Remarks On $\star$-$g\omega$-Closed Sets

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Article Info
Page Number: 1830-1838
Publication Issue:
Vol. 70 No. 2 (2021)

Abstract
In this paper, we introduce a new generalized class of sets called $\star$-$g\omega$-closed sets and further study is carried out using the established generalized classes. Apart from this, some new generalizations of ω-open sets and $\star$-g-closed sets are investigated.

Article History
***Article Received**: 05 September 2021*
***Revised**: 09 October 2021*
***Accepted**: 22 November 2021*
***Publication**: 26 December 2021*

1. Introduction

In 1982, the notions of ω-open sets, ω-closed sets and ω-closed mappings were introduced and investigated by Hdeib [2]. In 2005, Al-Zoubi [1] introduced and studied the concepts of $g\omega$-closed sets, $g\omega$-open sets and $g\omega$-continuity in topological spaces. In 2009, Noiri et al [8] introduced some weaker forms of ω-open sets and obtained some decompositions of continuity. In [7], Mandal and Mukherjee introduced and studied the notion of $\star$-g-closed sets in ideal topological spaces. In this paper, we introduce a new generalized class of sets called $\star$-$g\omega$-closed sets and further study is carried out using the established generalized classes. Apart from this, some new generalizations of ω-open sets and $\star$-g-closed sets are investigated.

2. Preliminaries

Throughout this paper, $\mathbb{R}$ (resp. $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}-\mathbb{Q}$) denotes the set of all real numbers (resp. the set of all natural numbers, the set of all rational numbers, the set of all irrational numbers).

2010 Mathematics Subject Classification: 54A05, 54A10
Key words and phrases. ω-open sets, $\star$-$g\omega$-closed sets, $\star$-g-closed sets and $\star$-g-open sets

By a space (X, τ), we always mean a topological space (X, τ) with no separation axioms assumed. For a subset H of a space (X, τ), cl(H) and int(H) denote the closure of H and the interior of H respectively.

Definition 2.1 [10] A subset H of a space (X,τ) is called regular open if $H = int(cl(H))$.

The complement of a regular open set is regular closed.

Definition 2.2 [2] Let H be a subset of a space (X, τ), a point p in X is called a condensation point of H if for each open set U containing p, U ∩ H is uncountable.

Definition 2.3 [2] A subset H of a space (X, τ) is called ω-closed if it contains all its condensation points.

The complement of an ω-closed set is called ω-open. The family of all ω-closed sets is denoted by $\omega C(X)$. The family of all ω-open sets is denoted by $\omega O(X)$. It is well known that a subset W of a space (X, τ) is ω-open [2] if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open sets, denoted by τ_ω, is a topology on X, which is finer than τ. The interior and closure operator in (X, τ_ω) are denoted by int_ω and cl_ω respectively.

Remark 2.4 [2] In a space (X, τ), every closed set is ω-closed but not conversely.

Lemma 2.5 [9] Let H be a subset of a space (X, τ). Then

1. H is ω-closed in X if and only if H = cl_ω(H).
2. cl_ω(X\H) = X\int_ω(H).
3. cl_ω(H) is ω-closed in H.
4. x ∈ cl_ω(H) if and only if H ∩ G $\neq \phi$ for each ω-open set G containing x.
5. cl_ω(H) ⊆ cl(H).
6. int(H) ⊆ int_ω(H).

Definition 2.6 [1] A subset H of a space (X, τ) is called generalized ω-closed (briefly, gω-closed) if $cl_\omega(H) \subseteq G$ whenever $H \subseteq G$ and $G \in \tau$.

The family of all $g\omega$-closed sets is denoted by $G\omega C(X)$.

An ideal [6] $\mathcal{J}$ on a nonempty set X is a nonempty collection of subsets of X which satisfies

1. $A \in \mathcal{J}$ and $B \subseteq A$ implies $B \in \mathcal{J}$ and
2. $A \in \mathcal{J}$ and $B \in \mathcal{J}$ implies $A \cup B \in \mathcal{J}$.

