BETWEEN ω-CLOSED SETS AND gω-CLOSED SETS

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Abstract

In this paper, further study is carried out using the established generalized classes of τ_ω. Apart from this, some new generalizations of ω-open sets are introduced and investigated. Also, using the generalized subsets of τ_ω , a new decomposition of ω-continuity in topological spaces is obtained

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Introduction

In 1961, Levine [8] obtained a decomposition of continuity. Tong [15] decomposed continuity into A-continuity and showed that his decomposition is independent of Levine's. In 1982, the notions of ω-open sets, ω-closed sets and ω-closed mappings were introduced and investigated by Hdeib [6]. In 2005, Al-Zoubi [2] introduced and studied the concepts of gω-closed sets and gω-open sets in topological spaces. In 2007, Al-Omari and Noorani [1] introduced and studied the concepts of rgω-closed sets and rgω-open sets in topological spaces. In 2009, Noiri et al [13] introduced some weaker forms of ω-open sets and obtained some decompositions of continuity. Quite Recently, Umamaheswari et al [14] studied gω- continuity and its topological properties. In this paper, further study is carried out using the established generalized classes of τ_ω. Apart from this, some new generalizations of ω-open sets are introduced and investigated. Also, using the generalized subsets of τ_ω , a new decomposition of ω-continuity in topological spaces is obtained.

PRELIMINARIES

Throughout this paper, R (resp. N, Q, R − Q) denotes the set of all real numbers (resp. the set of all natural numbers, the set of all rational numbers, the set of all irrational numbers). By a space (X, τ), we always mean a topological space (X, τ) with no separation axioms assumed.

For a subset H of a space (X, τ), cl(H) and int(H) denote the closure of H and the interior of H respectively.

Definition 2.1. A subset H of a space (X, τ) is called semi-open [10] if H ⊆ cl(int(H)).

The complement of a semi-open set is called semi-closed. The family of all semi-open sets is denoted by SO(X). scl(H) is the smallest semi-closed set containing H.

Definition 2.2. [2] Let H be a subset of a space (X, τ), a point p in X is called a condensation point of H if for each open set U containing p, $U \cap H$ is uncountable.
Definition 2.3. [2] A subset H of a space (X, τ) is called ω-closed if it contains all its condensation points.
The complement of an ω-closed set is called ω-open. The family of all ω-closed sets is denoted by $\omega C(X , \tau)$. The family of all ω-open sets is denoted by $\omega O(X)$. It is well known that a subset W of a space (X, τ) is ω-open [2] if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open sets, denoted by τ_ω, is a topology on X, which is finer than τ. The interior and closure operator in (X, τ_ω) are denoted by int_ω and cl_ω respectively.
Lemma 2.4. [14] Let H be a subset of a space (X, τ). Then
(a) H is ω-closed in X if and only if $H = cl_\omega(H)$.
(b) $cl_\omega(X\backslash H) = X\backslash int_\omega(H)$.
(c) $cl_\omega(H)$ is ω-closed in H.
(d) $x \in cl_\omega (H)$ if and only if $H \cap G \neq \varphi$ for each ω-open set G containing x.
(e) $cl_\omega (H) \subseteq cl(H)$.
(f) $int(H) \subseteq int_\omega (H)$.
Remark 2.5. [2] In a space (X, τ), every closed set is ω-closed but not conversely.
Definition 2.6. [2] A subset H of a space (X, τ) is called generalized ω-closed (briefly, gω-closed) if $cl_\omega (H) \subseteq G$ whenever $H \subseteq G$ and $G \in \tau$.
The family of all gω-closed sets is denoted by $G\omega C(X , \tau)$.
Definition 2.7. [3] A space (X, τ) is called $T_{1/2}$ -space if every singleton is open or closed.
Definition 2.8. [9] A subset H of a space (X, τ) is called generalized closed (briefly, g-closed) if $cl(H) \subseteq G$ whenever $H \subseteq G$ and $G \in \tau$.
The complement of a g-closed set is called g-open.
Theorem 2.9. [2] If (X, τ) is $T_{1/2}$ -space, then every gω-closed set in (X, τ) is ω-closed in (X, τ).
Lemma 2.10. [4] If H is an open set, then $cl(H \cap G) = cl(H \cap cl(G))$ and hence $H \cap cl(G) \subseteq cl(H \cap G)$ for any subset G.
Definition 2.11. [1] Two nonempty sets K and L of X are said to be separated if $cl(K) \cap L = \varphi = K \cap cl(L)$.
Definition 2.12. [5] A subset H of a space (X, τ) is called locally closed if $H = M \cap N$ where M is open and N is closed.
Definition 2.13. [2] A subset H of a space (X , τ) is called α-open if $H \subseteq int(cl(int(H)))$.
Definition 2.14. [11] A subset H of a topological space (X , τ) is called αgω-closed if $cl_\omega (H) \subseteq G$ whenever $H \subseteq G$ and G is α-open.
The family of all αgω-closed sets is denoted by $\alpha G\omega C(X , \tau)$.
Remark 2.15. [10] In a space (X , τ), every open set is semi-open but not conversely.
Theorem 2.16. [11] If (X , τ) is $T_{1/2}$ -space, then every αgω-closed set in (X , τ) is ω-closed in (X , τ).
Proposition 2.17. [11] In a space (X,τ), every αgω-closed set is gω-closed but not con- versely.

