

Comparative between Integrating Factor Method and Traditional Euler's Method to Solve Electrical Circuit

Neama Yahia Mohammed Haroun^{1*}

^{1*} Assistant Professor Department of Mathematics, College of Science, Tabuk University, Saudi Arabia

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Abstract

We provided two methods for selected ordinary differential equations representing initial value problems compared them. The aim is to investigate which solution is divergent and which of them is convergent when we connect the electrical circuit in parallel circuit or series one. The first solution is an analytical solution using the method of integrating factor; the second method is a numerical solution using the traditional Euler's Method. The two solutions have been applied to an illustrative example and an application problem from the field of electrical engineering. The application results have been analyzed and compared. We found that the numerical solution (Euler's method) when we solve the problem in series circuit the solution is convergent and the output is so smoothly, but in parallel connection there is big difference between the exact solution and the approximate one and the solution is divergent. When I replace the values of t from (1, 2, ... to 0.1, 0.2, ...) the solution is goes smoothly.

Keywords: Integrating Factor, Ordinary Differential Equations, Traditional Euler's Method, Electrical Circuit, Initial Value Problem.

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1. Introduction:

The differential equation has one dependent variable and derivatives of that dependent variable with regard to one or more independent variables. The largest derivative present determines the differential equation's order. When there is just one independent variable, an ordinary differential equation (ODE) exists, and all of its derivatives are ordinary derivatives.[1]

In a linear ordinary differential equation of order n , the variables y and x are both independent and dependent variables. The equation can be expressed in one of the following forms:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \quad (1)$$

$$\frac{d^4y}{dx^4} + x^2\frac{d^3y}{dx^3} + x^3\frac{dy}{dx} = xe^x \quad (2)$$

As seen in equations 3 and 4, an ordinary differential equation that is non-linear is known as a non-linear ordinary differential equation.[1]

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y^2 = 0 \quad (3)$$

Due to term $6y^2$ second-degree inclusion of the dependent variable y , equation 3 is nonlinear. Equation 4 below provides a further illustration of a nonlinear differential equation.

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 + 6y = 0 \quad (4)$$

Equation 4 is nonlinear because its expression, $5\left(\frac{dy}{dx}\right)^3$, involves the third power of the first derivative. Equation 5 is a linear 1st order ODE in the context of $y(x)$:

$$\frac{dy}{dx} + f(x)y = r(x) \quad (5)$$

where the functions $f(x)$ and $r(x)$ have been defined by the variable x .

If $r(x) = 0$ (i.e., $r(x)$ is exactly zero over the area of interest), the aforementioned linear ODE is said to be homogeneous.

As a result, Equation 6 depicts a homogeneous first order linear ODE.

$$\frac{dy}{dx} + f(x)y = 0 \quad (6)$$

A nonhomogeneous ODE is one that has not a zero values for a minimum of some values of x (within the range of interest) [1]. Equation 7 can be used to express the starting value problem for a differential equation of n^{th} order.

$$F\left(x, y, \frac{dy}{dx}, \dots, \frac{d^ny}{dx^n}\right) = 0 \quad (7)$$

It is necessary to utilize beginning conditions for the differential equation on an interval I that satisfy at x_0 in order to solve equation 7. The set of equation 8 can be used to characterize these initial conditions.

$$y(x_0) = y_0, \quad (8.1)$$

$$\frac{dy}{dx}(x_0) = y_1, \quad (8.2)$$

$$\frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_{n-1} \quad (8.3)$$

where y_0, y_1, \dots, y_{n-1} , are provided constants and x_0 belongs to I .

An approximate solution to an ordinary differential equation can be found using numerical methods if an exact solution cannot be found.

In this work, the Euler's Method—the earliest numerical technique for approximating the answer of a differential equation—was applied. Due to its simplicity and ease of use, the Euler's Method, which dates back to 1768, is still often utilized in this area. The Euler's Method is a useful starting point when researching numerical approximations for relatively straightforward differential equations because of how straightforward it is. [3].