If $A \subseteq X$, then $\mathcal{J} = \mathcal{P}(A)$ is an ideal on X and is called the principal ideal [13] on X generated by A. Given a topological space (X, τ) with an ideal $\mathcal{J}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^\star: \mathcal{P}(X) \to \mathcal{P}(X)$, called a local function [6] of A with respect to τ and $\mathcal{J}$ is defined as follows: for $A \subseteq X$, $A^\star(\mathcal{J}, \tau) = \{x \in X | U \cap A \notin \mathcal{J}$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau | x \in U\}$. We will make use of the basic facts about the local function [[4], Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $cl^\star(.)$ for a topology $\tau^\star(\mathcal{J}, \tau)$, called the $\star$-topology or extension topology, finer than τ is defined by $cl^\star(A) = A \cup A^\star(\mathcal{J}, \tau)$ [12]. $int^\star$ is the interior operator in $(X, \tau^\star)$. When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(\mathcal{J}, \tau)$ and $\tau^\star$ for $\tau^\star(\mathcal{J}, \tau)$. Clearly, A is $\tau^\star$-closed or $\star$-closed, if $A^\star \subseteq A$ [4]. If $\mathcal{J}$ is an ideal on X, then the space $(X, \tau, \mathcal{J})$ is called an ideal space or an ideal topological space. The complement of an $\star$-closed set is called $\star$-open.

Definition 2.7 [7] A subset H of an ideal space $(X, \tau, \mathcal{J})$ is called $\star$-g-closed if $cl(H) \subseteq G$ whenever $H \subseteq G$ and G is $\star$-open.

The complement of an $\star$-g-closed set is called $\star$-g-open.

Remark 2.8 [9] In an ideal space $(X, \tau, \mathcal{J})$, if $\mathcal{J} = \{\phi\}$ (resp. $\mathcal{P}(X)$) than $cl^\star(A) = cl(A)$ (resp. A) for any subset A of X.

3 $\star$-$g\omega$-closed sets

Definition 3.1 A subset H of an ideal space $(X,\tau,\mathcal{J})$ is called $\star$-$g\omega$-closed if $cl_\omega(H) \subseteq$ G whenever $H \subseteq G$ and G is $\star$-open.

The family of all $\star$-$g\omega$-closed sets is denoted by $\star\, G\omega C(X)$.

Example 3.2 If τ is any topology on a countable set X, then each subset is ω-open and (X,τ_ω) is a discrete space. Consequently $\tau_\omega = \mathcal{P}(X) = \omega C(X)$ where $\mathcal{P}(X)$ is the power set of X. It is clear that if $(X,\tau,\mathcal{J})$ is a countable ideal space, then $\mathcal{P}(X) = \omega C(X) = G\omega C(X) = \star\, G\omega C(X)$.

Proposition 3.3 Every $\star$-g-closed set is $\star$-$g\omega$-closed.

The proof follows immediately from the definitions. However the converse is not true as the following Example.

Example 3.4 In $\mathbb{R}$ with usual topology τ_u and $\mathcal{J} = \{\phi\}$, $H = (0,1) \cap \mathbb{Q}$ is ω-closed, being a countable set having no condensation points. Thus if G is any $\star$-open set such that $H \subseteq G$, then $cl_\omega(H) = H \subseteq G$. Hence H is $\star$-$g\omega$-closed. But $H \subseteq G = (0,1)$ where G is $\star$-open and $cl(H) = [0,1] \nsubseteq G$. Thus H is not $\star$-g-closed.

Example 3.5 In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{R} - \mathbb{Q}\}$ and $\mathcal{J} = \{\phi\}$, the set $H = \mathbb{R} - \mathbb{Q}$ is neither $\star$-$g\omega$-closed nor gω-closed.

Solution: H is open and $H \subseteq H$. But $cl_\omega(H) = \mathbb{R} \nsubseteq H$. Hence H is not $g\omega$-closed. Also H is $\star$-open, being open and $H \subseteq H$. But $cl_\omega(H) = \mathbb{R} \nsubseteq H$. Hence H is not $\star$-$g\omega$-closed also.

Proposition 3.6 Every ω-closed set is $\star$-$g\omega$-closed.

The proof follows immediately from the definitions. However the converse is not true as the following Example shows.

Example 3.7 In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \{1\}\}$ and $\mathcal{J} = \{\phi\}$, the set $H = \mathbb{R} - \mathbb{Q}$ is $\star$-$g\omega$-closed, since the only $\star$-open set containing H is $\mathbb{R}$. But H is not ω-closed, for 3 is a condensation point of H and $3 \notin H$.