SEMI-gω -CLOSED SETS

Definition 3.1. A subset H of a space (X, τ) is said to be semi-gω-closed if $cl_\omega (H) \subseteq G$ whenever $H \subseteq G$ and G is semi-open.
The family of all semi-gω-closed sets is denoted by $SG\omega C(X, \tau)$.
Example 3.2. If τ is any topology on a countable set X, then each subset is ω-open and (X, τ_ω) is a discrete space. Consequently $\tau_\omega = P(X) = \omega C(X , \tau)$ where $P(X)$ is the power set of X. It is clear that if (X, τ) is a countable space, then $P(X) = \omega C(X , \tau) = \alpha G\omega C(X, \tau) = G\omega C(X, \tau) = SG\omega C(X, \tau)$.
Proposition 3.3. In space (X, τ), a semi gω-closed subset is gω-closed. But the converse need not be true.

Proof. Let A be semi-gω-closed subset and $A \subseteq G$ where G is open. Since G is open, it is semi-open and A is semi-gω-closed implies $cl_\omega(H) \subseteq G$. This proves that A is gω-closed.
Example 3.4. In R with the topology $\tau = \{\varphi, R, R - Q\}$, the subset $H = R - \{1\}$ is gω-closed since H is not open and R is the only open set containing H. Also $cl(int(H)) = cl(R - Q) = R$ and $H \subseteq cl(int(H))$. Thus H is semi-open with $H \subseteq H$. But $cl_\omega(H) = R \nsubseteq H$ which verifies that H is not semi-gω-closed.
Example 3.5. In R with the topology $\tau = \{\varphi, R, R - Q\}$, the set $H = R - Q$ is neither semi-gω-closed nor gω-closed.
Solution: H is open and $H \subseteq H$. But $cl_\omega(H) = \boldsymbol{R} \nsubseteq H$. Hence H is not g$\omega$-closed. By Proposition 3.3, H is not semi-gω-closed.
Proposition 3.6. Every ω-closed set is semi-gω-closed.
The proof follows immediately from the definitions. However the converse need not be true as the following Example shows.
Example 3.7. In R if $F = \{A/0 \in A$ and A is non-empty open subset of R in the usual topology $\tau_u\}$ then $\tau = \{\varphi, R, F\}$ is a topology. $H = R-\{0\}$ is not semi-open in τ since $int(H) = \varphi$ for H contains no non-empty open subsets. Obviously R is the only semi-open subset containing H. Thus H is semi-gω-closed. But H is not ω-closed for 0 is a condensation point of H and $0 \notin H$.
Example 3.8. Let X be an uncountable set and let A, B, C and D be subsets of X such that each of them is uncountable and the family {A, B,C, D} is a partition of X. We define the topology $\tau = \{\varphi, X, \{A\},\{B\},\{A, B\},\{A, B,C\}\}$. Then for $H = \{A, C, D\}$, $cl_\omega(H) = H$. Hence H is ω-closed, gω-closed and semi-gω-closed.
Proposition 3.9. If $A = \{A_\alpha : \alpha \in I\}$ is a locally finite collection of semi-gω-closed sets of a space (X, τ), then $A = \cup_{\alpha\in I} A_\alpha$ is semi-gω-closed (in particular, a finite union of semi-gω-closed sets is semi-gω-closed).
Proof. Let H be a semi-open subset of (X, τ) such that $A \subseteq H$. Since $A_\alpha \in SG\omega C(X, \tau)$ and $A_\alpha \subseteq H$ for each $\alpha \in I$, $cl_\omega(A_\alpha) \subseteq H$. As τ_ω is a topology on X finer than τ, A is locally finite in (X, τ_ω). Therefore $cl_\omega(A) = cl_\omega(\cup_{\alpha\in I} A_\alpha) = \cup_{\alpha\in I} cl_\omega(A_\alpha) \subseteq H$. Thus, A is semi-g$\omega$-closed in (X, τ).
Proposition 3.10. Every semi-gω-closed set is αgω-closed.
The proof follows immediately from the definitions.
The converse of Proposition 3.10 need not be true. It is left to reader to find the example.
Example 3.11. It is left to reader to find the example.
Theorem 3.12 If (X, τ) is $T_{1/2}$-space, then every semi-gω-closed set (X, τ) is ω-closed in (X, τ)
The proof follows immediately from Proposition 3.10 and Theorem 2.16.
Remark 3.13 The following Example shows that a countable union of semi-gω-closed sets need not be semi-gω-closed.
Example 3.14 Consider X = R with the usual topology τ_u. For each $n \in N$ (the set of all natural numbers), put $A_n = [\frac{1}{n}, 1]$ and $A = \cup_{n\in N}^{\infty} A_n = (0, 1]$. Then A is a countable union of semi-gω-closed sets but A is not semi-gω-closed since $A = (0,1]$ is semi open and $A \subseteq A$ but $cl_\omega(A) = [0, 1] \nsubseteq A$.
Remark 3.15 The following Example shows that a finite intersection of semi-gω-closed sets need not be semi-gω-closed.
Example 3.16 This example is left to the reader to find.
Proposition 3.17 The intersection of a semi-open set and an open set is semi-open.
Proof. Let H be an open set and A be a semi-open set. Then $H = int(H)$ and $A \subseteq cl(int(A))$. We have $H \cap A \subseteq H \cap cl(int(A))$. We have, by Lemma 2.10, $H \cap A \subseteq cl[H \cap int(A)] = cl[int(H) \cap int(A)] = cl(int(H \cap A))$. This shows that $H \cap A$ is semi-open.
Proposition 3.18 The union of a semi-open set and an open set is semi-open.
Proof. Let H be an open set and A be a semi-open set. Then $H = int(H)$ and $A \subseteq cl(int(A))$. We have $H \cup A \subseteq H \cup cl(int(A))$. Since $H \subseteq cl(H)$, $H \cup A \subseteq cl(H) \cup cl(int(A)) = cl[H \cup int(A)] = cl[int(H) \cup int(A)] \subseteq cl(int(H \cup A))$. This shows that $H \cup A$ is semi-open.

