Unique Fixed-Point Theorem for H-Contraction Mapping in Complete Metric Space

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Abstract

This study introduces H-Contraction mapping, a unique kind of rational expressionbased contraction type mapping. In this mapping, we prove the uniqueness and existence of the fixed point. We further demonstrate that this mapping has a unique common fixed point.

Keywords-Unique Fixed points, Common fixed points, rational type contraction, Complete metric space

1. INTRODUCTION

The idea of fixed-point theory was initially put forth by H. Poincare in 1886. In 1906, M. Frechet went on to present the fixed-point theorem, which takes into consideration both the separation between the points and the associated images of the operator at those points in metric spaces. Banach developed a fixed theorem for a contraction mapping in complete metric space later in 1922. Numerous branches of mathematics depend on this concept. It is a great place to look for solutions to many nonlinear analysis results that have previously been discovered.

There are numerous different spaces to which the Banach contraction principle can be used, including rectangular metric space, generalised metric space, partial metric space, b-metric space, partial b-metric space, symmetric space, and quasi metric space.

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A contractive requirement of the rational type is satisfied by the theorem that Jaggi [1] in particular demonstrated in 1977. In 1975, Dass and Gupta created a novel contractive state they called a rational type contraction [7]. The definition of the notion of metric completeness is given by Kannan's theorem [2], which is distinct from the well-known Banach contraction principle. In Kannan's results on the metric fixed point theory, contractive type mappings were used, however they are not always continuous. A unique fixed point is said to exist for every mapping of a Kannan type on the entire metric space. The fixed point theorems involving these mappings are proved in this article using contractive type mapping on the complete metric space.

2. PRELIMINARIES

Theorem 2.1[7]

Let (Ξ, ϱ) be a complete metric space and $\Upsilon: \Xi \to \Xi$ a mapping such that there exists $\wp, \varphi \geq 0$ with $\wp + \varphi < 1$ satisfying

$$\varrho(\eta, \kappa) = \wp \frac{\varrho(\kappa, \Upsilon\kappa)[1 + \varrho(\eta, \Upsilon\eta)]}{1 + \varrho(\eta, \kappa)} + \varphi\, \varrho(\eta, \kappa)$$

For all $\eta, \kappa \in \Xi$. Then Υ has a unique fixed point.

Definition 2.2[2]

Let (Ξ, ϱ) be a metric space and $\Upsilon: \Xi \to \Xi$. The mapping Υ is said to be a K-contraction if there exists $\tau \in \left[0, \frac{1}{2}\right)$ such that for all $\eta, \kappa \in \Xi$ the following inequality holds:

$$\varrho(\Upsilon\eta, \Upsilon\kappa) \leq \tau, \varrho(\eta, \Upsilon\eta) + \varrho(\kappa, \Upsilon\kappa)\text{-}$$

Lemma 2.3[8]

If $\{\eta_\gamma\}$ is a sequence in a complete metric space (Ξ, ϱ) such that $\varrho(\eta_\gamma, \eta_{\gamma+1}) \leq \varpi\varrho(\eta_{\gamma-1}, \eta_\gamma)$ $\forall$ $\gamma \in \mathbb{N}$, where $\varpi \in ,0,1)$, then $\{\eta_\gamma\}$ is a Cauchy sequence

3. MAIN RESULTS

3.1. UNIQUE FIXED-POINT THEOREMS

Definition 3.1.1

A mapping $\Upsilon: \Xi \rightarrow \Xi$ be a rational type contraction, where (Ξ, ϱ) is a complete metric space, is called H-rational type Contraction if there exist positive real numbers $\wp$, φ, τ such that $0 \leq \wp + \varphi + 2\tau < 1$ for all $\eta, \kappa \in \Xi$, the following inequality holds,

$$\varrho(\Upsilon\eta, \Upsilon\kappa) \leq \wp\, \varrho(\eta, y) + \varphi \frac{\varrho(\kappa, \Upsilon\kappa)[1 + \varrho(\eta, \Upsilon\eta)]}{1 + \varrho(\eta, \kappa)} + \tau[\varrho(\eta, \Upsilon\eta) + \varrho(\kappa, \Upsilon\kappa)$$