Solution: If G is any $\star$-open set containing H, then $H \subseteq G \Rightarrow cl(H) \subseteq cl(G)$. It implies $\mathbb{R} - \{1\} = cl(H) \subseteq cl(G)$. If $G \neq \phi$, then $cl(G) \neq \phi$. It means that $G = \mathbb{R}$ or $\{1\}$. If $G = \{1\}$, this is a contradiction. Hence $G = \mathbb{R}$.

Proposition 3.8 Every ω-closed set is $g\omega$-closed.

The proof follows immediately from the definitions. However the converse is not true as the following Example shows.

Example 3.9 In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{R} - \{1\}\}$ and $\mathcal{J} = \{\phi\}$, the set $H = (\mathbb{R} - \mathbb{Q}) \cup \{1\}$ is $g\omega$-closed since $\mathbb{R}$ is the only open set containing H. But H is not ω-closed for 2 is a condensation point of H and $2 \notin H$.

Proposition 3.10 Every $\star$-$g\omega$-closed set is $g\omega$-closed.

The proof follows immediately from the definitions. However the converse is not true as the following Example shows.

Example 3.11 Let X be an uncountable set and let A, B, C and D be subsets of X such that each of them is uncountable and the family $\{A, B, C, D\}$ is a partition of X. We defined the topology $\tau = \{\phi, X, \{D\}, \{A, C\}, \{A, C, D\}\}$ where X is identified with $\{A, B, C, D\}$ and $\mathcal{J} = \{\phi, \{A\}, \{D\}, \{A, D\}\}$. Take $H = \{B, C\}$. Since the only open superset of H is X, H is $g\omega$-closed. On the other hand, $H \subseteq H$ and H is $\star$-open. But $cl_\omega(H) = X$. Hence H is not $\star$-$g\omega$-

closed.

Example 3.12 In $\mathbb{R}$ with usual topology τ_u and $\mathcal{J} = \{\phi\}$, $(-1,1)$ and (1, 3) are open sets and $G = (-1,1) \cup (1,3)$ is open and hence $\star$-open such that $K = [0,1) \cup (1,2] \subseteq G$, but $cl_\omega(K)$ = $[0,2] \nsubseteq G$ for $1 \notin G$. This shows that K is not $\star$-$g\omega$-closed.

Proposition 3.13 If $\mathcal{A} = \{A_\alpha : \alpha \in I\}$ is a locally finite collection of $\star$-$g\omega$-closed sets of an ideal space $(X, \tau, \mathcal{J})$, then $A = \cup_{\alpha \in I} A_\alpha$ is $\star$-$g\omega$-closed (in particular, a finite union of $\star$-$g\omega$-closed sets is $\star$-$g\omega$-closed).

Proof. Let H be an $\star$-open subset of $(X, \tau, \mathcal{J})$ such that $A \subseteq H$. Since $A_\alpha \in \star\, G\omega C(X)$ and $A_\alpha \subseteq H$ for each $\alpha \in I$, $cl_\omega(A_\alpha) \subseteq H$. As τ_ω is a topology on X finer than τ, $\mathcal{A}$ is locally finite in (X, τ_ω). Therefore $cl_\omega(A) = cl_\omega(\cup_{\alpha \in I} A_\alpha) = \cup_{\alpha \in I} cl_\omega(A_\alpha) \subseteq H$. Thus, A is $\star$-$g\omega$-closed in $(X, \tau, \mathcal{J})$.

Remark 3.14 The following Example shows that a countable union of $\star$-$g\omega$-closed sets need not be $\star$-$g\omega$-closed.

Example 3.15 Consider $X = \mathbb{R}$ with the usual topology τ_u and $\mathcal{J} = \{\phi\}$. For each $n \in \mathbb{N}$ (the set of all natural numbers), put $A_n = [\frac{1}{n}, 1]$ and $A = \cup_{n \in \mathbb{N}}^{\infty} A_n = (0,1]$. Then A is a countable union of $\star$-$g\omega$-closed sets but A is not $\star$-$g\omega$-closed since $H = (0,2)$ is $\star$-open, $A \subseteq H$ and $cl_\omega(A) = [0,1] \nsubseteq H$.

Theorem 3.16 Let $(X, \tau, \mathcal{J})$ be an ideal space. Then every subset of X is $\star$-$g\omega$-closed if and only if every $\star$-open set is ω-closed.

