Group Difference Cordial Labeling of some Snake Related Graphs

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Abstract

Let $G = (VG),\ E(G))$ be a graph. Let Γ be a group. For $u\,\epsilon\,\Gamma$, let $o(u)$ denotes the order of u in Γ. Let $f : V(G) \to \Gamma$ be a function. For each edge uv assign the label $\left|o\left(f(u)\right) - o\left(f(v)\right)\right|$. Let $v_f(i)$ denote the number of vertices of G having label i under f. Also $e_f(1)$, $e_f(0)$ respectively denote the number of edges labeled with 1 and not with 1.Now f is called a group difference cordial labeling if $\left|v_f(i) - v_f(j)\right| \leq 1$ for every $i, j\,\epsilon\Gamma$, $i \neq j$ and $\left|e_f(1) - e_f(0)\right| \leq 1$. A graph which admits a group difference cordial labeling is called group difference cordial graph. In this paper we fix the group Γ as the group $\{1, -1, i, -i\}$ which is the group of fourth roots of unity, that is cyclic with generators $i\ and - i$.

We prove that Quadrilateral snake QS_n , Alternate quadrilateral snake $A(QS_n)$ and further characterized Double quadrilateral snake $D(QS_n)$ and Alternate double quadrilateral snake $A(D(QS_n))$.

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1 Introduction

Graphs considered here are finite, undirected and simple. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. Labelled graphs serve as useful models for a broad range of applications such as : astronomy, circuit design, communication network addressing and models for constraint programming over finite domains.

Cahit [2] introduced the concept of cordial labeling.

Definition 1.1. [2] Let $f : V(G) \to \{0,1\}$ be any function. For each edge xy assign the label $|f(x) - f(y)|$. f is called a cordial labeling if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1. Also the number of edges labelled 0 and the number of edges labeled 1 differ by at most 1.

In[5] , Ponraj et al. introduced a new labeling called difference cordial labeling .

Definition 1.2. [5] Let G be a (p,q) graph .Let $f : V(G) \to \{1,2 \ldots p\}$ be a bijection. For each edge, assign the label $|f(u) - f(v)|$. f is called a difference cordial labeling if f is $1-1$ and $\left|e_f(0) - e_f(1)\right| = 1$ where $e_f(1)$ and $e_f(0)$ denote the number of edges with 1 and not labeled with 1 respectively. A graph with a difference cordial labeling is called a difference cordial graph.

Athisayanathan et al.[1] introduced the concept of group A cordial labeling.

Definition 1.3. [1] Let A be a group. We denote the order of an element $a \epsilon A$ by $o(a)$.Let$f: V(G) \to A$ be a function. For each edge uv assign the label 1 if $\left(o\big(f(u)\big), o\big(f(v)\big)\right) = 1$ or 0 otherwise . f is called a group A Cordial labeling if $\left|v_f(a) - v_f(b)\right| \leq 1$ and $\left|e_f(0) - e_f(1)\right| \leq 1$, where $v_f(x)$ and $e_f(n)$ respectively denote the number of vertices labeled with an element x and number of edges labeled with n $(n = 0,1)$ A graph which admits a group A cordial labeling is called a group A cordial graph.

2. Preliminaries

Definition 2.1. The Quadrilateral snake QS_n is obtained from a path $u_1, u_2 \ldots u_n$ by joining u_i , $u_{i+1}$to new vertices v_i, w_i respectively and then joining $v_i and$ w_i.

Definition 2.2. An Alternate Quadrilateral snake $A(QS_n)$ is obtained from a path $u_1, u_2 \ldots u_n$by joining u_i , u_{i+1} (Alternatively) to new vertices v_i, w_i respectively and then joining $v_i and$ w_i.

Definition 2.3. The Double Quadrilateral snake $D(QS_n)$ consists of two Quadrilateral snakes that have a common path.

Definition 2.4. The Alternate Double Quadrilateral snake $A(D(QS_n))$consists of two alternative Quadrilateral snakes that have a common path.

3. Group Difference cordial Graphs

Definition 3.1. Let $G = \big(V(G), E(G)\big)$ be a graph. Let Γ be a group. For $u\epsilon\Gamma$, let $o(u)$ denote the order of u in Γ . Let $f\colon V(G) \to \Gamma$ be a function .For each edge uv assign the label $\big|o\big(f(u)\big) - o(fv))\big|$. Let $v_f(i)$ denote the number of vertices of G having label i under f. Also e_f (1) , e_f (0) respectively denote the number of edges labeled with 1 and not with 1. Now f is called a group difference cordial labeling if $\left|v_f(i) - v_f(j)\right| \leq 1$ for every $i\,, j\ \epsilon\,\Gamma, i \neq j\ and\ \left|e_f(1) - e_f(0)\right| \leq 1$. A graph which admits a group difference cordial labeling is called group difference cordial graph .

