On Paired Double Domination Number of Grid Graphs

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Abstract
A Paired-dominating set of a graph G is a dominating set of vertices whose induced sub graph has a perfect matching and a double dominating set is a dominating set that dominates every vertex of G at least twice. A Paired-double dominating set of a graph G is a double dominating set of vertices whose induced sub graph has a perfect matching. In this paper, we characterize the Paired double domination number of the Cartesian product of path graphs. We determine the Paired double domination numbers of $P_n \square P_m$, where n is even and $m \geq 6$.

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Introduction

Let G = (V, E) be a graph with vertex set V and edge set E. We begin with some terminology. For a vertex v of a graph G, the open neighborhood of a vertex v ϵ V is $N(V) = \{u \epsilon V/ \ uv \epsilon E\}$ and the closed neighborhood is $N[v] = N(v) \cap \{v\}$.

A subset $S \subseteq$ V is a dominating set of G, if for every vertex v ϵ V, |N[v] ∩ S|≥ 1. The domination number γ(G) is the minimum cardinality of a dominating set of G. A subset S of V is a double dominating set of G if for every vertex v ϵ V,|N[v]∩S|≥ 2, that is v is in S and has at least one neighbour in S or v is in V-S and has at least two neighbours in S [3]. A set S is called paired-dominating set if it dominates V and <S> contains at least one perfect matching. A paired-dominating set S with matching M is a dominating set S={$v_1,v_2,\ldots,v_{2t-1},v_{2t}$} with independent edge set M = {$e_1,e_2,\ldots,e_t$} where each edge e_i joins two elements of S, that is M is a perfect matching in <S>. If $v_jv_k = e_i$ ϵ M, we say that v_j and v_k are paired in S [6]. The double domination number γ_{dd} (G) is the minimum cardinality of a double dominating set of G, and the paired-domination number γ_{pr} (G) is the minimum cardinality of a paired-dominating set of G.A paired (respectively, double) dominating set of minimum cardinality is called a γ_{pr} (G) set (respectively γ_{dd}(G) set). The Cartesian Product of graphs G and H, denoted by G □ H, is a graph such that $V(\mathrm{G} \,\square\, \mathrm{H}) = V(G) \times V(H)$ and $E(\mathrm{G} \,\square\, \mathrm{H}) = \{(u_1, v_1)(u_2, v_2): u_1 = u_2 \ and \ v_1v_2 \epsilon \ E(H\ ) \ or \ v_1 = v_2 \ and \ \ u_1u_2 \epsilon \ E(G\ )\}$. Grid graph is the Cartesian product of two paths. A set S is called a paired-double dominating set if it is a double dominating set and the induced sub graph $< S >$ contains at least one perfect matching. The minimum cardinality taken over all paired-double dominating sets is called the paired-double domination number and is denoted by γ_{prdd}. Any paired-double dominating set with γ_{prdd}vertices is called a γ_{prdd} set of G.

In this paper, we characterize the Paired double domination number of grid graphs . We determine the Paired double domination numbers of $P_n \square P_m$, where n is even and $m \geq 6$. We take the set s_1, s_2, s_3 and s_4 are disjoint sets whose intersection is empty.

Theorem 1.1. [8] For any path P_n , $\gamma_{prdd}(P_n)$ = $\left\{\begin{array}{ll} 2 & if n = 2 \\ does\ not\ exist & if\ n = 3 \\ 2\left\lfloor\frac{n}{3}\right\rfloor + 2 & other\ wise \end{array}\right\}$

Theorem 1.2. [8] For any cycle C_n $\gamma_{prdd}(C_n)$= $2\left\lceil\frac{n}{3}\right\rceil$

Theorem 1.3. [8] For any path P_n , n≠3 , $\gamma_{dd}(P_n) \leq \gamma_{prdd}(P_n)$

Theorem 1.4. [8] For any cycle C_n , $\gamma_{dd}(C_n) \leq \gamma_{prdd}(C_n)$

Theorem 1.5. [8] If n = 3k+2 where k ϵ N, then$\gamma_{prdd}(P_n) = \gamma_{prdd}(C_n)$.

2. Main Results

Theorem 2.1

Let P_4 be a path of length 4 and P_m be a path of length m and $m \geq 4$. Then Paired double domination number of product of these two paths $\gamma_{prdd}(P_4 \square P_m) = 2m + 2$.

Proof:

Case (i) $m \equiv 0\ (mod\ 2)$

Let $S_1 = \left\{(a_1, 1 + 2b_1), (a_2, 1 + 2b_1)/a_1 = 1{,}3\ , a_2 = 2{,}4\ and\ b_1 = 0{,}1{,}2 \ldots \left(\frac{m}{2}\right) - 1\right\}$, $S_2 = \{(2, m), (3, m)\}$. Then $S = S_1 \cup S_2$ is a double dominating set and $\langle S \rangle$has a perfect matching S is a Paired double dominating set of $P_4 \square P_m$ with minimum cardinality $|S| = |S_1| + |S_2| = 2m + 2$.