2. Euler's Method Equation:

Suppose that we have **D.E.** as follows : $\frac{dy}{dt} = f(t, y)$ (9)

So the formula of Euler's Method the equation is

$$y_{n+1} = y_n + hf(x_n, y_n), \quad n = 1, 2, 3, \dots \quad \text{Where is : } x_{n+1} = x_n + h. \quad (10)$$

R L C Circuit [2]

A resistor, an inductor, and a capacitor are linked either in series or parallel to form an RLC circuit (the letters R, L, and C can be arranged differently). The letters RLC, the traditional electrical symbols for resistance, inductance, and capacitance, stand in for the eponymous component's RLC.

There are numerous topologies that can be used to combine the three circuit components. The simplest in concept and easiest to analyze are the three elements in series or parallel. There are other designs, some of which are better for real circuits. One issue that commonly comes up is the necessity of taking into account inductor resistance. The resistance of coils of wire used to make inductors is often undesirable, yet it frequently has a substantial impact on the circuit.[2]

Therefore, the main objective of the present study is to compare two solutions of the same ODE that describes variation of electric current with time in an electric circuit and to determine which one is divergent or convergent in an important problem to electrical engineering. The two solutions are the

exact analytical, the Euler numerical and the exact solution. The basic building blocks of the electric circuit used to illustrate the two solutions are a resistor, an inductor, and a capacitor. The resistor is a component that maintains a safe level of electric current flow in the circuit. A coil of wire, typically copper, acts as an inductor and stores energy in a magnetic field. The capacitor stores energy between two plates that are electrically charged in opposition to one another.

3. Methodology:

The methodology followed to obtain the exact analytical solution for initial value problems such as the one described in equation 7 is the Method of Integrating Factor. The method of integrating factor for solving initial value problems, as described in equation 11, involves the following 5 steps: [3]

$$y' + p(t)y = g(t) \quad (11)$$

1. The equation is put in standard form: $y' + p(t)y = g(t)$.
2. Making the integrating factor calculation $\mu(t) = e^{\int p(t) dt}$.
3. The equation is multiplied by $\mu(t)$, and it is written as follows: $[\mu(t)y]' = \mu(t)g(t)$.
4. using equation integration to derive the function $\mu(t)y = \int \mu(t)g(t)dt + c$.
5. When given functions p and g, one must solve for y.

The required solution interval of the independent variable is divided into manageable steps to produce the Euler numerical solution. The corresponding values of the dependent variable are then discovered by substituting the derivative in the formula.

4. Results and Discussion:

Illustrative Examples

Example (4.1) : [4]

The problem with initial values $y' = x + 2y$, $y(0) = 0$ is used here to demonstrate the above methodology. A numerical solution for the problem is required at $x = 1$, for a step of size $h = 0.25$ for comparison purposes.

Firstly the analytical solution for the equation is obtained using the method of integrating factor. The given initial problem equation can be put in the form of equation 12:

$$y' - 2y = x \quad (12)$$

Equation 13 can be produced by multiplying both sides of equation 12 by the integrating factor: $\mu(t) = e^{-2x}$ and rearrangement.

$$(e^{-2x} y)' = x e^{-2x} \quad (13)$$

Now by integrating both sides of equation 13 and using C as a constant, the resultant in equation 14 can be obtained:

$$y = -\frac{1}{2}x - \frac{1}{4} + e^{2x}C \quad (14)$$

Equation 15 describes the analytical solution for the starting value problem example. Using the stated condition of $y(0) = 0$, the value of constant C is found to be 1/4.