In this paper we take the group Γ as the group $\{1, -1, i, -i\}$ which is the group of fourth roots of unity, that is cyclic with generators $i\ and - i$.

$Theorem$ 3.2 . The Quadrilateral snake QS_n is a group difference cordial graph if $and\ only\ if$ $n \not\equiv 2\ (mod\ 4)$.

Proof: Let $G = QS_n$ be a quadrilateral snake of 3n-2 vertices and 4n-4 edges, f be the group difference cordial labeling of G. Let $V(G) = \{u_1, u_2 \ldots u_n\}$ be the vertices in the path and $\{v_i, w_i\ \ /\ 1 \leq i \leq n-1\}$ be the vertices joined by $u_i\,, u_{i+1}$ respectively.

Assume $n \not\equiv 2\ (mod\ 4)$, To prove G is a group difference cordial graph

$Case\ (i)$: $n \equiv 0\ (mod\ 4), Let\ \ n = 4k, k \geq 1$ and $define\, f\colon\ V(G) \to \{1, -1, i, -i\}$ as follows

$$f(u_{2i-1}) = \{ \begin{smallmatrix} 1 & if\ 1\leq i \leq k \\ i & if\ k+1 \leq i \leq 2k \end{smallmatrix}$$

$$f(u_{2i}) = \{ \begin{smallmatrix} -1 & if\ 1\leq i\leq k \\ -i & if\ k+1\leq i\leq 2k \end{smallmatrix}$$

$$f(v_{2i-1}) = \{ \begin{smallmatrix} -1 & if\ 1\leq i \leq k \\ -i & if\ k+1 \leq i \leq 2k \end{smallmatrix}$$

$$f(v_{2i}) = \{ \begin{smallmatrix} 1 & if\ 1\leq i\leq k \\ i & if\ k+1\leq i\leq 2k-1, for\ all\ \ k\geq 2 \end{smallmatrix}$$

$$f(w_{2i-1}) = \{ \begin{smallmatrix} 1 & if\ 1\leq i \leq k \\ i & if\ k+1 \leq i \leq 2k \end{smallmatrix}$$

$$f(w_{2i}) = \{ \begin{smallmatrix} -1 & if\ 1\leq i\leq k \\ -i & if\ k+1\leq i\leq 2k-1, for\ all\ k\geq 2 \end{smallmatrix}$$

Clearly $V_f(1) = V_f(-1) = 3k, V_f(i) = V_f(-i) = 3k - 1$ also $e_f(1) = e_f(0) = 8k - 2$. Therefore, f is a group difference cordial labeling of G.

Case (ii): $n \equiv 1 \ (mod\ 4)$, Let $n = 4k + 1, k \geq 1$.

$$f(u_{2i-1}) = \begin{cases} 1 & if \ \ 1 \leq i \leq k+1 \\ i & if \ \ k+2 \leq i \leq 2k+1 \end{cases}$$

$$f(u_{2i}) = \begin{cases} -1 & if \ \ 1 \leq i \leq k \\ -i & if \ \ k+1 \leq i \leq 2k \end{cases}$$

$$f(v_{2i-1}) = \begin{cases} -1 & if \ 1 \leq i \leq k \\ -i & if \ k+1 \leq i \leq 2k \end{cases}$$

$$f(v_{2i}) = \begin{cases} 1 & if \ 1 \leq i \leq k \\ i & if \ k+1 \leq i \leq 2k-1 \end{cases}$$

$$f(w_{2i-1}) = \begin{cases} 1 & if \ 1 \leq i \leq k \\ i & if \ k+1 \leq i \leq 2k \end{cases}$$

$$f(w_{2i}) = \begin{cases} -1 & if \ 1 \leq i \leq k \\ -i & if \ k+1 \leq i \leq 2k-1 \end{cases}$$

Clearly $V_f(1) = 3k + 1$, $V_f(-1) = V_f(i) = V_f(-i) = 3k$ also $e_f(1) = 8k = e_f(0)$. Therefore, f is a group difference cordial labeling of G.

Case (iii): $n \equiv 3 \ (mod\ 4)$,Let $n = 4k + 3, k \geq 1$.

$$f(u_{2i-1}) = \begin{cases} 1 & if \ \ 1 \leq i \leq k+1 \\ i & if \ \ k+2 \leq i \leq 2k+2 \end{cases}$$

$$f(u_{2i}) = \begin{cases} -1 & if \ \ 1 \leq i \leq k+1 \\ -i & if \ \ k+2 \leq i \leq 2k+1 \end{cases}$$

$$f(v_{2i-1}) = \begin{cases} -1 & if \ 1 \leq i \leq k+1 \\ -i & if \ k+2 \leq i \leq 2k+1 \end{cases}$$

$$f(v_{2i}) = \begin{cases} 1 & if \ 1 \leq i \leq k \\ i & if \ k+1 \leq i \leq 2k+1 \end{cases}$$

$$f(w_{2i-1}) = \begin{cases} 1 & if \ 1 \leq i \leq k+1 \\ i & if \ k+2 \leq i \leq 2k+1 \end{cases}$$

$$f(w_{2i}) = \begin{cases} -1 & if \ 1 \leq i \leq k \\ -i & if \ k+1 \leq i \leq 2k+1 \end{cases}$$

Clearly $V_f(1) = V_f(-1) = V_f(i) = 3k + 2$, $V_f(-i) = 3k + 1$. Also $e_f(1) = 8k + 4 = e_f(0)$

Therefore, f is a group difference cordial labeling of G.