Case (i) $m \equiv 1\ (mod\ 2)$

Let $S_1 = \left\{(a_1, 1 + 2b_1), (a_2, 1 + 2b_1)/a_1 = 1{,}3\ , a_2 = 2{,}4\ and\ b_1 = 0{,}1{,}2 \ldots \left\lfloor\frac{m}{2}\right\rfloor\right\}$, $S_2 = \{(2, m), (3, m)\}$. Then S is a double dominating set and $\langle S \rangle$has a perfect matching S is a Paired double dominating set of $P_4 \square P_m$ with minimum cardinality $|S| = 2(m + 1) = 2m + 2$.

Theorem 2.2

Let P_6 be a path of length 6 and P_m be a path of length m and $m \geq 4$. Then Paired double domination number of product of these two paths $\gamma_{prdd}(P_6 \square P_m) = \left\{\begin{array}{ll} 3m + 2 & if\ m\ is\ even \\ 3m + 1 & if\ m\ is\ odd \end{array}\right\}$.

Proof:

We consider the following two cases.

Case (i) m is even.

Let $S_1 = \left\{(4a_1 + 1, 1 + 2b_1), (4a_1 + 2, 1 + 2b_1)/a_1 = 0,1 \; and \; b_1 = 0,1,2 \ldots \left(\frac{m}{2}\right) - 1\right\}$, $S_2 = \left\{(3, 2b_2), (4, 2b_2)/ \; b_2 = 1,2 \ldots \left(\frac{m}{2}\right) - 1\right\}$ and $S_3 = \{(2,m),(3,m),(5,m),(6,m)\}$. Then $S = S_1 \cup S_2 \cup S_3$ is a double dominating set and $\langle S \rangle$ has a perfect matching S is a Paired double dominating set of $P_6 \square P_m$ with cardinality $|S| = |S_1| + |S_2| + |S_3| = 2m + (m-2) + 4 = 3m + 2$.

Case (ii) m is odd.

Let$S_1 = \left\{(4a_1 + 1, 1 + 2b_1), (4a_1 + 2, 1 + 2b_1)/a_1 = 0,1 \; and \; b_1 = 0,1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor\right\}$, $S_2 = \left\{(4a_2 + 3, 2b_2), (4a_2 + 4, 2b_2)/a_2 = 0,1,2 \ldots \left\lfloor\frac{n}{4}\right\rfloor - 1 \; and \; b_2 = 1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor\right\}$.

Then $S = S_1 \cup S_2$ is a double dominating set and $\langle S \rangle$has a perfect matching S is a Paired double dominating set of $P_6 \square P_m$ with cardinality $|S| = |S_1| + |S_2| = 2(m+1) + m - 1 = 3m + 1$.

Theorem 2.3

Let P_8 be a path of length 8 and P_m be a path of length m and $m \geq 4$. Then Paired double domination number of product of these two paths $\gamma_{prdd}(P_8 \square P_m) = \begin{cases} 4(m+1) & if \; m \; is \; even \\ 4m + 2 & if \; m \; is \; odd \end{cases}$.

Proof:

We consider the following two cases.

Case (i) m is even.

Let $S_1 = \left\{(4a_1 + 1, 1 + 2b_1), (4a_1 + 2, 1 + 2b_1)/a_1 = 0,1 \; and \; b_1 = 0,1,2 \ldots \left(\frac{m}{2}\right) - 1\right\}$, $S_2 = \left\{(3, 2b_2), (4, 2b_2)/ \; b_2 = 1,2 \ldots \left(\frac{m}{2}\right) - 1\right\}$, $S_3 = \{(2,m),(3,m),(5,m),(6,m),(8,m),(9,m)\} \; and \; S_4 = \left\{(7, 1 + 2b_1), (8, 1 + 2b_1)/ \; b_1 = 0,1,2 \ldots \left(\frac{m}{2}\right) - 1\right\}$. Then $S = S_1 \cup S_2 \cup S_3 \cup S_4$ is a double dominating set and $\langle S \rangle$ has a perfect matching S is a Paired double dominating set of $P_8 \square P_m$ with cardinality $|S| = |S_1| + |S_2| + |S_3| + |S_4| = 2m + (m-2) + 6 + m = 4m + 4 = 4(m+1)$. .

Case (ii) m is odd.