$$y = \frac{1}{4} e^{2x} - \frac{1}{2}x - \frac{1}{4} \quad (15)$$

Second, the same initial value problem is pursued numerically using Euler's Method. The formula $f(x, y)$ required by the Euler Method can be derived from the preceding differential equation as $f(x, y) = x + 2y$, and the initial condition provides the coordinate values of our starting point: $x_0 = 0$, $y_0 = 0$. Using the Euler method formulas to generate values for x_1 and y_1 , the numerical along the x-axis can be evaluated using the formula: $x_1 = x_0 + h$, with h representing the step size. Corresponding value of

y_1 can be evaluated using the formula: $y_1 = y_0 + h \cdot f(x_0, y_0)$. As stated in the example, the solution is required at $x = 1$, so there will be four steps of size 0.25 between $x = 0$ and $x = 1$. Table 1 below show the corresponding values of y for these four points. Corresponding values of y , for the x values used in the numerical solution, were also obtained from the analytical solution and inserted in the last column of Table 1.

Table 1: Euler's numerical solution vs analytical solution

n	x	y (Euler numerical solution)	y (Analytical solution)
0	0.00	0.000000	0.000000
1	0.25	0.000000	0.037180
2	0.50	0.062500	0.179570
3	0.75	0.218750	0.495422
4	1.00	0.515625	1.097264

Figure 1 shows plots for the y values, obtained from the two solutions, against the independent variable x . It could be clearly seen from the plots that there is a difference between the values of y yielded by the two solutions, which increases with an increase in the value of x .

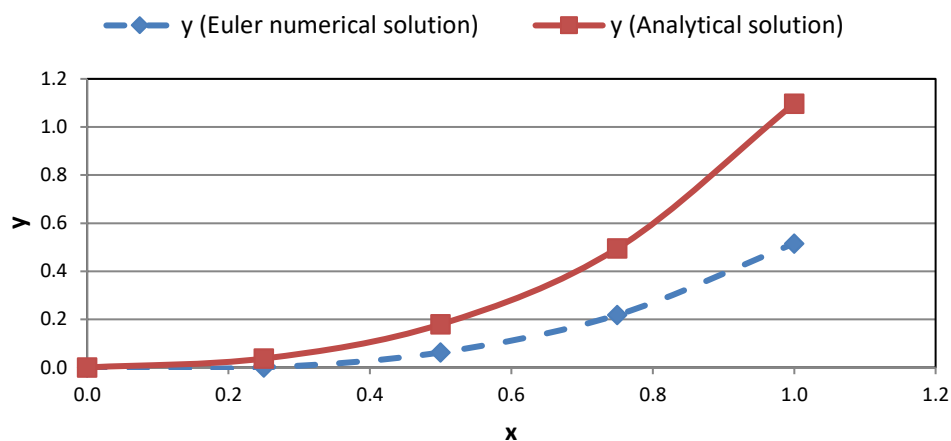


Figure 1 Numerical versus analytical solutions (Illustrative Example)

The electrical characteristics of series and parallel resistor arrangements are quite different. Here, we'll examine the characteristics of each configuration:

- ⌚ A series circuit has every component connected end to end, giving electrons only one path to follow.
- ⌚ In a parallel circuit, one component is connected across the other to produce precisely two sets of electrically related points.
- ⌚ A parallel circuit's "branch" is a current path made by one of the load elements, such as a resistor.[6]

g Electrical Applications :gineerin5.E

1. Series electrical circuit [5]

After demonstrating the simplicity and accuracy of the solutions obtained for initial value problem, a real world problem is pursued in this section to demonstrate their usefulness in the field of electrical engineering. A basic electrical circuit shown in Figure 2 (Reference), is used for this purpose.