Conversly, Assume G is a group difference cordial graph and f is a group difference cordial labeling of G.

Claim $n \not\equiv 2 \ (mod\ 4)$.Suppose $n \equiv 2 \ (mod\ 4)$, Let $n = 4k + 2, k \geq 1$.

So $V_f(1) = V_f(-1) = V_f(i) = V_f(-i) = 3k + 1$ for any group difference cordial labeling f

To get an edge $e = uv$ with label 1, we need either $f(u) = 1, f(v) = -1$ or vice versa. The maximum number of vertices labeled with 1 and -1 alternatively in the path are $2k + 1$. Therefore $2k$ edges have labeling as 1.Also for the set of vertices $\{v_i \ /1 \le i \le n-1\}, 2k+1$ vertices are labeled as 1 and -1 alternatively and for the set of vertices $\{w_i \ / \le i \le n-1\}, 2k$ vertices are labeled as 1 and -1 alternatively. Therefore, totally we get $4k + 1$ vertices are labeled as 1 and -1 and thus we get $4k + 1$ edges labeled as 1.Also by the definition of QS_n the edges connecting v_i and w_i, we get $2k$ edges labeling with 1 .Thus $e_f(1) = 8k + 1$ and $e_f(0) = 8k + 3$, which is contradiction.

Theorem 3.3. The double quadrilateral snake $D(QS_n)$ is a group difference cordial graph if and only if $n \not\equiv 0 \ (mod\ 4)$.

Proof: Let $G = D(QS_n)$ be a double quadrilateral snake obtained by a path $u_1, u_2 \ldots u_n$ by joining u_i and u_{i+1} to new vertices v_i, x_i and w_i, y_i respectively and adding edges v_i, w_i and x_i, y_i where $1 \le i \le n-1$. Thus $V(G)$ has $5n - 4$ vertices and $E(G)$ has $7n - 7$ edges, f be the group difference cordial labeling of G. Assume $n \not\equiv 0 \ (mod\ 4)$, To prove G is a group difference cordial graph

Case (i): $n \equiv 1 \ (mod\ 4)$, Let $n = 4k + 1, k \ge 1$ and define $f : V(G) \to \{1, -1, i, -i\}$ as follows

$$f(u_{2i-1}) = \begin{cases} 1 & if\ 1 \le i \le k+1 \\ i & if\ k+2 \le i \le 2k+1 \end{cases}$$

$$f(u_{2i}) = \begin{cases} -1 & if\ 1 \le i \le k \\ -i & if\ k+1 \le i \le 2k \end{cases}$$

$$f(v_{2i-1}) = f(x_{2i-1}) = \begin{cases} -1 & if\ 1 \le i \le k \\ -i & if\ k+1 \le i \le 2k \end{cases}$$

$$f(v_{2i}) = f(x_{2i}) = \begin{cases} 1 & if\ 1 \le i \le k \\ i & if\ k+1 \le i \le 2k \end{cases}$$

$$f(w_{2i-1}) = f(y_{2i-1}) = \begin{cases} 1 & if\ 1 \le i \le k \\ i & if\ k+1 \le i \le 2k \end{cases}$$

$$f(w_{2i}) = f(y_{2i}) = \begin{cases} -1 & if\ 1 \le i \le k \\ -i & if\ k+1 \le i \le 2k \end{cases}$$

Clearly $V_f(1) = 5k + 1, V_f(-1) = V_f(i) = V_f(-i) = 5k$ also $e_f(1) = e_f(0) = 14k$.

Therefore, f is a group difference cordial labeling of G.

Case (ii): $n \equiv 2 \ (mod\ 4)$, Let $n = 4k + 2, k \ge 1$.