Theorem 3.1.2

Let Υ be a self-mapping on a complete metric space (Ξ, ϱ). If Υ is a rational type contraction, there exist $\wp, \varphi, \tau \in [0,1)$, where $\wp + \varphi + 2\tau < 1$ such that

$$\varrho(\Upsilon\eta, \Upsilon\kappa) \leq \wp\, \varrho(\eta, \kappa) + \varphi \frac{\varrho(\kappa, \Upsilon\kappa)[1 + \varrho(\eta, \Upsilon\eta)]}{1 + \varrho(\eta, \kappa)} + \tau[\varrho(\eta, \Upsilon\eta) + \varrho(\kappa, \Upsilon\kappa) \qquad (1)$$

for all $\eta, \kappa \in \Xi$, then Υ has a unique fixed point in Ξ.

Proof:

Let $\eta_{\gamma+1} = \Upsilon\eta_\gamma$ for all $\gamma = 1,2, \ldots$

Then,

$$\begin{aligned}\varrho(\eta_\gamma, \eta_{\gamma+1}) &= \varrho(\Upsilon\eta_{\gamma-1}, \Upsilon\eta_\gamma) \\ &\leq \wp\varrho(\eta_{\gamma-1}, \eta_\gamma) \\ &\quad + \varphi \frac{\varrho(\eta_\gamma, \Upsilon\eta_\gamma)[1 + \varrho(\eta_{\gamma-1}, \Upsilon\eta_{\gamma-1})]}{1 + \varrho(\eta_{\gamma-1}, \eta_\gamma)} + \tau[\varrho(\eta_{\gamma-1}, \Upsilon\eta_{\gamma-1}) + \varrho(\eta_\gamma, \Upsilon\eta_\gamma)] \\ &= \wp\, \varrho(\eta_{\gamma-1}, \eta_\gamma) \\ &\quad + \varphi \frac{\varrho(\eta_\gamma, \eta_{\gamma+1})[1 + \varrho(\eta_{\gamma-1}, \eta_\gamma)]}{1 + \varrho(\eta_{\gamma-1}, \eta_\gamma)} + \tau[\varrho(\eta_{\gamma-1}, \eta_\gamma) + \varrho(\eta_\gamma, \eta_{\gamma+1})]\end{aligned}$$

$$\varrho(\eta_\gamma, \eta_{\gamma+1}) \leq \frac{\wp+\tau}{[1-\varphi-\tau]}\, \varrho(\eta_{\gamma-1}, \eta_\gamma)$$

Since $\frac{(\wp+\tau)}{[1-\varphi-\tau]} < 1$

Let $\varpi = \frac{(\wp+\tau)}{[1-\varphi-\tau]} < 1$

$$\varrho(\eta_\gamma, \eta_{\gamma+1}) \leq \varpi\varrho(\eta_{\gamma-1}, \eta_\gamma)$$

$$\leq \varpi^2 \varrho(\eta_{\gamma-1}, \eta_{\gamma-2})$$

$$\dots \dots$$

$$\leq \varpi^{\gamma} \varrho(\eta_0, \eta_1)$$

For $\delta \geq \gamma$, by triangle inequality,

$$\varrho(\eta_\delta, \eta_\gamma) \leq \varrho(\eta_\delta, \eta_{\delta-1}) + \varrho(\eta_{\delta-1}, \eta_{\delta-2}) + \cdots + \varrho(\eta_{\gamma+1}, \eta_\gamma)$$

$$\leq (\varpi^{\delta-1} + \varpi^{\delta-2} + \cdots + \varpi^{\gamma}) \varrho(\eta_1, \eta_0)$$

$$\leq \left(\frac{\varpi^{\gamma}}{1-\varpi}\right) \varrho(\eta_1, \eta_0)$$

As $\gamma, \delta \to \infty$, $\varrho(\eta_\delta, \eta_\gamma) \to 0$

Thus $\{\eta_\gamma\}$ is a Cauchy sequence.