Let $S_1 = \left\{(4a_1 + 1, 1 + 2b_1), \quad (4a_1 + 2, 1 + 2b_1)/a_1 = 0,1 \; and \; b_1 = 0,1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor\right\}$, $S_2 = \left\{(4a_2 + 3, 2b_2), (4a_2 + 4, 2b_2)/a_2 = 0 \; and \; b_2 = 1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor\right\} \; and \; S_4 = \left\{(7, 1 + 2b_1), (8, 1 + 2b_1)/ \; b_1 = 0,1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor\right\}$. Then $S = S_1 \cup S_2 \cup S_3$ is a double dominating set and $\langle S \rangle$has a perfect matching S is a Paired double dominating set of $P_8 \square P_m$ with cardinality $|S| = |S_1| + |S_2| + |S_3| = 2(m+1) + m - 1 + m + 1 = 4m + 2$.

Theorem 2.4

Let P_{10} be a path of length 10 and P_m be a path of length m and $m \geq 4$. Then Paired double domination number of product of these two paths $\gamma_{prdd}(P_{10} \square P_m) = \begin{cases} 5m + 2 & if \; m \; is \; even \\ 5m + 1 & if \; m \; is \; odd \end{cases}$.

Proof:

We consider the following two cases.

Case (i) m is even.

Let $S_1 = \left\{(4a_1 + 1, 1 + 2b_1),\ (4a_1 + 2, 1 + 2b_1)/a_1 = 0,1,2\ and\ b_1 = 0,1,2 \ldots \left(\frac{m}{2}\right) - 1\right\}$, $S_2 = \left\{(4a_2 + 3, 2b_2), (4a_2 + 4, 2b_2)/\ a_2 = 0,1,2\ and\ b_2 = 1,2 \ldots \left(\frac{m}{2}\right) - 1\right\}$ and $S_3 = \{(2,m), (3,m), (5,m), (6,m), (8,m), (9,m)\}$. Then $S = S_1 \cup S_2 \cup S_3$ is a double dominating set and $\langle S \rangle$ has a perfect matching S is a Paired double dominating set of $P_{10}\square P_m$ with cardinality $|S| = |S_1| + |S_2| + |S_3| = 3m + 2m - 4 + 6 = 5m + 2$.

Case (ii) m is odd.

Let $S_1 = \left\{(4a_1 + 1, 1 + 2b_1),\ (4a_1 + 2, 1 + 2b_1)/a_1 = 0,1,2\ and\ b_1 = 0,1,2, .. \left\lfloor\frac{m}{2}\right\rfloor\right\}$, $S_2 = \left\{(4a_2 + 3, 2b_2), (4a_2 + 4, 2b_2)/a_2 = 0,1\ and\ b_2 = 1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor\right\}$. Then $S = S_1 \cup S_2$ is a double dominating set and $\langle S \rangle$has a perfect matching S is a Paired double dominating set of $P_6\square P_m$ with cardinality $|S| = |S_1| + |S_2| = 3(m + 1) + 2(m - 1) = 5m + 1$.

Theorem 2.5

Let P_n be a path of length n and $n \equiv 2\ (mod 12), n\ \geq 12$ and P_m be a path of length m and $m \geq 6$. Then Paired double domination number of product of these two paths

$$\gamma_{prdd}\,(P_n\square P_m) = \begin{Bmatrix} \frac{mn}{2} + 4 & if\ m\ is\ even \\ \frac{mn}{2} + 1 & if\ m\ is\ odd \end{Bmatrix}.$$

Proof:

We consider the following two cases.

Case (i) m is even.

Let $S_1 = \left\{(4a_1 + 1, 1 + 2b_1), (4a_1 + 2, 1 + 2b_1)/a_1 = 0,1,2 \ldots \left\lfloor\frac{n}{4}\right\rfloor\ and\ b_1 = 0,1,2 \ldots \left(\frac{m}{2}\right) - 1\right\}$, $S_2 = \left\{(4a_2 + 3, 2b_2), (4a_2 + 4, 2b_2)/a_2 = 0,1,2 \ldots \left\lfloor\frac{n}{4}\right\rfloor - 1\ and\ b_2 = 1,2 \ldots \left(\frac{m}{2}\right) - 1\right\}$ and $S_3 = \left\{(2,m), (3,m), (5,m), (6,m), (8,m), (9,m), (11,m), (12,m),\ (n - 1,m)\ , (n,m), (4p - 1,m), (4p,m)/p = 4,5, \ldots \left\lfloor\frac{n}{4}\right\rfloor\right\}$. Then $S = S_1 \cup S_2 \cup S_3$ is a double dominating set and $\langle S \rangle$has a perfect matching S is a Paired double dominating set of $P_n\square P_m$ with cardinality $|S| = |S_1| + |S_2| + |S_3| = m\left\lceil\frac{n}{4}\right\rceil + 2\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{4}\right\rfloor + 2\left\lceil\frac{n}{4}\right\rceil + 2 = 2k_1\left\lceil\frac{12k+2}{4}\right\rceil + 2\left\lfloor\frac{2k_1-1}{2}\right\rfloor\left\lfloor\frac{12k+2}{4}\right\rfloor + 2\left\lceil\frac{12k+2}{4}\right\rceil + 2 = 2k_1(3k + 1) + 2(k_1 - 1)(3k) + 2(3k + 1) + 2 = 12k_1k + 2k_1 + 4 = 12\left(\frac{m}{2}\right)\left(\frac{n-2}{12}\right) + 2\left(\frac{m}{2}\right) + 4 = \frac{mn}{2} + 4.$

Case (ii) m is odd.