$$f(u_{2i-1}) = \begin{cases} 1 & if \quad 1\leq i \leq k+1 \\ i & if \quad k+2 \leq i \leq 2k+1 \end{cases}$$

$$f(u_{2i}) = \begin{cases} -1 & if \quad 1\leq i\leq k+1 \\ -i & if \quad k+2\leq i\leq 2k+1 \end{cases}$$

$$f(v_{2i-1}) = \begin{cases} -1 & if \ 1\leq i \leq k+1 \\ -i & if \ k+2 \leq i \leq 2k+1 \end{cases}$$

$$f(v_{2i}) = \begin{cases} 1 & if \ 1\leq i\leq k \\ i & if \ k+1\leq i\leq 2k \end{cases}$$

$$f(w_{2i-1}) = \begin{cases} 1 & if \ 1\leq i \leq k+1 \\ i & if \ k+2 \leq i \leq 2k+1 \end{cases}$$

$$f(w_{2i}) = \begin{cases} -1 & if \ 1\leq i\leq k \\ -i & if \ k+1\leq i\leq 2k \end{cases}$$

$$f(x_{2i-1}) = \begin{cases} -1 & if \quad 1\leq i \leq k \\ -i & if \quad k+1 \leq i \leq 2k+1 \end{cases}$$

$$f(x_{2i}) = \begin{cases} 1 & if \quad 1\leq i\leq k \\ i & if \quad k+1\leq i\leq 2k \end{cases}$$

$$f(y_{2i-1}) = \begin{cases} 1 & if \ 1\leq i \leq k \\ i & if \ k+1 \leq i \leq 2k+1 \end{cases}$$

$$f(y_{2i}) = \begin{cases} -1 & if \ 1\leq i\leq k \\ -i & if \ k+1\leq i\leq 2k \end{cases}$$

Clearly $V_f(1) = 5k + 2 = V_f(-1), V_f(i) = V_f(-i) = 5k + 1$ also $e_f(1) = 14k + 4$ *and* $e_f(0) = 14k + 3$. Therefore, f is a group difference cordial labeling of G.

Case (iii): $n \equiv 3 \ (mod\ 4)$,Let $n = 4k + 3, k \geq 1$.

$$f(u_{2i-1}) = \begin{cases} 1 & if \quad 1\leq i \leq k+1 \\ i & if \quad k+2 \leq i \leq 2k+2 \end{cases}$$

$$f(u_{2i}) = \begin{cases} -1 & if \quad 1\leq i\leq k+1 \\ -i & if \quad k+2\leq i\leq 2k+1 \end{cases}$$

$$f(v_{2i-1}) = f(x_{2i-1}) = \begin{cases} -1 & if \ 1\leq i \leq k+1 \\ -i & if \ k+2 \leq i \leq 2k+1 \end{cases}$$

$$f(v_{2i}) = f(x_{2i}) = \begin{cases} 1 & if \ 1\leq i\leq k \\ i & if \ k+1\leq i\leq 2k+1 \end{cases}$$

$$f(w_{2i-1}) = f(y_{2i-1}) = \begin{cases} 1 & if \ 1\leq i \leq k+1 \\ i & if \ k+2 \leq i \leq 2k+1 \end{cases}$$

$$f(w_{2i}) = f(y_{2i}) = \begin{cases} -1 & if \ 1\leq i\leq k \\ -i & if \ k+1\leq i\leq 2k+1 \end{cases}$$

Clearly $V_f(1) = V_f(-1) = V_f(i) = 5k + 3$, $V_f(-i) = 5k + 2$. Also $e_f(1) = 14k + 7 = e_f(0)$.

Conversly, Assume G is a group difference cordial graph and f is a group difference cordial labeling of G.

Claim $n \not\equiv 0\ (mod\ 4)$.Suppose $n \equiv 0(mod\ 4)$,Let $n = 4k, k \geq 1$.

So $V_f(1) = V_f(-1) = V_f(i) = V_f(-i) = 5k - 1$ for any group difference cordial labeling f

To get an edge $e = uv$ with label 1,we need either$f(u) = 1, f(v) = -1$ or vice versa. The maximum number of vertices labeled with 1 and -1 alternatively in the path are $2k$.

Thus we get $2k - 1$ edges of labeling 1.Also for the set of vertices$\{x_i, y_i\ /1 \leq i \leq n - 1\}, 2k - 1$ vertices are labeled as 1 and -1 alternatively and for the set of vertices $\{v_i, w_i\ / \leq i \leq n - 1\}, 2k$ vertices are labeled as 1 and -1 alternatively. Therefore, totally we get $4k - 2$ vertices for $\{x_i, y_i\ /1 \leq i \leq n - 1\}, 4k$ vertices for $\{v_i, w_i\ / \leq i \leq n - 1\}$are labeled as 1 and -1 and thus we get $8k - 3$ edges labeled as 1.Also by the definition of $D(QS_n)$ the edges connecting $v_i\ and\ w_i$; $x_i\ and\ y_i$ we get $2k\ and\ 2k - 1$ edges labeling with 1 respectively. Thus$e_f(1) = 14k - 5\ and\ e_f(0) = 14k - 2$ which is a contradiction .

Theorem 3.4. The alternate quadrilateral snake $A(QS_n)$ is a group difference cordial graph for all "n" if it starts with an edge and if $n \not\equiv 2(mod\ 4)$ if it starts with cycle.