Proof. Suppose every subset of X is $\star$-$g\omega$-closed. If U is $\star$-open, U is $\star$-$g\omega$-closed implies $cl_\omega(U) \subseteq$ U. Hence $cl_\omega(U)$ = U implies U is ω-closed. Conversely, suppose that every $\star$-open set is ω-closed. If H $\subseteq$ X and W is an $\star$-open set such that H $\subseteq$ W, then cl_ω(H) $\subseteq cl_\omega$(W) = W and so H is $\star$-$g\omega$-closed.

Proposition 3.17 Let H be an $\star$-$g\omega$-closed subset of an ideal space $(X, \tau, \mathcal{J})$ and $G \subseteq X$. Then the following hold.

(a) $cl_\omega(H) - H$ contains no non-empty $\star$-closed set.

(b) If $H \subseteq G \subseteq cl_\omega(H)$, then $G \in \star\, G\omega C(X)$.

Proof. (a) If $cl_\omega(H) - H$ contains an $\star$-closed set C then $H \subseteq X - C$ and $X - C$ is $\star$-open in $(X, \tau, \mathcal{J})$. Then cl_ω(H) $\subseteq X - C$ or equivalently, $C \subseteq X - cl_\omega$(H). Therefore, $C \subseteq (X - cl_\omega(H)) \cap (cl_\omega(H) - H)$ = $(X - cl_\omega(H)) \cap cl_\omega$(H) $\cap$ $(X - H)$ = ϕ.

(b) Let U be $\star$-open and $G \subseteq$ U. Then $H \subseteq G \subseteq$ U. Since $H \in \star\, G\omega C(X)$, $cl_\omega(H) \subseteq$ U. But $cl_\omega(G) \subseteq cl_\omega(cl_\omega(H)) = cl_\omega(H) \subseteq$ U and the result follows.

Lemma 3.18 If H is an $\star$-open and $\star$-$g\omega$-closed subset of an ideal space (X, τ, $\mathcal{J}$), then H is ω-closed in X.

The proof is obvious.

Theorem 3.19 Let H be an $\star$-$g\omega$-closed subset of (X, τ, $\mathcal{J}$). Then H = $cl_\omega(int_\omega(H))$ if and only if $cl_\omega(int_\omega(H)) - H$ is $\star$-closed.

Proof. If H = $cl_\omega(int_\omega(H))$, then $cl_\omega(int_\omega(H)) - H$ = ϕ and hence $cl_\omega(int_\omega(H)) - H$ is $\star$-closed. Conversely, let $cl_\omega(int_\omega(H)) - H$ be $\star$-closed, since $cl_\omega(H) - H$ contains the $\star$-closed set $cl_\omega(int_\omega(H)) - H$. By Proposition 3.17 (a), $cl_\omega(int_\omega(H)) - H$ = ϕ and hence H = $cl_\omega(int_\omega(H))$.

Theorem 3.20 Every $\star$-$g\omega$-closed set and $\star$-g-closed set are $g\omega$-closed.

The proof is obvious.

Corollary 3.21 Let $(X,\tau,\mathcal{J})$ be an ideal space and H be an $\star$-$g\omega$-closed set. The following are equivalent.

(a) H is an ω-closed set.

(b) $cl_\omega(H) - H$ is an $\star$-closed set.

Proof. (a) $\Rightarrow$ (b) If H is ω-closed, then $cl_\omega(H) - H = \phi$ and so $cl_\omega(H) - H$ is $\star$-closed.

(b) $\Rightarrow$ (a) since $cl_\omega(H) - H$ is $\star$-closed and H is $\star$-$g\omega$-closed by assumption, using Proposition 3.17 (a) we have $cl_\omega(H) - H = \phi$ and so H is ω-closed.

Theorem 3.22 Let $(X,\tau,\mathcal{J})$ be an ideal space and $H \subseteq X$. If H is $\star$-$g\omega$-closed then H = $G - N$ where G is ω-closed and N contains no nonempty $\star$-closed set.

Proof. If H is $\star$-$g\omega$-closed, then by Proposition 3.17 (a), $cl_\omega(H) - H$ = N (say) contains no nonempty $\star$-closed set. If G = $cl_\omega(H)$, then G is ω-closed such that $G - N$ = $cl_\omega(H) - (cl_\omega(H) - H)$ = H.