Since Ξ is complete and the subsequence $\{\eta_{\gamma_\varpi}\}$ of $\{\eta_\gamma\}$ converges to $\eta \in \Xi$.

$$\lim_{\gamma \to \infty} \eta_\gamma = \eta$$

To prove fixed Point:

$$\varrho(\eta, \Upsilon\eta) \leq \varrho(\eta, \Upsilon\eta_\gamma) + \varrho(\Upsilon\eta_\gamma, \Upsilon\eta)$$

$$\leq \varrho(\eta, \Upsilon\eta_\gamma) + \wp\varrho(\eta_\gamma, \eta)$$

$$+\varphi \frac{\varrho(\eta, \Upsilon\eta)[1 + \varrho(\eta_\gamma, \Upsilon\eta_\gamma)]}{1 + \varrho(\eta_\gamma, \eta)} + \tau[\varrho(\eta_\gamma, \Upsilon\eta_\gamma) + \varrho(\eta, \Upsilon\eta)]$$

$$= \varrho(\eta, \eta_{\gamma+1}) + \wp\, \varrho(\eta_\gamma, \eta)$$

$$+\varphi \frac{\varrho(\eta, \Upsilon\eta)[1 + \varrho(\eta_\gamma, \eta_{\gamma+1})]}{1 + \varrho(\eta_\gamma, \eta)} + \tau[\varrho(\eta_\gamma, \eta_{\gamma+1}) + \varrho(\eta, \Upsilon\eta)]$$

$$\Rightarrow \varrho(\eta, \Upsilon\eta) \leq \left[\frac{\varrho(\eta_\gamma, \eta)}{(1-\varphi-\tau) + \varrho(\eta_\gamma, \eta)[1-\tau] - \varphi\, \varrho(\eta_\gamma, \eta_{\gamma+1})}\right]$$

$$[\varrho(\eta, \eta_{\gamma+1}) + \wp\, \varrho(\eta_\gamma, \eta) + \tau\varrho(\eta_\gamma, \eta_{\gamma+1})]$$

As $\gamma \to \infty$, $\varrho(\eta, \Upsilon\eta) \to 0$

$$\Rightarrow \ \Upsilon\eta = \eta$$

To Prove Uniqueness:

Let $\Upsilon\eta = \eta$ and $\Upsilon\kappa = \kappa$

$\varrho(\eta, \kappa) = \varrho(\Upsilon\eta, \Upsilon\kappa)$

$$\leq \wp\, d(\eta, \kappa) + \varphi \frac{\varrho(\kappa, \Upsilon\kappa)[1 + \varrho(\eta, \Upsilon\eta)]}{1 + \varrho(\eta, \kappa)} + \tau[\varrho(\eta, \Upsilon\eta) + \varrho(\kappa, \Upsilon\kappa)]$$

$$= \wp\, \varrho(\eta, \kappa)$$

Since, $\wp < 1$

$\Rightarrow \ \varrho(\eta, \kappa) = 0$

$$\Rightarrow \eta = \kappa$$

Hence, the theorem.

Example 3.1.3

Let $\Xi = \{a, b, c\}$ and ϱ be a metric defined on Ξ as follows:

(i) $\varrho(a, b) = 4; \varrho(b, c) = 3; \varrho(c, a) = 1$
$\Upsilon(a) = a; \Upsilon(b) = c; \Upsilon(c) = a.$
$\wp = \frac{2}{3}; \varphi = \frac{1}{4}; \tau = \frac{1}{30}$ and
Such that $\wp + \varphi + 2\tau = \frac{59}{60} < 1$

$$\varrho(\Upsilon a, \Upsilon b) \leq \wp\, \varrho(a, b) + \varphi \frac{\varrho(b, \Upsilon b)[1 + \varrho(a, \Upsilon a)]}{1 + \varrho(a, b)} + \tau[\varrho(a, \Upsilon a) + \varrho(b, \Upsilon b)$$

$$= \wp\, \varrho(a, b) + \varphi \frac{\varrho(b, c)[1 + \varrho(a, a)]}{1 + \varrho(a, b)} + \tau[\varrho(a, a) + \varrho(b, c)]$$

Therefore,

$$\varrho(a, c) = 1 \leq \frac{2}{3}[4] + \frac{1}{4}\left[\frac{3 \times (1 + 0)}{1 + 4}\right] + \frac{1}{30}[0 + 3$$

$$= 2\frac{11}{12}$$

The inequality holds. Likewise, it might be examined for $\varrho(\Upsilon b, \Upsilon c)$ and $\varrho(\Upsilon c, \Upsilon a)$. Thus Υ is a H- rational type contraction. Further Ξ is complete and in what follows is a unique fixed point a for Υ.