Proof: Let $G = A(QS_n)$ be the alternate quadrilateral snake obtained by a path $u_1, u_2 \dots u_n$ by joining $u_i\ and\ u_{i+1}$ (alternatively) with new vertices $v_j\ and\ w_j$,$1 \leq i \leq n\ and\ 1 \leq j \leq \lfloor \frac{n}{2} \rfloor$. (quadrilateral starts from u_1) $1 \leq j \leq \frac{n}{2} - 1$ (quadrilateral starts from u_2).Note that $V(G) = 2n$ and $E(G) = \frac{5n}{2} - 1$ if n is even starts from u_1 ; $V(G) = 2n - 2$ and $E(G) = \frac{5n}{2} - 4$ if n is even starts from u_2; $V(G) = 2n - 1$ and $E(G) = \frac{5n-5}{2}$ if n is odd

To prove G is a group difference cordial graph

Case (I): Alternate quadrilateral snake $A(QS_n)$ starts from u_2

Sub case (i) $n \equiv 0\ (mod\ 4), Let\ n = 4k, k \geq 1$ and $define\ f : V(G) \rightarrow \{1, -1, i, -i\}$ as follows

$$f(u_{2i-1}) = \begin{cases} 1 & if\ 1 \leq i \leq k \\ i & if\ k+1 \leq i \leq 2k \end{cases}$$

$$f(u_{2i}) = \begin{cases} -1 & if\ 1 \leq i \leq k \\ -i & if\ k+1 \leq i \leq 2k \end{cases}$$

$$f(v_i) = \begin{cases} 1 & if\ 1 \leq i \leq k \\ i & if\ k+1 \leq i \leq 2k-1, for\ all\ k \geq 2 \end{cases}$$

$$f(w_i) = \begin{cases} -1 & if\ 1 \leq i \leq k \\ -i & if\ k+1 \leq i \leq 2k-1\ for\ all\ k \geq 2 \end{cases}$$

Clearly $V_f(1) = V_f(-1) = 2k$,$V_f(i) = V_f(-i) = 2k - 1$ also $e_f(1) = e_f(0) = 5k - 2$.

Therefore, f is a group difference cordial labeling of G.

Subcase (ii): $n \equiv 1\ (mod\ 4)$, Let $n = 4k + 1, k \geq 1$.

$$f(u_{2i-1}) = \begin{cases} 1 & if \quad 1 \leq i \leq k+1 \\ i & if \quad k+2 \leq i \leq 2k+1 \end{cases}$$

$$f(u_{2i}) = \begin{cases} -1 & if \quad 1 \leq i \leq k \\ -i & if \quad k+1 \leq i \leq 2k \end{cases}$$

$$f(v_i) = \begin{cases} 1 & if\ 1 \leq i \leq k \\ i & if\ k+1 \leq i \leq 2k \end{cases}$$

$$f(w_i) = \begin{cases} -1 & if\ 1 \leq i \leq k \\ -i & if\ k+1 \leq i \leq 2k \end{cases}$$

Clearly $V_f(1) = 2k + 1, V_f(-1) = V_f(i) = V_f(-i) = 2k$ also $e_f(1) = 5k = e_f(0)$. Therefore, f is a group difference cordial labeling of G.

Subcase (iii): $n \equiv 2\ (mod\ 4)$,Let $n = 4k + 2, k \geq 1$.

$$f(u_{2i-1}) = \begin{cases} 1 & if \quad 1 \leq i \leq k+1 \\ i & if \quad k+2 \leq i \leq 2k+1 \end{cases}$$

$$f(u_{2i}) = \begin{cases} -1 & if \quad 1 \leq i \leq k+1 \\ -i & if \quad k+2 \leq i \leq 2k+1 \end{cases}$$

$$f(v_i) = \begin{cases} 1 & if\ 1 \leq i \leq k \\ i & if\ k+1 \leq i \leq 2k \end{cases}$$

$$f(w_i) = \begin{cases} -1 & if\ 1 \leq i \leq k \\ -i & if\ k+1 \leq i \leq 2k \end{cases}$$

Clearly $V_f(1) = V_f(-1) = 2k + 1, V_f(i) = V_f(-i) = 2k$. Also $e_f(1) = 5k + 1, e_f(0) = 5k$. Therefore, f is a group difference cordial labeling of G.

Subcase (iv): $n \equiv 3\ (mod\ 4)$,Let $n = 4k + 3, k \geq 1$.

$$f(u_{2i-1}) = \begin{cases} 1 & if \quad 1 \leq i \leq k+1 \\ i & if \quad k+2 \leq i \leq 2k+2 \end{cases}$$

$$f(u_{2i}) = \begin{cases} -1 & if \quad 1 \leq i \leq k+1 \\ -i & if \quad k+2 \leq i \leq 2k+1 \end{cases}$$

$$f(v_i) = \begin{cases} 1 & if\ 1 \leq i \leq k+1 \\ i & if\ k+2 \leq i \leq 2k+1 \end{cases}$$

$$f(w_i) = \begin{cases} -1 & if\ 1 \leq i \leq k \\ -i & if\ k+1 \leq i \leq 2k+1 \end{cases}$$

Clearly $V_f(1) = 2k + 2, V_f(-1) = V_f(i) = V_f(-i) = 2k + 1$. Also $e_f(1) = 5k + 2, e_f(0) = 5k + 3$.Therefore, f is a group difference cordial labeling of G.