Theorem 3.23 If K and L are $\star$-$g\omega$-closed sets, then K ∪ L is $\star$-$g\omega$-closed.

Proof. Suppose that K ∪ L $\subseteq U$, where U is $\star$-open. Then $K \subseteq U$ and $L \subseteq U$. Since K and L are $\star$-$g\omega$-closed sets, $cl_\omega(K) \subseteq U$ and $cl_\omega(L) \subseteq U$. Now $cl_\omega(K \cup L)$ = $cl_\omega(K) \cup cl_\omega(L) \subseteq U$. It shows $K \cup L$ is $\star$-$g\omega$-closed.

Theorem 3.24 Let K and L be subsets of an ideal space $(X,\tau,\mathcal{J})$ such that $K \subseteq L \subseteq cl_\omega(K)$ and K is an $\star$-$g\omega$-closed set. Then L is also $\star$-$g\omega$-closed.

Proof. Let $L \subseteq U$, where U is $\star$-open. Then $K \subseteq L \subseteq U$ and K is $\star$-$g\omega$-closed implies that $cl_\omega(K) \subseteq U \Rightarrow cl_\omega(L) \subseteq cl_\omega(cl_\omega(K)) = cl_\omega(K) \subseteq U$. Therefore L is $\star$-$g\omega$-closed.

Definition 3.25 An ideal space (X, τ, $\mathcal{J}$) is called $\star$-$g\omega$-$T_{\frac{1}{2}}$ if every $\star$-$g\omega$-closed set in $(X,\tau,\mathcal{J})$ is ω-closed in $(X,\tau,\mathcal{J})$.

Example 3.26 Every countable ideal space is $\star$-$g\omega$- $T_{\frac{1}{2}}$.

Example 3.27 In Example 3.7, the set $H = \mathbb{R} - \mathbb{Q}$ is an $\star$-$g\omega$-closed set which is not ω-closed. Therefore $(\mathbb{R},\tau,\mathcal{J})$ is not $\star$-$g\omega$-$T_{\frac{1}{2}}$.

Theorem 3.28 For an ideal space (X, τ, $\mathcal{J}$), the following are equivalent. [(a)]

(a) X is an $\star$-$g\omega$-$T_{\frac{1}{2}}$ space.

(b) Every singleton is either $\star$-closed or ω-open.

Proof. (a) $\Rightarrow$ (b): Suppose $\{x\}$ is not an $\star$-closed subset for some $x \in X$. Then $X - \{x\}$ is not $\star$-open and hence X is the only $\star$-open set containing $X - \{x\}$. Therefore $X - \{x\}$ is $\star$-$g\omega$-closed. Since $(X,\tau,\mathcal{J})$ is an $\star$-$g\omega$-$T_{\frac{1}{2}}$ space, $X - \{x\}$ is ω-closed and thus $\{x\}$ is ω-open.

(b) $\Rightarrow$ (a): Let H be an $\star$-$g\omega$-closed subset of $(X,\tau,\mathcal{J})$ and $x \in cl_\omega(H)$. We show that $x \in H$.

Case (i) If $\{x\}$ is $\star$-closed and $x \notin H$, then $x \in (cl_\omega(H) - H)$. Thus $cl_\omega(H) - H$ contains a nonempty $\star$-closed set {x}, a contradiction to Proposition 3.17 (a). So $x \in H$.

Case (ii) If $\{x\}$ is ω-open, since $x \in cl_\omega(H)$, then for every ω-open set U containing x, we have $U \cap H \neq \phi$. But $\{x\}$ is ω-open, then $\{x\} \cap H \neq \phi$. Hence $x \in H$.

So in both cases, we have $x \in H$. Therefore H is ω-closed.

4 $\star$-$g\omega$-open sets

Definition 4.1 A subset H of an ideal space $(X, \tau, \mathcal{I})$ is called $\star$-$g\omega$-open if its complement $X - H$ is $\star$-$g\omega$-closed in $(X, \tau, \mathcal{I})$.

Theorem 4.2 Let $(X, \tau, \mathcal{I})$ be an ideal space and $H \subseteq X$. Then H is $\star$-$g\omega$-open if and only if $G \subseteq int_{\omega}(H)$ whenever G is $\star$-closed and $G \subseteq H$.