(ii) $\varrho(a,b)=1; \varrho(b,c)=4; \varrho(c,a)=3$

$\Upsilon(a)=b; \Upsilon(b)=b; \Upsilon(c)=a.$

$\wp=\frac{1}{3}; \varphi=\frac{1}{4}; \tau=\frac{1}{6}$ and

Such that $\wp+\varphi+2\tau=\frac{11}{12}<1$

$$\varrho(\Upsilon a,\Upsilon b)\le \wp\,\varrho(a,b)+\varphi\frac{\varrho(b,\Upsilon b)[1+\varrho(a,\Upsilon a)]}{1+\varrho(a,b)}+\tau[\varrho(a,\Upsilon a)+\varrho(b,\Upsilon b)$$

$$=\wp\,\varrho(a,b)+\varphi\frac{\varrho(b,b)[1+\varrho(a,b)]}{1+\varrho(a,b)}+\tau[\varrho(a,b)+\varrho(b,b)$$

Therefore,

$$\varrho(b,b)=0\le\frac{1}{3}[1]+\frac{1}{4}\left[\frac{0\times(1+1)}{1+1}\right]+\frac{1}{6}[1+0$$

$$=\frac{1}{2}$$

The inequality holds. Likewise, it might be examined for $\varrho(\Upsilon b,\Upsilon c)$ and $\varrho(\Upsilon c,\Upsilon a)$. Thus Υ is a H- rational type contraction. Further Ξ is complete and in what follows is a unique fixed point b for Υ.

(iii) $\varrho(a,b)=3; \varrho(b,c)=1; \varrho(c,a)=2$ and

$\Upsilon(a)=b; \Upsilon(b)=c; \Upsilon(c)=c$

$\wp=\frac{2}{3}; \varphi=\frac{1}{8}; \tau=\frac{1}{12}$.

Such that $\wp+\varphi+2\tau=\frac{23}{24}<1$

$$\varrho(\Upsilon a,\Upsilon b)\le \wp\,\varrho(a,b)+\varphi\frac{\varrho(b,\Upsilon b)[1+\varrho(a,\Upsilon a)]}{1+\varrho(a,b)}+\tau[\varrho(a,\Upsilon a)+\varrho(b,\Upsilon b)$$

$$=\wp\,\varrho(a,b)+\varphi\frac{\varrho(b,c)[1+\varrho(a,b)]}{1+\varrho(a,b)}+\tau[\varrho(a,b)+\varrho(b,c)]$$

Therefore,

$$\varrho(b,c)=1\le\frac{2}{3}[3]+\frac{1}{8}\left[\frac{1\times(1+3)}{1+3}\right]+\frac{1}{12}[3+1$$

$$=\frac{59}{24}$$

The inequality holds. Likewise, it might be examined for $\varrho(\Upsilon b,\Upsilon c)$ and $\varrho(\Upsilon c,\Upsilon a)$. Thus Υ is a H- rational type contraction. Further Ξ is complete and in what follows is a unique fixed point c Υ for . That is $\Upsilon c=c$.

Remark 3.1.4

(i) If $\gamma = 0$, then $\wp + \varphi < 1$ equation (1) becomes $\varrho(\Upsilon x, \Upsilon y) \leq \wp\, \varrho(a,b) + \varphi \frac{\varrho(b,\Upsilon b)[1+\varrho(a,\Upsilon a)]}{1+\varrho(a,b)}$, which is Dass and Gupta contraction for rational type

(ii) If $\wp = \varphi = 0$, then $2\,\tau < 1$ implies $\tau < \frac{1}{2}$, hence equation (1) becomes $\varrho(\Upsilon x, \Upsilon y) \leq \tau[\varrho(x,\Upsilon x) + \varrho(y,\Upsilon y)$- , which is Kannan Mapping.