$Case\ (II)$: Alternate quadrilateral snake $A(QS_n)$ starts from u_1

$Sub\ case\ (i)\ n \equiv 0\ (mod\ 4), Let\ n = 4k, k \geq 1$ and $define\ f:\ V(G) \rightarrow \{1, -1, i, -i\}$ as follows $f(u_{2i-1}) = \{ \begin{smallmatrix} 1 & if\ 1\leq i \leq k \\ i & if\ k+1 \leq i \leq 2k \end{smallmatrix}$

$$f(u_{2i}) = \{ \begin{smallmatrix} -1 & if\ 1\leq i\leq k \\ -i & if\ k+1\leq i\leq 2k \end{smallmatrix}$$

$$f(v_i) = \{ \begin{smallmatrix} -1 & if\ 1\leq i \leq k \\ -i & if\ k+1 \leq i \leq 2 \end{smallmatrix}$$

$$f(w_i) = \{ \begin{smallmatrix} 1 & if\ 1\leq i\leq k \\ i & if\ k+1\leq i\leq 2k \end{smallmatrix}$$

Clearly $V_f(1) = V_f(-1) = V_f(i) = V_f(-i) = 2k$ also $e_f(1) = 5k - 1$, $e_f(0) = 5k$. Therefore, f is a group difference cordial labeling of G.

Subcase (ii): $n \equiv 1\ (mod\ 4)$, Let $n = 4k + 1, k \geq 1$.

$$f(u_{2i-1}) = \{ \begin{smallmatrix} 1 & if & 1\leq i \leq k+1 \\ i & if & k+2 \leq i \leq 2k+1 \end{smallmatrix}$$

$$f(u_{2i}) = \{ \begin{smallmatrix} -1 & if & 1\leq i\leq k \\ -i & if & k+1\leq i\leq 2k \end{smallmatrix}$$

$$f(v_i) = \{ \begin{smallmatrix} -1 & if\ 1\leq i\leq k \\ -i & if\ k+1\leq i\leq 2k \end{smallmatrix}$$

$$f(w_i) = \{ \begin{smallmatrix} 1 & if\ 1\leq i\leq k \\ i & if\ k+1\leq i\leq 2k \end{smallmatrix}$$

Clearly $V_f(1) = 2k + 1, V_f(-1) = V_f(i) = V_f(-i) = 2k$ also $e_f(1) = 5k = e_f(0)$. Therefore, f is a group difference cordial labeling of G.

Subcase (iii): $n \equiv 3\ (mod\ 4)$,Let $n = 4k + 3, k \geq 1$.

$$f(u_{2i-1}) = \{ \begin{smallmatrix} 1 & if & 1\leq i \leq k+1 \\ i & if & k+2 \leq i \leq 2k+2 \end{smallmatrix}$$

$$f(u_{2i}) = \{ \begin{smallmatrix} -1 & if & 1\leq i\leq k \\ -i & if & k+1\leq i\leq 2k+1 \end{smallmatrix}$$

$$f(v_i) = \{ \begin{smallmatrix} -1 & if\ 1\leq i \leq k+1 \\ -i & if\ k+2 \leq i \leq 2k+1 \end{smallmatrix}$$

$$f(w_i) = \{ \begin{smallmatrix} 1 & if\ 1\leq i\leq k+1 \\ i & if\ k+2\leq i\leq 2k+1 \end{smallmatrix}$$

Clearly $V_f(1) = 2k + 2, V_f(-1) = V_f(i) = V_f(-i) = 2k + 1$. Also $e_f(1) = 5k + 2, e_f(0) = 5k + 3$.Therefore, f is a group difference cordial labeling of G.

Remark:

$A(QS_n)$ is not a group difference cordial graph for $n \not\equiv 2(mod\ 4)$.Because to satisfy the vertex condition for group difference cordial labeling it is essential to assign $V_f(1) = V_f(-1) = V_f(i) = V_f(-i) = 2k + 1\ for\ 8k + 4$ vertices.The vertices with labeling 1 and -1 gives rise to the edges of labeling 1 as $5k$+1and the vertices with other labeling gives rise to the edges of labeling 0 as $5k$+3,that is $e_f(1) = 5k + 1, e_f(0) = 5k + 3$ Therefore, $\left|e_f(0) - e_f(1)\right| \geq 2$.