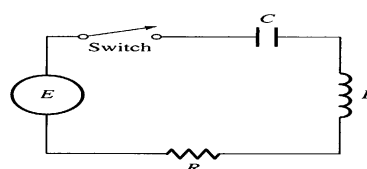


Figure 2: Series Electric circuit

An electromotive force (usually a battery or generator) generates both a voltage of volts (V) and a current of amperes (A) in the electrical circuit depicted in Figure 2. The circuit consists of an inductor with an inductance L of henries (H) and a resistor with a resistance of R ohms. According to Ohm's Law, the voltage drop caused by a resistor is represented by the letters RI , whereas the voltage drop caused by an inductor is represented by the letters (LdI/dt) . Kirchhoff's law states that the total voltage drop and the applied voltage $E(t)$ are equal. Thus, here we are

$$L \frac{dI}{dt} + RI = E(t) \quad (16)$$

A first-order linear differential equation is Equation 16. The answer reveals the present I at time t . [5]

Example(5.1) : Assume that the resistance in Figure 2's simple circuit is 12 and the inductance is 4 H. Finding the current $I(t)$ at any t is necessary if the battery provides a constant voltage of 60 V and the switch is closed at $t = 0$, causing the current to start with $I(0) = 0$. This is the solution to equation 16 in this case. If the values for E , R and L are substituted equation 16 will reduce to:

$$4 \frac{dI}{dt} + 12I = 60 \quad \text{or} \quad \frac{dI}{dt} + 3I = 15 \quad (17)$$

Following the same steps described in the illustrative example two solutions were obtained for equation 17 as follows.

First, the initial value problem's precise analytical solution is discovered using the integrating factor approach by multiplying both sides of equation 17 by the integrating factor $\mu(t) = e^{3t}$ to produce equation 18.

$$e^{3t} \frac{dI}{dt} + e^{3t}(3I) = 15e^{3t} \quad (18)$$

Equation 19 can be used to represent the integration result after rearranging, integrating, and utilizing a constant C on both sides of equation 18.

$$I(t) = 5 + Ce^{-3t} \quad (19)$$

The value of C is evaluated as -5, using the initial conditions given above, and equation (19) will reduce to equation 20, which is the exact analytical solution for the current in t time.

$$I(t) = 5 - 5e^{-3t} \quad (20)$$

Second, using a step size h of 0.1 seconds, the $f(t,I)$ function from equation 17 is first evaluated to yield the Euler's numerical solution. The initial values of $t_0=0$ and $I_0=0$ are used to find the values of t_1 and I_1 using the formulas: $t_1 = t_0 + h$; and $I_1 = I_0 + hf(t_0, I_0)$. Table 2 below shows the values of t and I using these formulae. Corresponding values of I , for the t values used in the numerical solution, were also obtained from the analytical solution (equation 20) and inserted in the second column of Table 2.

Table 2: Euler's numerical solution vs analytical solution (Electric circuit)

t	Exact	Euler with $h = 0.1$
1	4.7511	1.50000
2	4.9876	2.55000
3	4.9994	3.28500
4	4.99997	3.79950
5	4.99999	4.15965

Figure 3 shows plots for the $I(t)$ values, obtained from the two solutions, against the independent variable t . It could be clearly observed that there is a difference between the values of I yielded by the two solutions. This led to the claim that Euler's approach merely provides approximations and does not provide an exact solution to an initial-value problem.

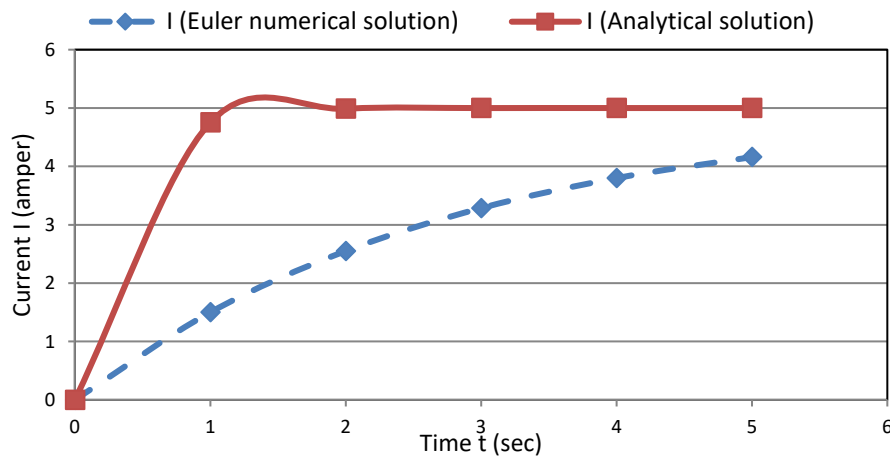


Figure 3 Numerical versus analytical solutions (Electric Circuit)

So the solution is divergent because when we use big values or small one the solution of analytical and approximate solution there is no difference between them.

