k)\omega^k \qquad (18)$$

Note that, by the Caratheodory lemma $|c_k|\le 2$ and $|d_k|\le 2$. Comparing the corresponding coefficients of (15) and (17), for any $k\ge 2$, yields

$$[1+\gamma\mu k(k-1)+(\gamma-\mu)(k-1)]\left([1+(k-1)\delta]^n\frac{(\lambda+k-1)!}{\lambda!(k-1)!}\right)a_k =$$

$$(1-\alpha)\mathrm{K}_{k-1}^1(c_1,c_2,...,c_{k-1}) \qquad (19)$$

and similarly, from (16) and (18) we find

$$[1+\gamma\mu k(k-1)+(\gamma-\mu)(k-1)]\left([1+(k-1)\delta]^n\frac{(\lambda+k-1)!}{\lambda!(k-1)!}\right)A_k =$$

$$(1-\alpha)\mathrm{K}_{k-1}^1(d_1,d_2,...,d_{k-1}) \qquad (20)$$

If $a_j=0(2\le j\le k-1)$, we have $A_k=a_k$ and also

$$[1+\gamma\mu k(k-1)+(\gamma-\mu)(k-1)]\left([1+(k-1)\delta]^n\frac{(\lambda+k-1)!}{\lambda!(k-1)!}\right)a_k=(1-\alpha)c_{k-1} \qquad (21)$$

$$-[1+\gamma\mu k(k-1)+(\gamma-\mu)(k-1)]\left([1+(k-1)\delta]^n\frac{(\lambda+k-1)!}{\lambda!(k-1)!}\right)A_k=(1-\alpha)d_{k-1} \qquad (22)$$

Taking absolute value of the above equalities, we obtain

$$|a_k|=\frac{(1-\alpha)|c_{k-1}|}{[1+\gamma\mu k(k-1)+(\gamma-\mu)(k-1)]M_k}$$

$$=\frac{(1-\alpha)|d_{k-1}|}{[1+\gamma\mu k(k-1)+(\gamma-\mu)(k-1)]M_k}$$

$$|a_k|\le\frac{2(1-\alpha)}{[1+\gamma\mu k(k-1)+(\gamma-\mu)(k-1)]M_k},$$

where $M_K=\left([1+(k-1)\delta]^n\frac{(\lambda+k-1)!}{\lambda!(k-1)!}\right)$

Hence Completes the theorem. Suitable choice of parameters for the above result we obtain the following corollaries.

2.2 Corollary

[18] For $\lambda\ge 1$and $0\le\alpha<1$, let the function $f\in\mathrm{B}_\Sigma(\alpha,\lambda)$ be given by (1). If

$a_k = 0(2 \le k \le n-1),$ then $|a_k| \le \frac{2(1-\alpha)}{1+(\mathrm{n}-1)\lambda}, (\mathrm{n} \ge 4).$

2.3 Corollary

[19] For $\delta \ge 0$ and $0 \le \alpha < 1,$ let the function $f \in \mathrm{N}_{\Sigma}^{(\alpha,\lambda)}$ be given by (1). If

$a_k = 0(2 \le k \le n-1),$ then $|a_k| \le \frac{2(1-\alpha)}{n[1+(\mathrm{n}-1)\delta]}, (\mathrm{n} \ge 4).$

2.4 Theorem

For $0 \le \mu \le \gamma \le 1, \delta \ge 0, \lambda > -1$ and $0 \le \alpha < 1,$ let the function $f \in R_{\delta,\Sigma}^{n,\lambda}$ be given by (1). Then one has the following

$$|a_2| \le \begin{cases} \sqrt{\dfrac{4(1-\alpha)}{(1+6\gamma\mu+2\gamma-2\mu)(1+2\delta)^n(\lambda^2+3\lambda+2)}}; 0 \le \alpha < 1 - \dfrac{\left((1+2\gamma\mu+\gamma-\mu)(1+\delta)^n(\lambda+1)\right)^2}{(1+6\gamma\mu+2\gamma-2\mu)(1+2\delta)^n(\lambda^2+3\lambda+2)}, \\ \dfrac{2(1-\alpha)}{(1+2\gamma\mu+\gamma-\mu)(1+\delta)^n(\lambda+1)}; 1 - \dfrac{\left((1+2\gamma\mu+\gamma-\mu)(1+\delta)^n(\lambda+1)\right)^2}{(1+6\gamma\mu+2\gamma-2\mu)(1+2\delta)^n(\lambda^2+3\lambda+2)} \le \alpha < 1 \end{cases}$$

$$|a_3| \le \frac{4(1-\alpha)}{(1+6\gamma\mu+2\gamma-2\mu)(1+2\delta)^n(\lambda^2+3\lambda+2)},$$

$$|a_3 - a_2^2| \le \frac{4(1-\alpha)}{(1+6\gamma\mu+2\gamma-2\mu)(1+2\delta)^n(\lambda^2+3\lambda+2)}.$$

By the suitable choice of parameters in the above theorem we get the following corollaries.

Corollary 2.5:

[18] For $\lambda \ge 1$ and $0 \le \alpha < 1,$ let the function $f \in \mathrm{B}_{\Sigma}(\alpha,\lambda,\delta)$ be given by (1). If

$a_k = 0(2 \le k \le n-1),$ then one has the following $|a_2| \le \begin{cases} \sqrt{\dfrac{2(1-\alpha)}{1+2\lambda}}; 0 \le \alpha < \dfrac{1+2\lambda-\lambda^2}{2(1+2\lambda)} \\ \dfrac{2(1-\alpha)}{1+\lambda}; \dfrac{1+2\lambda-\lambda^2}{2(1+2\lambda)} \le \alpha < 1 \end{cases},$

$$|a_3| \le \frac{2(1-\alpha)}{1+2\lambda},$$

$$|a_3 - 2a_2^2| \le \frac{2(1-\alpha)}{1+2\lambda}.$$

Corollary 2.6 [19] For $\delta \ge 0$ and $0 \le \alpha < 1,$ let the function $f \in \mathrm{N}_{\Sigma}^{(\alpha,\lambda)}$ be given by (1). Then one has the following

$$|a_2| \le \begin{cases} \sqrt{\dfrac{2(1-\alpha)}{3(1+2\delta)}}; 0 \le \alpha < \dfrac{1+2\delta-2\delta^2}{3(1+2\delta)} \\ \dfrac{(1-\alpha)}{1+\delta}; \dfrac{1+2\delta-\delta^2}{3(1+2\delta)} \le \alpha < 1 \end{cases},$$

$$|a_3| \leq \frac{2(1-\alpha)}{3(1+2\delta)}.$$

3 Conclusions

In this article we estimate the Faber polynomial for new subclasses by using two linear operators. Furthermore,this work motivated the researchers to extend the results of this article into some new subclass of q-calculus to estimate the Faber polynomial.

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