Proof. Suppose H is $\star$-$g\omega$-open. If G is $\star$-closed and $G \subseteq H$, then $X - H \subseteq X - G$ and so $cl_{\omega}(X - H) \subseteq X - G$. Therefore $G \subseteq int_{\omega}(H)$. Conversely, Suppose the condition holds. Let U be an $\star$-open set such that $X - H \subseteq$ U. Then $X - U \subseteq H$ and so by assumption $X - U \subseteq int_{\omega}(H)$ which implies that $cl_{\omega}(X - H) \subseteq$ U. Therefore $X - H$ is $\star$-$g\omega$-closed and so H is $\star$-$g\omega$-open.

Theorem 4.3 Let K and L be subsets of an ideal space (X, τ, $\mathcal{I}$) such that $int_{\omega}(K) \subseteq L \subseteq K$. If K is $\star$-$g\omega$-open, then L is also $\star$-$g\omega$-open.

The proof is similar to the proof of Theorem 3.24.

Theorem 4.4 If H is an $\star$-$g\omega$-closed subset of $(X, \tau, \mathcal{I})$, then $cl_{\omega}(H) - H$ is $\star$-$g\omega$-open.

Proof. Let H be an $\star$-$g\omega$-closed subset of $(X, \tau, \mathcal{I})$ and let U be an $\star$-closed subset such that $U \subseteq cl_{\omega}(H) - H$. By Proposition 3.17 (a), $U = \phi$ and thus $U \subseteq int_{\omega}(cl_{\omega}(H) - H)$. By Theorem 4.2 $cl_{\omega}(H) - H$ is $\star$-$g\omega$-open.

Lemma 4.5 If every $\star$-open set is closed, then all subsets of $(X, \tau, \mathcal{I})$ are $\star$-$g\omega$-closed (and hence all are $\star$-$g\omega$-open).

Proof. Let A be any subset of X such that $A \subseteq U$ and U is $\star$-open, then $cl_{\omega}(A) \subseteq cl_{\omega}(U) \subseteq cl(U) = U$. Therefore A is $\star$-$g\omega$-closed.

5 $\star$-$g\omega$-normal spaces

Definition 5.1 An ideal space $(X, \tau, \mathcal{I})$ is said to be $\star$-normal if for every pair of disjoint $\star$-closed sets P and Q, there exist disjoint open sets K and L such that $P \subseteq K$ and Q $\subseteq$ L.

Definition 5.2 An ideal space $(X, \tau, \mathcal{I})$ is said to be $\star$-$g\omega$-normal space if for every pair of disjoint $\star$-closed sets P and Q, there exist disjoint $\star$-$g\omega$-open sets K and L such that $P \subseteq K$ and Q $\subseteq$ L.

Remark 5.3 Every $\star$-normal space is $\star$-$g\omega$-normal.

Proof. Since every open set is ω-open and every ω-open set is $\star$-$g\omega$-open, every $\star$-normal space is $\star$-$g\omega$-normal.

The following Example shows that a $\star$-$g\omega$-normal space is not necessarily a $\star$-normal space.

Example 5.4 Let $X = \{a, b, c\}$ and τ = $\{\phi$, X, $\{b\}$, $\{a, b\}$, $\{b, c\}\}$ and $\mathcal{I} = \{\phi, \{b\}\}$. Then $\omega O(X) = \mathcal{P}(X)$. Here every open set is $\star$-closed and every $\star$-open set is ω-closed and so Theorem 3.16, every subset of X is $\star$-$g\omega$-closed and hence every subset of X is $\star$-$g\omega$-open. This implies $(X, \tau, \mathcal{I})$ is $\star$-$g\omega$-normal. $\{a\}$ and $\{c\}$ are disjoint $\star$-closed subsets of X which are not separated by disjoint open sets and so $(X, \tau, \mathcal{I})$ is not $\star$-normal.

Theorem 5.5 Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the following are equivalent. [(a)]

(a) X is $\star$-$g\omega$-normal

(b) For every pair of disjoint $\star$-closed sets P and Q, there exist disjoint $\star$-$g\omega$-open sets K and L such that $P \subseteq K$ and $Q \subseteq L$.

(c) For every $\star$-closed set P and an $\star$-open set M containing P, there exists an $\star$-$g\omega$-open set N such that $P \subseteq N \subseteq cl_\omega(N) \subseteq M$.

Proof. (a) ⇒ (b): The proof follows from the definition of $\star$-$g\omega$-normal space.