2. Parallel electrical circuit

Example (5.2) :

Use a basic electric parallel circuit with the following parts to create a table of estimated values for the initial-value problem: resistance $10\ \Omega$, capacitor $4\ \text{F}$, using the Euler's approach with step size 0.2 . A battery produces a voltage of $60\ \text{V}$ consistently.

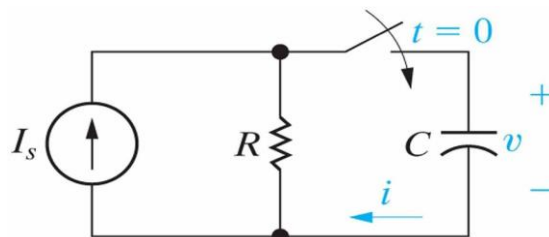


Figure 4: Parallel electrical circuit

An electromotive force (often a battery or generator) generates a voltage of volts (V) and a current of amperes (A) simultaneously in the parallel electric circuit of Figure 4. The circuit has an inductor with a capacitor C of farad (F) and a resistor with a resistance of R ohms (Ω). Ohm's Law expresses the voltage drop caused by the resistor as RI . (LdI/dt) is the voltage drop caused by the inductor. According to one of Kirchhoff's rules, the total voltage drop equals the applied voltage $E(t)$. So, here we are

$$\frac{dV}{dt} = \frac{1}{C} [I_s - \frac{V}{R}] \quad (21)$$

First, we must utilize the approach of integrating factors as demonstrated here to determine the precise equation of the solution.

The equation can be written in the form : $\frac{dV}{dt} + \frac{V}{CR} = \frac{I_s}{C}$ (22)

If both sides of equation 22 are multiplied by the integrating factor : $\mu(t) = e^{\frac{1}{RC}t}$ and re-arranged then equation 23 can be obtained.

$$(e^{\frac{1}{RC}t} V)' = \int \frac{I_S}{C} e^{\frac{1}{RC}t} dt \quad (23)$$

Now by integrating both sides of equation 23 and using C_1 as a constant, the resultant in equation 24 can be obtained:

$$e^{\frac{1}{RC}t} V = I_S \frac{RC}{C} e^{\frac{1}{RC}t} + C_1 \quad (24)$$

The value of C_1 is evaluated as $-I_S R$, using the initial conditions given above, and equation (24) will reduce to equation 25, which is the exact analytical solution for the current in t time.