$Theorem$ 3.5. The alternate double quadrilateral snake $A(D(QS_n))$ is a group difference cordial graph for all "n"

Proof: Let $G = A(D(QS_n))$ be the alternate quadrilateral snake obtained by a path $u_1, u_2 \dots u_n$ by joining u_i and u_{i+1} (alternatively) to get four new vertices v_j, w_j and x_j, y_j, by the edges $u_i v_j$, $u_{i+1} w_j$, v_j, w_j, $u_i x_j$, $u_{i+1} y_j$ and x_j, y_j where $1 \leq i \leq n$ and $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$. (quadrilateral starts from u_1) $1 \leq j \leq \frac{n}{2} - 1$ (quadrilateral starts from u_2).Note that $V(G) = 3n$ and $E(G) = 4n - 1$ if n is even and starts from u_1 ; $V(G) = 3n - 4$ and $E(G) = 4n - 7$ if n is even starts from u_2; $V(G) = 3n - 2$ and $E(G) = 4n - 4$ if n is odd

To prove G is a group difference cordial graph

$Case$ (I): Alternate double quadrilateral snake $A(D(QS_n))$ starts from u_1

$Sub\ case$ (i) $n \equiv 0\ (mod\ 4), Let\ n = 4k, k \geq 1$ and $define\ f:\ V(G) \to \{1, -1, i, -i\}$ as follows $f(u_{2i-1}) = \{ \begin{smallmatrix} 1 & if\ 1 \leq i \leq k \\ i & if\ k+1 \leq i \leq 2k \end{smallmatrix}$

$$f(u_{2i}) = \{ \begin{smallmatrix} -1 & if\ 1 \leq i \leq k \\ -i & if\ k+1 \leq i \leq 2k \end{smallmatrix}$$

$$f(v_i) = f(x_i) = \{ \begin{smallmatrix} -1 & if\ 1 \leq i \leq k \\ -i & if\ k+1 \leq i \leq 2k \end{smallmatrix}$$

$$f(w_i) = f(y_i) = \{ \begin{smallmatrix} 1 & if\ 1 \leq i \leq k \\ i & if\ k+1 \leq i \leq 2k \end{smallmatrix}$$

Clearly $V_f(1) = V_f(-1) = V_f(i) = V_f(-i) = 3k$ also $e_f(1) = 8k - 1$, $e_f(0) = 8k$.

Therefore, f is a group difference cordial labeling of G.

Subcase (ii): $n \equiv 1\ (mod\ 4)$, Let $n = 4k + 1, k \geq 1$.

$$f(u_{2i-1}) = \{ \begin{smallmatrix} 1 & if & 1 \leq i \leq k+1 \\ i & if & k+2 \leq i \leq 2k+1 \end{smallmatrix}$$

$$f(u_{2i}) = \{ \begin{smallmatrix} -1 & if & 1 \leq i \leq k \\ -i & if & k+1 \leq i \leq 2k \end{smallmatrix}$$

$$f(v_i) = f(x_i) = \{ \begin{smallmatrix} -1 & if\ 1 \leq i \leq k \\ -i & if\ k+1 \leq i \leq 2k \end{smallmatrix}$$

$$f(w_i) = f(y_i) = \{ \begin{smallmatrix} 1 & if\ 1 \leq i \leq k \\ i & if\ k+1 \leq i \leq 2k \end{smallmatrix}$$

Clearly $V_f(1) = 3k + 1, V_f(-1) = V_f(i) = V_f(-i) = 3k$ also $e_f(1) = 8k = e_f(0)$. Therefore, f is a group difference cordial labeling of G.

Subcase (iii): $n \equiv 2\ (mod\ 4)$,Let $n = 4k + 2, k \geq 1$.

$$f(u_{2i-1}) = \begin{cases} 1 & if \quad 1\leq i \leq k+1 \\ i & if \quad k+2 \leq i \leq 2k+1 \end{cases}$$

$$f(u_{2i}) = \begin{cases} -1 & if \quad 1\leq i\leq k+1 \\ -i & if \quad k+2\leq i\leq 2k+1 \end{cases}$$

$$f(v_i) = \begin{cases} -1 & if\ 1\leq i \leq k+1 \\ -i & if\ k+2 \leq i \leq 2k+1 \end{cases}$$

$$f(w_i) = \begin{cases} 1 & if\ 1\leq i\leq k+1 \\ i & if\ k+2\leq i\leq 2k+1 \end{cases}$$

$$f(x_i) = \begin{cases} -1 & if\ 1\leq i \leq k \\ -i & if\ k+1 \leq i \leq 2k+1 \end{cases}$$

$$f(y_i) = \begin{cases} 1 & if\ 1\leq i\leq k \\ i & if\ k+1\leq i\leq 2k+1 \end{cases}$$

Clearly $V_f(1) = V_f(-1) = 3k + 2, V_f(i) = V_f(-i) = 3k + 1$. Also $e_f(1) = 8k + 4, e_f(0) = 8k + 3$.Therefore, f is a group difference cordial labeling of G.