(iii) If $\varphi = \tau = 0$, then $\wp < 1$ and equation (1) becomes $\varrho(\Upsilon x, \Upsilon y) \leq \wp\varrho(x,y)$, which is Banach Contraction principle

(iv) If $\varphi = 0$, then $\wp + 2\tau < 0$, equation (1) becomes $\varrho(\Upsilon x, \Upsilon y) \leq \wp\varrho(x,y) + \tau[\varrho(x,\Upsilon x) + \varrho(y,\Upsilon y)$-, which is Banach contraction + Kannan map.

3.2.UNIQUE COMMON FIXED-POINT THEOREM

Theorem 3.2.1

Let Φ, Υ be continuous self-maps on a complete metric space (Ξ,ϱ). Let $\eta_0 \in \Xi$ and define a sequence $\{\eta_\gamma\}$ such that $\eta_{2\gamma+1} = \Phi\eta_{2\gamma}$ and $\eta_{2\gamma+2} = \Upsilon\eta_{2\gamma+1}$ for $\gamma = 0,1,2,\ldots$ in which all the terms are distinct. Suppose there exists $\wp, \varphi, \tau \in [0,1)$, where $\wp + \varphi + 2\tau < 1$ such that

$$\varrho(\Phi\eta, \Upsilon\kappa) \leq \wp\, \varrho(\eta,\kappa) + \varphi \frac{\varrho(\kappa,\Upsilon\kappa)[1+\varrho(\eta,\Phi\eta)]}{1+\varrho(\eta,\kappa)} + \tau[\varrho(\eta,\Phi\eta) + \varrho(\kappa,\Upsilon\kappa)\text{-}$$

For all $\eta, \kappa \in \Xi$, then Υ and Φ have a unique common fixed point.

Proof:

Let $\eta_0 \in \Xi$ and define a sequence $\{\eta_\gamma\}$ such that $\eta_{2\gamma+1} = \Phi\eta_{2\gamma}$ and $\eta_{2\gamma+2} = \Upsilon\eta_{2\gamma+1}$ for $\gamma = 0,1,2,\ldots$

If there is i_0 such that $\eta_{\gamma_0} = \eta_{\gamma_0+1} = \eta_{\gamma_0+2}$, then η_0 is a common fixed point of Φ and Υ. So, assume that there is no such consecutive identical elements in $\{\eta_\gamma\}$ and that $\eta_0 \neq \eta_1$.

$$\varrho(\eta_{2\gamma+1},\eta_{2\gamma+2}) = \varrho(\Phi\eta_{2\gamma},\Upsilon\eta_{2\gamma+1})$$

$$\leq \wp\varrho(\eta_{2\gamma},\eta_{2\gamma+1}) + \frac{\varrho(\eta_{2\gamma+1},\Upsilon\eta_{2\gamma+1})[1+\varrho(\eta_{2\gamma},\Phi\eta_{2\gamma})]}{1+\varrho(\eta_{2\gamma},\eta_{2\gamma+1})}$$

$$+\tau[\varrho(\eta_{2\gamma},\Phi\eta_{2\gamma}) + \varrho(\eta_{2\gamma+1},\Upsilon\eta_{2\gamma+1})]$$

$$= \wp\varrho(\eta_{2\gamma},\eta_{2\gamma+1})$$

$$\varrho(\eta_{2\gamma+1},\eta_{2\gamma+2}) \leq \frac{(\wp+\gamma)}{[1-\varphi-\tau]}\varrho(\eta_{2\gamma},\eta_{2\gamma+1})$$

Let $\mathfrak{J} = \frac{(\wp+\tau)}{[1-\varphi-\tau]} < 1$

$$\varrho(\eta_{2\gamma+1},\eta_{2\gamma+2}) \leq \varpi\varrho(\eta_{2\gamma},\eta_{2\gamma+1}) \tag{2}$$