$$V(t) = IR(1 - e^{-\frac{t}{RC}}) \quad (25)$$

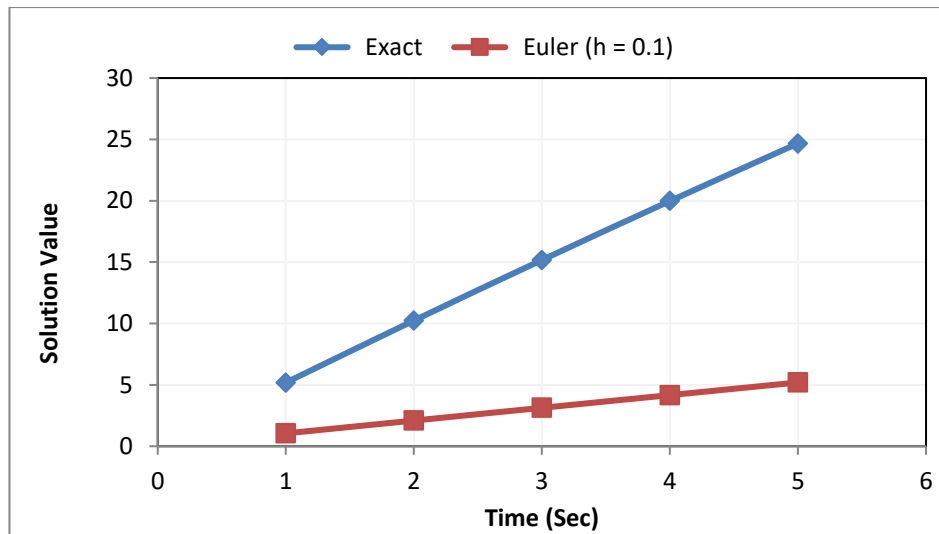
Secondly, the Euler's numerical solution is obtained by first evaluating the $f(t, I)$ function from equation 21 as: $\frac{dV}{dt} = \frac{1}{C} [I_S - \frac{V}{R}]$ and using at step size h of 0.2 second. Using the initial values of $t_0 = 0$ and $I_0 = 0$ the values of t_1 and I_1 are found from the formulae: $t_1 = t_0 + h$; and $I_1 = I_0 + hf(t_0, I_0)$. Table 3 below shows the values of t and I using these formulae. Corresponding values of I , for the t values used in the numerical solution, were also obtained from the analytical solution (equation 25) and inserted in the second column of Table 3.

Table 3: Euler's numerical solution vs analytical solution (Electric circuit)

t	Exact	Euler's with h = 0.2	Error
1	5.18492	1.05	79.75%
2	10.24182	2.09475	79.55%
3	15.173868	3.13427625	79.34%
4	19.984114	4.1686086	79.14%
5	24.675651	5.197765557	78.94%

Figure 5 shows plots for the $v(t)$ values, obtained from the two solutions, against the independent variable t . It could be clearly observed that there is a big difference between the values of two solutions so its divergent.

The level of accuracy required will, of course, depend on the intended use of the results, but the inaccuracies in the Table are too big for the majority of scientific or engineering applications. One might either try further smaller values or limit the computations to a relatively limited distance from the starting point to improve the issue.

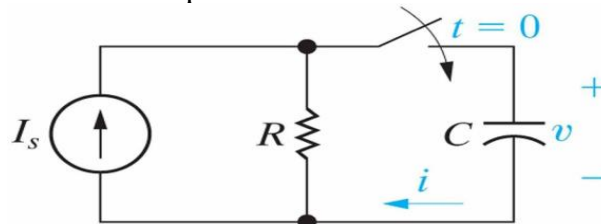


Here is the graph solution of Euler's Method and the Exact solution what a big difference between the two solutions. So we use small values as ($t = 0.1, 0.2, 0.3, \dots$) instead of $t = 1, 2, 3, \dots$ so as to decrease the space between the two solutions.

Example(5.3) :

Solve for $i(t)$ for the circuit given that $v(t) = 60$ V, $R = 15 \Omega$, and $C = 8$ F. Take the circuit as it is at $t = 0$ when the switch is closed.

using 0.1-step Euler's method to solve a parallel circuit.



We use Kirchhoff's Law as written in the equation 26

$$\frac{dv}{dt} = \frac{1}{C} \left[I_s - \frac{v}{R} \right] \quad (26)$$

The equation sum of the component is: $I_s = I_R + I_C$ (27)

Using the values above and substituting in the equations taking in account Euler's Method as follows: $I_1 = I_0 + hf(t_0, I_0)$, we can get the first point as: $(0.1, 0.14375)$. With the same manner we can draw Table 4 below shows the corresponding values of I for these four points.