Let $S_1 = \left\{(4a_1 + 1, 1 + 2b_1), (4a_1 + 2, 1 + 2b_1)/a_1 = 0,1,2 \ldots \left\lfloor\frac{n}{4}\right\rfloor\ and\ b_1 = 0,1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor\right\}$, $S_2 = \left\{(4a_2 + 3, 2b_2), (4a_2 + 4, 2b_2)/a_2 = 0,1,2 \ldots \left\lfloor\frac{n}{4}\right\rfloor - 1\ and\ b_2 = 1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor\right\}$.

Then $S = S_1 \cup S_2$ is a double dominating set and $\langle S \rangle$has a perfect matching S is a Paired double dominating set of $P_n \square P_m$ with cardinality $|S| = |S_1| + |S_2| = 2\left\lceil\frac{m}{2}\right\rceil\left\lceil\frac{n}{4}\right\rceil + 2\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{4}\right\rfloor = 2\left\lceil\frac{2k_1+1}{2}\right\rceil\left\lceil\frac{12k+2}{4}\right\rceil + 2\left\lfloor\frac{2k_1-1}{2}\right\rfloor\left\lfloor\frac{12k+2}{4}\right\rfloor = 2(k_1+1)(3k+1) + 2(k_1)(3k) = 12k_1k + 6k + 2k_1 + 2 = 12\left(\frac{m-1}{2}\right)\left(\frac{n-2}{12}\right) + 2\left(\frac{m-1}{2}\right) + 6\left(\frac{n-2}{12}\right) + 2 = \frac{mn}{2} + 1.$

Theorem 2.6

Let P_n be a path of length n and $n \equiv 6\ (mod 12), n \geq 12$ and P_m be a path of length m and $m \geq 6.$ Then Paired double domination number of product of these two paths

$$\gamma_{prdd}(P_n \square P_m) = \begin{Bmatrix} \frac{mn}{2} + 4 & if\ m\ is\ even \\ \frac{mn}{2} + 1 & if\ m\ is\ odd \end{Bmatrix}.$$

Proof:

We consider the following two cases.

Case (i) m is even.

Let $S_1 = \left\{(4a_1+1, 1+2b_1), (4a_1+2, 1+2b_1)/a_1 = 0,1,2 \ldots \left\lfloor\frac{n}{4}\right\rfloor\ and\ b_1 = 0,1,2 \ldots \left(\frac{m}{2}\right) - 1\right\}$, $S_2 = \left\{(4a_2+3, 2b_2), (4a_2+4, 2b_2)/a_2 = 0,1,2 \ldots \left\lfloor\frac{n}{4}\right\rfloor - 1\ and\ b_1 = 1,2 \ldots \left(\frac{m}{2}\right) - 1\right\}$ and $S_3 = \left\{(2,m), (3,m), (5,m), (6,m), (8,m)(9,m), (11,m), (12,m), (n-1,m), (n,m), (4p-1,m), (4p,m)/p = 4,5, \ldots \left\lfloor\frac{n}{4}\right\rfloor\right\}.$

Then $S = S_1 \cup S_2 \cup S_3$ is a double dominating set and $\langle S \rangle$has a perfect matching S is a Paired double dominating set of $P_n \square P_m$ with cardinality $|S| = |S_1| + |S_2| + |S_3| = m\left\lceil\frac{n}{4}\right\rceil + 2\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{4}\right\rfloor + 2\left\lceil\frac{n}{4}\right\rceil + 2 = 2k_1\left\lceil\frac{12k+6}{4}\right\rceil + 2\left\lfloor\frac{2k_1-1}{2}\right\rfloor\left\lfloor\frac{12k+6}{4}\right\rfloor + 2\left\lceil\frac{12k+6}{4}\right\rceil + 2 = 2k_1(3k+2) + 2(k_1-1)(3k+1) + 2(3k+2) + 2 = 12k_1k + 6k_1 + 4 = 12\left(\frac{m}{2}\right)\left(\frac{n-6}{12}\right) + 6\left(\frac{m}{2}\right) + 4 = \frac{mn}{2} + 4.$

Case (ii) m is odd.