(b) ⇒ (c): Let P be an $\star$-closed set and M be an $\star$-open set containing P. Since P and $X - M$ are disjoint $\star$-closed sets, there exist disjoint $\star$-$g\omega$-open sets N and W such that $P \subseteq N$ and $X - M \subseteq W$. Again N ∩ W = ϕ implies that N ∩ $int_\omega(W) = \phi$ and so $cl_\omega(N) \subseteq X - int_\omega(W)$. Since $X - M$ is $\star$-closed and W is $\star$-$g\omega$-open, $X - M \subseteq W$ implies that $X - M \subseteq int_\omega(W)$ and so $X - int_\omega(W) \subseteq M$. Thus we have $P \subseteq N \subseteq cl_\omega(N) \subseteq X - int_\omega(W) \subseteq M$ which proves (c).

(c) ⇒ (a): Let P and Q be two disjoint $\star$-closed subsets of X. By hypothesis, there exists an $\star$-$g\omega$-open set K such that $P \subseteq K \subseteq cl_\omega(K) \subseteq X - Q$. If $L = X - cl_\omega(K)$, then K and L are the required disjoint $\star$-$g\omega$-open sets containing P and Q respectively. So $(X, \tau, \mathcal{I})$ is $\star$-$g\omega$-normal.

Theorem 5.6 Let $(X, \tau, \mathcal{I})$ be an $\star$-$g\omega$-normal space. If F is $\star$-closed and A is an $\star$-g-closed set such that $A \cap F = \phi$, then there exist disjoint $\star$-$g\omega$-open sets K and L such that $A \subseteq K$ and $F \subseteq L$.

Proof. Since $A \cap F = \phi$, $A \subseteq X - F$ where $X - F$ is $\star$-open. Therefore, by hypothesis, $cl(A) \subseteq X - F$. Since $cl(A) \cap F = \phi$, $cl(A)$ is $\star$-closed. Since X is $\star$-$g\omega$-normal, there exist disjoint $\star$-$g\omega$-open sets K and L such that $A \subseteq cl(A) \subseteq K$ and $F \subseteq L$.

Theorem 5.7 Let $(X, \tau, \mathcal{I})$ be a space which is $\star$-$g\omega$-normal. Then the following hold.

(a) For every $\star$-closed set P and every $\star$-g-open set Q containing P, there exists an $\star$-$g\omega$-open set U such that $P \subseteq int_\omega(U) \subseteq U \subseteq Q$.

(b) For every $\star$-g-closed set P and every $\star$-open set Q containing P, there exists $\star$-$g\omega$-closed set U such that $P \subseteq U \subseteq cl_\omega(U) \subseteq Q$.

Proof. (a) Let P be an $\star$-closed set and Q be an $\star$-g-open set containing P. Then $P \cap (X - Q) = \phi$, where P is $\star$-closed and $X - Q$ is $\star$-g-closed. By Theorem 5.6, there exist disjoint $\star$-$g\omega$-open sets U and V such that $P \subseteq U$ and $X - Q \subseteq V$. Since $U \cap V = \phi$, we have $U \subseteq X - V$. By Theorem 4.2, $P \subseteq int_\omega(U)$. Therefore $P \subseteq int_\omega(U) \subseteq U \subseteq X - V \subseteq Q$. This proves (a).

(c) Let P be an $\star$-g-closed set and Q be an $\star$-open set containing P. Then $X - Q$ is an $\star$-closed set contained in the $\star$-g-open set $X - P$. By (a), there exists an $\star$-$g\omega$-open set V such that $X - Q \subseteq int_\omega(V) \subseteq V \subseteq X - P$. Therefore $P \subseteq X - V \subseteq cl_\omega(X - V) \subseteq Q$. If $U = X - V$, then $P \subseteq U \subseteq cl_\omega(U) \subseteq Q$ and so U is the required $\star$-$g\omega$-closed set.

(d)

6. Applications to Digital Topology

The digital line or so called the Khalimsky line is the set of integers $\mathbb{Z}$, equipped with the topology κ generated by $\{\{2m-1, 2m, 2m+1\}/m \in \mathbb{Z}\}$. The concept of the digital line $(\mathbb{Z}, \kappa)$ is initiated by Khalimsky. In $(\mathbb{Z}, \kappa)$, each singleton $\{2n\}$ is closed and each singleton $\{2n+1\}$ is open where $n \in \mathbb{Z}$. If $U(x)$ is the smallest open set containing x, then $U(2m) = \{2m-1, 2m, 2m+1\}$ and $U(2m+1) = \{2m+1\}$ where $m \in \mathbb{Z}$. For a subset H of a digital line $(\mathbb{Z}, \kappa)$, $cl_\kappa(H)$ and $int_\kappa(H)$ denote the closure of H and the interior of H respectively.