Subcase (iv): $n \equiv 3\ (mod\ 4)$,Let $n = 4k + 3, k \geq 1$.

$$f(u_{2i-1}) = \begin{cases} 1 & if \quad 1\leq i \leq k+1 \\ i & if \quad k+2 \leq i \leq 2k+2 \end{cases}$$

$$f(u_{2i}) = \begin{cases} -1 & if \quad 1\leq i\leq k+1 \\ -i & if \quad k+2\leq i\leq 2k+1 \end{cases}$$

$$f(v_i) = \begin{cases} -1 & if\ 1\leq i \leq k+1 \\ -i & if\ k+2 \leq i \leq 2k+1 \end{cases}$$

$$f(w_i) = \begin{cases} 1 & if\ 1\leq i\leq k+1 \\ i & if\ k+2\leq i\leq 2k+1 \end{cases}$$

$$f(x_i) = \begin{cases} -1 & if\ 1\leq i \leq k \\ -i & if\ k+1 \leq i \leq 2k+1 \end{cases}$$

$$f(y_i) = \begin{cases} 1 & if\ 1\leq i\leq k \\ i & if\ k+1\leq i\leq 2k+1 \end{cases}$$

Clearly $V_f(1) = V_f(-1) = V_f(i) = 3k + 2$, $V_f(-i) = 3k + 1$. Also $e_f(1) = 8k + 4 = e_f(0)$. Therefore, f is a group difference cordial labeling of G.

$Case\ (II)$: Alternate double quadrilateral snake $A(D(QS_n))$ starts from u_2

$Sub\ case\ (i)\ n \equiv 0\ (mod\ 4), Let\ \ n = 4k, k \geq 1$ and $define\ f :\ V(G) \to \{1, -1, i, -i\}$ as follows ,Fix the labeling of the vertex $f(u_1) = -i$

$$f(u_{2i+1}) = \begin{cases} -1 & if\ 1\leq i \leq k\ for\ all\ \ k\geq 2 \\ -i & if\ k+1 \leq i \leq 2k for\ all\ k\geq 2 \end{cases}$$

$$f(u_{2i}) = \begin{cases} 1 & if\ 1\le i\le k \\ i & if\ k+1\le i\le 2k \end{cases}$$

$$f(v_i) = \begin{cases} -1 & if\ 1\le i\le k \\ -i & if\ k+1\le i\le 2-1\ for\ all\ k\ge 2 \end{cases}$$

$$f(w_i) = \begin{cases} 1 & if\ 1\le i\le k \\ i & if\ k+1\le i\le 2k-1 for\ all\ k\ge 2 \end{cases}$$

$$f(x_i) = \begin{cases} -1 & if\ 1\le i\le k-1 for\ all\ k\ge 2 \\ -i & if\ k\le i\le 2k-1 \end{cases}$$

$$f(y_i) = \begin{cases} 1 & if\ 1\le i\le k-1 for\ all\ k\ge 2 \\ i & if\ k\le i\le 2k-1 \end{cases}$$

Clearly $V_f(1) = V_f(-1) = V_f(i) = V_f(-i) = 3k - 1$ also $e_f(1) = 8k - 4$, $e_f(0) = 8k - 3$. Therefore, f is a group difference cordial labeling of G.

Subcase (ii): $n \equiv 1\ (mod\ 4)$, Let $n = 4k + 1, k \ge 1$ is same as subcase(ii) in case(I)

Subcase (iii): $n \equiv 2\ (mod\ 4)$,Let $n = 4k + 2, k \ge 1$.

$$f(u_{2i-1}) = \begin{cases} 1 & if\ 1\le i\le k+1 \\ i & if\ k+2\le i\le 2k+2 \end{cases}$$

$$f(u_{2i}) = \begin{cases} -1 & if\ 1\le i\le k+1 \\ -i & if\ k+2\le i\le 2k+1 \end{cases}$$

$$f(v_i) = f(x_i) = \begin{cases} 1 & if\ 1\le i\le k \\ i & if\ k+1\le i\le 2k \end{cases}$$

$$f(w_i) = f(y_i) = \begin{cases} -1 & if\ 1\le i\le k \\ -i & if\ k+1\le i\le 2k \end{cases}$$

Clearly $V_f(1) = V_f(-1) = 3k + 1, V_f(i) = V_f(-i) = 3k$. Also $e_f(1) = 8k + 1, e_f(0) = 8k$. Therefore, f is a group difference cordial labeling of G.

Subcase (iv): $n \equiv 3\ (mod\ 4)$, Let $n = 4k + 3, k \ge 1$ is same as subcase(iv) in case(i).

Conclusion

In this we have found group difference cordial labeling of different types of graphs such as Quadrilateral snake QS_n ,Double quadrilateral snake $D(QS_n)$, Alternate quadrilateral snake $A(QS_n)$,Alternate double quadrilateral snake $A(D(QS_n))$. Investigating group difference cordial labeling in other classes of graphs for future work.

Conflict of interest

The authors declare that they have no conflict of interest.

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