Let $S_1 = \left\{(4a_1+1, 1+2b_1), (4a_1+2, 1+2b_1)/a_1 = 0,1,2 \ldots \left\lfloor\frac{n}{4}\right\rfloor\ and\ b_1 = 0,1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor\right\}$, $S_2 = \left\{(4a_2+3, 2b_2), (4a_2+4, 2b_2)/a_2 = 0,1,2 \ldots \left\lfloor\frac{n}{4}\right\rfloor - 1\ and\ b_1 = 1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor\right\}$

Then $S = S_1 \cup S_2$ is a double dominating set and $\langle S \rangle$has a perfect matching S is a Paired double dominating set of $P_n \square P_m$ with cardinality $|S| = |S_1| + |S_2| = 2\left\lceil\frac{m}{2}\right\rceil\left\lceil\frac{n}{4}\right\rceil + 2\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{4}\right\rfloor = 2\left\lceil\frac{2k_1+1}{2}\right\rceil\left\lceil\frac{12k+6}{4}\right\rceil + 2\left\lfloor\frac{2k_1-1}{2}\right\rfloor\left\lfloor\frac{12k+6}{4}\right\rfloor = 2(k_1+1)(3k+2) + 2(k_1)(3k+1) = 12k_1k + 6k + 6k_1 + 4 = 12\left(\frac{m-1}{2}\right)\left(\frac{n-6}{12}\right) + 6\left(\frac{m-1}{2}\right) + 6\left(\frac{n-6}{12}\right) + 4 = \frac{mn}{2} + 1.$

Theorem 2.7

Let P_n be a path of length n and $n \equiv 10\ (mod 12), n\ \geq 12$ and P_m be a path of length m and $m \geq 6$. Then Paired double domination number of product of these two paths $\gamma_{prdd}\,(P_n\square P_m) = \begin{Bmatrix} \frac{mn}{2} + 4 & if\ m\ is\ even \\ \frac{mn}{2} + 1 & if\ m\ is\ odd \end{Bmatrix}$.

Proof:

We consider the following two cases.

Case (i) m is even.

Let $S_1 = \left\{(4a_1 + 1,1 + 2b_1), (4a_1 + 2,1 + 2b_1)/a_1 = 0,1,2 \dots \left\lfloor\frac{n}{4}\right\rfloor and\ b_1 = 0,1,2 \dots \left(\frac{m}{2}\right) - 1\right\}$, $S_2 = \left\{(4a_2 + 3, 2b_2), (4a_2 + 4,2b_2)/a_2 = 0,1,2 \dots \left\lfloor\frac{n}{4}\right\rfloor - 1\ and\ b_2 = 1,2 \dots \left(\frac{m}{2}\right) - \ 1\right\}$ and $S_3 = \left\{(2, m), (3, m), (5, m), (6, m), (8, m)(9, m), (11, m), (12, m), (n - 1, m)\ , (n, m), (4p - 1, m), (4p, m)/p = 4,5, \dots \left\lfloor\frac{n}{4}\right\rfloor\right\}$.

Then $S = S_1 \cup S_2 \cup S_3$ is a double dominating set and $\langle S\rangle$has a perfect matching S is a Paired double dominating set of $P_n\square P_m$ with cardinality $|S| = |S_1| + |S_2| + |S_3| = m\left\lceil\frac{n}{4}\right\rceil + 2\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{4}\right\rfloor + 2\left\lceil\frac{n}{4}\right\rceil + 2 = 2k_1\left\lceil\frac{12k+10}{4}\right\rceil + 2\left\lfloor\frac{2k_1-1}{2}\right\rfloor\left\lfloor\frac{12k+10}{4}\right\rfloor + 2\left\lceil\frac{12k+10}{4}\right\rceil + 2 = 2k_1(3k + 3) + 2(k_1 - 1)(3k + 2) + 2(3k + 3) + 2 = 12k_1k + 10k_1 + 4 = 12\left(\frac{m}{2}\right)\left(\frac{n-10}{12}\right) + 10\left(\frac{m}{2}\right) + 4 = \frac{mn}{2} + 4.$

Case (ii) m is odd.

Let $S_1 = \left\{(4a_1 + 1,1 + 2b_1), (4a_1 + 2,1 + 2b_1)/a_1 = 0,1,2 \dots \left\lfloor\frac{n}{4}\right\rfloor and\ b_1 = 0,1,2 \dots \left\lfloor\frac{m}{2}\right\rfloor\right\}$, $S_2 = \left\{(4a_2 + 3, 2b_2), (4a_2 + 4,2b_2)/a_2 = 0,1,2 \dots \left\lfloor\frac{n}{4}\right\rfloor - 1\ and\ b_1 = 1,2 \dots \left\lfloor\frac{m}{2}\right\rfloor\right\}$