Lemma 6.1 [3, 5, 11] Let H be a subset of $\mathbb{Z}$. Then [(i)]

i. H is open if and only if for every $x \in H$, the following holds: (x is odd) or (x is even with $x-1, x+1 \in H$).
ii. H is closed if and only if for every $x \in H$, the following holds: (x is even) or (x is odd with $x-1, x+1 \in H$).

Let $\mathfrak{N}(x)$ denote the smallest neighbourhood of x in $(\mathbb{Z}, \kappa)$. Then

$$\mathfrak{N}(x) = \begin{cases} \{x\} & \text{if } x \text{ is odd} \\ \{x-1, x, x+1\} & \text{if } x \text{ is even.} \end{cases}$$

Let $\mathfrak{C}(x)$ denote the smallest closed set containing x in $(\mathbb{Z}, \kappa)$. Then

$$\mathfrak{C}(x) = \begin{cases} \{x\} & \text{if } x \text{ is even} \\ \{x-1, x, x+1\} & \text{if } x \text{ is odd.} \end{cases}$$

The topology satisfying the properties in Lemma 6.1 is exactly the Khalimsky line topology on $\mathbb{Z}$. Based on the concepts of "open" and "closed" in Lemma 6.1, we obtain the topology on $\mathbb{Z}$, which is referred to as the Khalimsky line topology on $\mathbb{Z}$.

Remark 6.2 [11] Let $A = \{2n-1, 2n, 2n+1\}$ and $B = \{2m-1, 2m, 2m+1\}$ where m and n are integers. Then

$$A \cap B = \begin{cases} A & if\, n = m \\ \{2n+1\} & if\, n = m-1 \\ \{2n-1\} & if\, n = m+1 \\ \phi & otherwise \end{cases}$$

Example 6.3 In $(\mathbb{Z}, \kappa)$, (1) the set $H = \{0,4,7\}$ is neither regular open nor open since (1) $int_\kappa(cl_\kappa(H)) = int_\kappa(\{0,4,6,7,8\}) = \{7\} \neq H$ and (2) $int_\kappa(H) = \{7\} \neq H$.

(2) the set $H = \{1,2,3,4\}$ is neither open nor closed for $int_\kappa(H) = \{1,2,3\} \neq H$ and $cl_\kappa(H) = \{0,1,2,3,4\} \neq H$.

Remark 6.4 In $(\mathbb{Z}, \kappa, \mathcal{J})$, each singleton even is $\star$-closed.

Example 6.5 In $(\mathbb{Z}, \kappa, \mathcal{J})$, the set $H = \{2,3,4\}$ is regular closed since $cl_\kappa(int_\kappa(\{2,3,4\})) = cl_\kappa(\{3\}) = \{2,3,4\} = H$. Hence H is closed and $\star$-closed. Also the set $G = \{6,7,8\}$ is regular closed since $cl_\kappa(int_\kappa(\{6,7,8\})) = cl_\kappa(\{7\}) = \{6,7,8\} = G$. Hence G is closed and $\star$-closed. Since $H \cap G = \phi$, H and G are disjoint $\star$-closed sets. We know that $P = \{1,2,3,4,5\}$ is the smallest open set containing H, $Q = \{5,6,7,8,9\}$ is the smallest open set containing G and $P \cap Q \neq \phi$. Hence $(\mathbb{Z}, \kappa, \mathcal{J})$ is not $\star$-normal.

Example 6.6 In $(\mathbb{Z}, \kappa)$, since $\mathbb{Z}$ is countable, every infinite subset of $\mathbb{Z}$ is countable. Therefore $\mathbb{Z}$ is ω-closed and every infinite subset of $\mathbb{Z}$ is ω-closed.

Example 6.7 Let $(\mathbb{Z}, \kappa)$ be the digital line. Then $(\mathbb{Z}, \kappa, \mathcal{J})$ is $\star$-$g\omega$-normal but it is not $\star$-normal.

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