Also, we have,

$$\begin{aligned}
\varrho(\eta_{2\gamma},\eta_{2\gamma+1}) &= \varrho(\Phi\eta_{2\gamma-1},\Upsilon\eta_{2\gamma}) \\
&\leq \wp\varrho(\eta_{2i-1},\eta_{2i}) \\
&\quad +\varphi\frac{\varrho(\eta_{2\gamma},\Upsilon\eta_{2\gamma})[1+\varrho(\eta_{2\gamma-1},\Phi\eta_{2\gamma-1})]}{1+\varrho(\eta_{2\gamma-1},\eta_{2\gamma})} + \tau[\varrho(\eta_{2\gamma-1},\Phi\eta_{2\gamma-1}) + \varrho(\eta_{2\gamma},\Upsilon\eta_{2\gamma})] \\
&= \wp\,\varrho(\eta_{2\gamma-1},\eta_{2\gamma}) \\
&\quad +\varphi\frac{\varrho(\eta_{2\gamma},\eta_{2\gamma+1})[1+\varrho(\eta_{2\gamma-1},\eta_{2\gamma})]}{1+\varrho(\eta_{2\gamma-1},\eta_{2\gamma})} + \tau[\varrho(\eta_{2\gamma-1},\eta_{2\gamma}) + \varrho(\eta_{2\gamma},\eta_{2\gamma+1})]
\end{aligned}$$

$$\varrho(\eta_{2\gamma},\eta_{2\gamma+1}) \leq \frac{(\wp+\tau)}{[1-\varphi-\tau]}\varrho(\eta_{2\gamma-1},\eta_{2\gamma})$$

$$\varrho(\eta_{2\gamma},\eta_{2\gamma+1}) \leq \varpi\varrho(\eta_{2\gamma-1},\eta_{2\gamma}) \tag{3}$$

Where, $\varphi = \frac{(\wp+\tau)}{[1-\varphi-\tau}$

From (2) and (3)

$$\varrho(\eta_\gamma,\eta_{\gamma+1}) \leq \varpi\varrho(\eta_{\gamma-1},\eta_\gamma)\ \forall\ \gamma \in \mathbb{N}$$

By **lemma 2.3**, we conclude that $\{\eta_\gamma\}$ is a Cauchy sequence. Since, (Ξ, d) is complete there exists $\eta \in \Xi$ such that $\eta_\gamma \to \eta$ as $\gamma \to \infty$

$$\Longrightarrow \lim_{\gamma\to\infty}\eta_\gamma = \eta$$

Further,

Since, Υ is continuous,

$$\Upsilon\eta = \Upsilon\left(\lim_{\gamma\to\infty}\eta_\gamma\right) = \lim_{\gamma\to\infty}\Upsilon\eta_\gamma = \lim_{\gamma\to\infty}\Upsilon\eta_{2\gamma+1} = \lim_{\gamma\to\infty}\eta_{2\gamma+2} = \eta$$

Also, since, Φ is continuous,

$$\Phi\eta = \Phi\left(\lim_{\gamma\to\infty}\eta_\gamma\right) = \lim_{\gamma\to\infty}\Phi\eta_\gamma = \lim_{\gamma\to\infty}\Phi\eta_{2\gamma} = \lim_{\gamma\to\infty}\eta_{2\gamma+1} = \eta$$

Therefore Υ and Φ have common fixed point.

To prove uniqueness of common fixed point, suppose that κ is a common fixed point of Φ and Υ, then

$$d(\eta,\kappa) = d(\Phi\eta, \Upsilon\kappa)$$

$$\leq \wp\, \varrho(\eta,\kappa) + \varphi \frac{\varrho(\kappa,\Upsilon\kappa)[1+\varrho(\eta,\Phi\eta)]}{1+\varrho(\eta,\kappa)} + \tau[\varrho(\eta,\Phi\eta) + \varrho(\kappa,\Upsilon\kappa)]$$
$$= \wp\, \varrho(\eta,\kappa) + \varphi \frac{\varrho(\kappa,\kappa)[1+\varrho(\eta,\eta)]}{1+\varrho(\eta,\kappa)} + \tau[\varrho(\eta,\eta) + \varrho(\kappa,\kappa)$$

$$\varrho(\eta,\kappa) \leq \wp\, \varrho(\eta,\kappa)$$

Since, $\wp < 1$

$\Longrightarrow \varrho(\eta,\kappa) = 0$

$\Longrightarrow \eta = \kappa$

Hence, η is the unique common fixed point of Φ and Υ.

Remark 3.2.2

If $\Phi = \Upsilon$, theorem 3.2.1 reduces to theorem 3.1.2.

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