Then $S = S_1 \cup S_2$ is a double dominating set and $\langle S\rangle$has a perfect matching S is a Paired double dominating set of $P_n\square P_m$ with cardinality $|S| = |S_1| + |S_2| = 2\left\lceil\frac{m}{2}\right\rceil\left\lceil\frac{n}{4}\right\rceil + 2\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{4}\right\rfloor = 2\left\lceil\frac{2k_1+1}{2}\right\rceil\left\lceil\frac{12k+10}{4}\right\rceil + 2\left\lfloor\frac{2k_1-1}{2}\right\rfloor\left\lfloor\frac{12k+10}{4}\right\rfloor = 2(k_1 + 1)(3k + 3) + 2(k_1)(3k + 2) = 12k_1k + 6k + 10k_1 + 6 = 12\left(\frac{m-1}{2}\right)\left(\frac{n-10}{12}\right) + 10\left(\frac{m-1}{2}\right) + 6\left(\frac{n-10}{12}\right) + 6 = \frac{mn}{2} + 1.$

Theorem 2.8

Let P_n be a path of length n and $n \equiv 0\ (mod 12), n\ \geq 12$ and P_m be a path of length m and $m \geq 6$. Then Paired double domination number of product of these two paths $\gamma_{prdd}\,(P_n\square P_m) = \begin{Bmatrix} \frac{mn}{2} + 4 & if\ m\ is\ even \\ \frac{mn}{2} + 2 & if\ m\ is\ odd \end{Bmatrix}$.

Proof:

We consider the following two cases.

Case (i) m is even.

Let $S_1 = \left\{(4a_1+1,1+2b_1),(4a_1+2,1+2b_1)/a_1 = 0,1,2\ldots\left(\frac{n}{4}\right)-1 and\ b_1 = 0,1,2\ldots\left(\frac{m}{2}\right)-1\right\}$, $S_2 = \left\{(4a_2+3,2b_2),(4a_2+4,2b_2)/a_2 = 0,1,2\ldots\left(\frac{n}{4}\right)-2\ and\ b_1 = 1,2\ldots\left(\frac{m}{2}\right)-1\right\}$, $S_3 = \left\{(2,m),(3,m),(5,m),(6,m),(8,m),(9,m),(11,m),(12,m),(n-1,m)),(n,m),(4p-1,m),(4p,m)/p = 4,5,\ldots\left(\frac{n}{4}\right)\right\}$ and $S_4 = \left\{(n-1,1+2b_3),(n,1+2b_3)/b_3 = 0,1,2\ldots\left(\frac{m}{2}\right)-1\right\}$.

Then $S = S_1 \cup S_2 \cup S_3 \cup S_4$ is a double dominating set and $\langle S\rangle$has a perfect matching S is a Paired double dominating set of $P_n \square P_m$ with cardinality $|S| = |S_1|+|S_2|+|S_3|+|S_4| = m\left(\frac{n}{4}\right)+2\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n-1}{4}\right\rfloor+2\left(\frac{n}{4}\right)+2+m = 2k_1\left(\frac{12k}{4}\right)+2\left\lfloor\frac{2k_1-1}{2}\right\rfloor\left\lfloor\frac{12k-1}{4}\right\rfloor+2\left(\frac{12k}{4}\right)+2+m = 2k_1(3k)+2(k_1-1)(3k-1)+6k+2+2k_1 = 12k_1k+2k_1-2k_1+4 = 12\left(\frac{m}{2}\right)\left(\frac{n}{12}\right)+4 = \frac{mn}{2}+4$.

Case (ii) m is odd.

Let $S_1 = \left\{(4a_1+1,1+2b_1),(4a_1+2,1+2b_1)/a_1 = 0,1,2\ldots\left(\frac{n}{4}\right)-1\ and\ b_1 = 0,1,2\ldots\left\lfloor\frac{m}{2}\right\rfloor\right\}$, $S_2 = \left\{(4a_2+3,2b_2),(4a_2+4,2b_2)/a_2 = 0,1,2\ldots\left(\frac{n}{4}\right)-2\ and\ b_2 = 1,2\ldots\left\lfloor\frac{m}{2}\right\rfloor\right\}$, and $S_3 = \left\{(n-1,1+2b_3),(n,1+2b_3)/b_3 = 0,1,2\ldots\left\lfloor\frac{m}{2}\right\rfloor\right\}$.

Then $S = S_1 \cup S_2 \cup S_3$ is a double dominating set and $\langle S\rangle$has a perfect matching S is a Paired double dominating set of $P_n \square P_m$ with cardinality $|S| = |S_1|+|S_2|+|S_3| = 2\left\lceil\frac{m}{2}\right\rceil\left(\frac{n}{4}\right)+2\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n-1}{4}\right\rfloor+2\left\lfloor\frac{m}{2}\right\rfloor+2 = 2\left\lceil\frac{2k_1+1}{2}\right\rceil\left(\frac{12k}{4}\right)+2\left\lfloor\frac{2k_1+1-1}{2}\right\rfloor\left\lfloor\frac{12k-1}{4}\right\rfloor+2\left\lfloor\frac{m}{2}\right\rfloor+2 = 2(k_1+1)(3k)+2(k_1)(3k-1)+2k_1+2 = 12k_1k+6k+2 = 12\left(\frac{m-1}{2}\right)\left(\frac{n}{12}\right)+6\left(\frac{n}{12}\right)+2 = \frac{mn}{2}+2$.