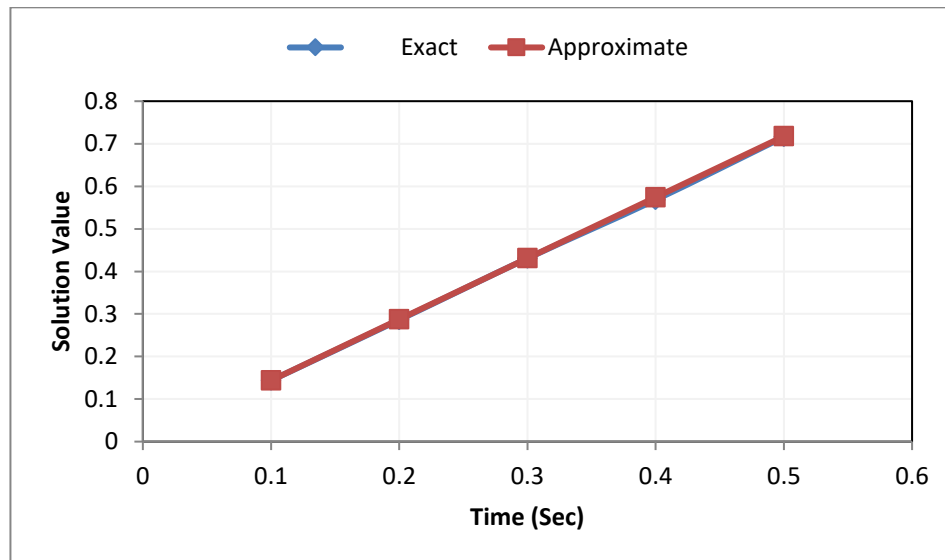
Corresponding values of I , for the t values used in the numerical solution, were also obtained from the analytical solution and inserted in the second column of Table 4

Time, t_n	Approximation	Exact	Error
$t_1 = 0.1$	$I_1 = 0.14375$	$V(0.1) = 0.1436326$	0.08 %
$t_2 = 0.2$	$I_2 = 0.28738$	$V(0.2) = 0.2861124$	0.44 %
$t_3 = 0.3$	$I_3 = 0.430891$	$V(0.3) = 0.4307114$	0.042 %
$t_4 = 0.4$	$I_4 = 0.574282$	$V(0.4) = 0.5683177$	0.05 %
$t_5 = 0.5$	$I_5 = 0.7175534$	$V(0.5) = 0.7161095$	0.20 %

The mistake has also been expressed as a percentage. If you look at percentages, it's frequently simpler to see how an approximation performs.

The formula of the Percent error = $\frac{|\text{exact} - \text{approximate}|}{\text{exact}} \times 100$

The absolute value of the error for each recorded value of I and each value of t is plotted in the table. The lines in this graph are inserted to improve the plot's visual appeal but may not accurately depict the inaccuracy between the data points.



Here is the plot of drawing two curves Euler (Approximate) , Exact Solution and we see what a relationship between the curves they are close to each other , as if there is no difference between the two curves.

Results:

Finally we can say that the analytical numerical for initial value problems were presented using an illustrative example and a real electrical engineering problem. The two solutions are common for initial value problems .When we solve the problem of electrical series circuit either using small values or big values the solution is divergent and there is no difference between the analytical or numerical solutions . But when we solve the problem of electrical parallel circuit either with small values or big values there is big difference between the analytical and numerical solutions and it is convergent as we have seen in an illustrative example, to release this problem I replace and substitute the values of $t = 1, 2, 3, \dots$ by $t = 0.1, 0.2, 0.3, \dots$ as we see in example ④ there is no difference between the analytical or numerical solutions and it is divergent.

Conclusions:

As has been demonstrated in the present study, solving initial value problem analytically using the method of integrating factor had helped in ensuring that it is the exact solution for the same problem which can be easily obtained. Thus, it can be concluded here that solving ODE by using the Numerical Method (Euler's Method), is very easy and simple to make comparative between the two methods .

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