Theorem 2.9

Let P_n be a path of length n and $n \equiv 4\ (mod 12), n \geq 12$ and P_m be a path of length m and $m \geq 6$. Then Paired double domination number of product of these two paths

$$\gamma_{prdd}\,(P_n \square P_m) = \begin{Bmatrix} \frac{mn}{2}+4 & if\ m\ is\ even \\ \frac{mn}{2}+2 & if\ m\ is\ odd \end{Bmatrix}.$$

Proof:

We consider the following two cases.

Case (i) m is even.

Let $S_1 = \left\{(4a_1+1,1+2b_1),(4a_1+2,1+2b_1)/a_1 = 0,1,2\ldots\left(\frac{n}{4}\right)-1 and\ b_1 = 0,1,2\ldots\left(\frac{m}{2}\right)-1\right\}$, $S_2 = \left\{(4a_2+3,2b_2),(4a_2+4,2b_2)/a_2 = 0,1,2\ldots\left(\frac{n}{4}\right)-2\ and\ b_1 = 1,2\ldots\left(\frac{m}{2}\right)-1\right\}$, $S_3 = \left\{(2,m),(3,m),(5,m),(6,m),(8,m),(9,m),(11,m),(12,m),(n-1,m)),(n,m),(4p-1,m),(4p,m)/p = 4,5,\ldots\left(\frac{n}{4}\right)\right\}$ and $S_4 = \left\{(n-1,1+2b_3),(n,1+2b_3)/b_3 = 0,1,2\ldots\left(\frac{m}{2}\right)-1\right\}$.

Then $S = S_1 \cup S_2 \cup S_3 \cup S_4$ is a double dominating set and $\langle S \rangle$has a perfect matching S is a Paired double dominating set of $P_n \square P_m$ with cardinality $|S| = |S_1| + |S_2| + |S_3| + |S_4| = m\left(\frac{n}{4}\right) + 2\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n-1}{4}\right\rfloor + 2\left(\frac{n}{4}\right) + 2 + m = 2k_1\left(\frac{12k+4}{4}\right) + 2\left\lfloor\frac{2k_1-1}{2}\right\rfloor\left\lfloor\frac{12k+4-1}{4}\right\rfloor + 2\left(\frac{12k+4}{4}\right) + 2 + m = 2k_1(3k+1) + 2(k_1-1)(3k) + 2(3k+1) + 2 + 2k_1 = 12k_1k + 4k_1 + 4 = 12\left(\frac{m}{2}\right)\left(\frac{n-4}{12}\right) + 4\left(\frac{m}{2}\right) + 4 = \frac{mn}{2} + 4$.

Case (ii) m is odd.

Let $S_1 = \left\{(4a_1+1, 1+2b_1), (4a_1+2, 1+2b_1) / a_1 = 0,1,2 \ldots \left(\frac{n}{4}\right) - 1 \ and \ b_1 = 0,1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor\right\}$, $S_2 = \left\{(4a_2+3, 2b_2), (4a_2+4, 2b_2) / a_2 = 0,1,2 \ldots \left(\frac{n}{4}\right) - 2 \ and \ b_2 = 1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor\right\}$, and $S_3 = \left\{(n-1, 1+2b_3), (n, 1+2b_3) / b_3 = 0,1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor\right\}$.

Then $S = S_1 \cup S_2 \cup S_3$ is a double dominating set and $\langle S \rangle$has a perfect matching S is a Paired double dominating set of $P_n \square P_m$ with cardinality $|S| = |S_1| + |S_2| + |S_3| = 2\left\lceil\frac{m}{2}\right\rceil\left(\frac{n}{4}\right) + 2\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n-1}{4}\right\rfloor + 2\left\lceil\frac{m}{2}\right\rceil + 2 = 2\left\lceil\frac{2k_1+1}{2}\right\rceil\left(\frac{12k+4}{4}\right) + 2\left\lfloor\frac{2k_1+1-1}{2}\right\rfloor\left\lfloor\frac{12k+4-1}{4}\right\rfloor + 2\left\lfloor\frac{2k_1+1}{2}\right\rfloor + 2 = 2(k_1+1)(3k+1) + 2(k_1)(3k) + 2k_1 + 2 = 12k_1k + 6k + 4k_1 + 4 = 12\left(\frac{m-1}{2}\right)\left(\frac{n-4}{12}\right) + 6\left(\frac{n-4}{12}\right) + 4\left(\frac{m-1}{2}\right) + 4 = \frac{mn}{2} + 2$.

Theorem 2.10

Let P_n be a path of length n and $n \equiv 8\ (mod 12), n \geq 12$ and P_m be a path of length m and $m \geq 6$. Then Paired double domination number of product of these two paths

$$\gamma_{prdd}(P_n \square P_m) = \begin{Bmatrix} \frac{mn}{2} + 4 & if\ m\ is\ even \\ \frac{mn}{2} + 2 & if\ m\ is\ odd \end{Bmatrix}.$$

Proof:

We consider the following two cases.

Case (i) m is even.

Let $S_1 = \left\{(4a_1+1, 1+2b_1), (4a_1+2, 1+2b_1) / a_1 = 0,1,2 \ldots \left(\frac{n}{4}\right) - 1 and \ b_1 = 0,1,2 \ldots \left(\frac{m}{2}\right) - 1\right\}$, $S_2 = \left\{(4a_2+3, 2b_2), (4a_2+4, 2b_2) / a_2 = 0,1,2 \ldots \left(\frac{n}{4}\right) - 2 \ and \ b_1 = 1,2 \ldots \left(\frac{m}{2}\right) - 1\right\}$, $S_3 = \left\{(2,m), (3,m), (5,m), (6,m), (8,m), (9,m), (11,m), (12,m), (n-1,m)), (n,m), (4p-1,m), (4p,m) / p = 4,5, \ldots \left(\frac{n}{4}\right)\right\}$ and $S_4 = \left\{(n-1, 1+2b_3), (n, 1+2b_3) / b_3 = 0,1,2 \ldots \left(\frac{m}{2}\right) - 1\right\}$.

Then $S = S_1 \cup S_2 \cup S_3 \cup S_4$ is a double dominating set and $\langle S \rangle$has a perfect matching S is a Paired double dominating set of $P_n \square P_m$ with cardinality $|S| = |S_1| + |S_2| + |S_3| + |S_4| = m\left(\frac{n}{4}\right) + 2\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n-1}{4}\right\rfloor + 2\left(\frac{n}{4}\right) + 2 + m = 2k_1\left(\frac{12k+8}{4}\right) + 2\left\lfloor\frac{2k_1-1}{2}\right\rfloor\left\lfloor\frac{12k+8-1}{4}\right\rfloor + 2\left(\frac{12k+8}{4}\right) + 2 + m = 2k_1(3k+2) + 2(k_1-1)(3k+1) + 2(3k+2) + 2 + 2k_1 = 12k_1k + 8k_1 + 4 = 12\left(\frac{m}{2}\right)\left(\frac{n-8}{12}\right) + 8\left(\frac{m}{2}\right) + 4 = \frac{mn}{2} + 4$.

Case (ii) m is odd.

Let $S_1 = \left\{(4a_1 + 1,1 + 2b_1), (4a_1 + 2,1 + 2b_1)/a_1 = 0,1,2 \ldots \left(\frac{n}{4}\right) - 1 \text{ } and \text{ } b_1 = 0,1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor\right\}$, $S_2 = \left\{(4a_2 + 3, 2b_2), (4a_2 + 4,2b_2)/a_2 = 0,1,2 \ldots \left(\frac{n}{4}\right) - 2 \text{ } and \text{ } b_2 = 1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor\right\}$, and $S_3 = \left\{(n - 1,1 + 2b_3), (n, 1 + 2b_3)/b_3 = 0,1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor\right\}$.

Then $S = S_1 \cup S_2 \cup S_3$ is a double dominating set and $\langle S\rangle$has a perfect matching S is a Paired double dominating set of $P_n \square P_m$ with cardinality $|S| = |S_1| + |S_2| + |S_3| = 2\left\lceil\frac{m}{2}\right\rceil\left(\frac{n}{4}\right) + 2\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n-1}{4}\right\rfloor + 2\left\lfloor\frac{m}{2}\right\rfloor + 2 = 2\left\lceil\frac{2k_1+1}{2}\right\rceil\left(\frac{12k+8}{4}\right) + 2\left\lfloor\frac{2k_1+1-1}{2}\right\rfloor\left\lfloor\frac{12k+8-1}{4}\right\rfloor + 2\left\lfloor\frac{2k_1+1}{2}\right\rfloor + 2 = 2(k_1 + 1)(3k + 2) + 2(k_1)(3k + 1) + 2k_1 + 2 = 12k_1k + 8k + 6k_1 + 2 + 4 = 12\left(\frac{m-1}{2}\right)\left(\frac{n-8}{12}\right) + 6\left(\frac{n-8}{12}\right) + 8\left(\frac{m-1}{2}\right) + +2 + 4 = \frac{mn}{2} + 2.$

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