Proposition 3.19 If $H \in SG\omega C(X, \tau)$ and G is closed in (X, τ), then $H \cap G \in SG\omega C(X, \tau)$.

Proof. Let U be a semi-open set in (X, τ) such that $H \cap G \subseteq U$. Put $W = X - G$. Then $H \subseteq U \cup W$ and by Proposition 3.16, $U \cup W$ is semi-open. Since $H \in SG\omega C(X, \tau)$, $cl_\omega(H) \subseteq U \cup W$. Now, $cl_\omega(H \cap G) \subseteq cl_\omega(H) \cap cl_\omega(G) \subseteq cl_\omega(H) \cap cl(G) = cl_\omega(H) \cap G \subseteq (U \cup W) \cap G = (U \cap G) \cup \varphi = U \cap G \subseteq U$.

Theorem 3.20 Let (X, τ) be a space. Then every subset of X is semi-gω-closed if and only if every semi-open set is ω-closed.

Proof. Suppose every subset of X is semi-gω-closed. If U is semi-open, U is semi-gω-closed implies $cl_\omega(U) \subseteq U$. Hence $cl_\omega(U) = U$ implies U is ω-closed. Conversely, suppose that every semi-open set is ω-closed. If $H \subseteq X$ and W is a semi-open set such that $H \subseteq W$, then $cl_\omega(H) \subseteq cl_\omega(W) = W$ and so H is semi-gω-closed.

Proposition 3.21 Let H be a semi-gω-closed subset of a space (X, τ) and $G \subseteq X$. Then the following hold.

(a) $cl_\omega(H) - H$ contains no non-empty semi-closed set.

(b) If $H \subseteq G \subseteq cl_\omega(H)$, then $G \in SG\omega C(X, \tau)$.

Proof. (a) If $cl_\omega(H) - H$ contains a semi-closed set C then $H \subseteq X - C$ and $X - C$ is semi- open in (X, τ). Then $cl_\omega(H) \subseteq X - C$ or equivalently, $C \subseteq X - cl_\omega(H)$. Therefore, $C \subseteq (X - cl_\omega(H)) \cap (cl_\omega(H) - H) = (X - cl_\omega(H)) \cap cl_\omega(H) \cap (X - H) = \varphi$.

(b) Let U be semi-open and $G \subseteq U$. Then $H \subseteq G \subseteq U$. Since $H \in SG\omega C(X, \tau)$, $cl_\omega(H) \subseteq U$. But $cl_\omega(G) \subseteq cl_\omega(cl_\omega(H)) = cl_\omega(H) \subseteq U$ and the result follows.

Lemma 3.22 If H is a semi-open and semi-gω-closed subset of a space (X, τ), then H is ω-closed in X.

The proof is obvious.

Theorem 3.23 Let H be a semi-gω-closed subset of (X, τ). Then $H = cl_\omega(int_\omega(H))$ if and only if $cl_\omega(int_\omega(H)) - H$ is semi-closed.

Proof. If $H = cl_\omega(int_\omega(H))$, then $cl_\omega(int_\omega(H)) - H = \varphi$ and hence $cl_\omega(int_\omega(H)) - H$ is semi-closed. Conversely, let $cl_\omega(int_\omega(H)) - H$ be semi-closed since $cl_\omega(H) - H$ contains the semi-closed set $cl_\omega(int_\omega(H)) - H$ by Proposition 3.21 (a), $cl_\omega(int_\omega(H)) - H = \varphi$. Hence $H = cl_\omega(int_\omega(H))$.

Theorem 3.24 Every semi-gω-closed set and every g-closed set is gω-closed.

The proof is obvious.

Corollary 3.25 Let (X, τ) be a space and H be a semi-gω-closed set. The following are equivalent.

(a) H is an ω-closed set.

(b) $cl_\omega(H) - H$ is a semi-closed set.

Proof. (a) $\Rightarrow$ (b) If H is ω-closed, then $cl_\omega(H) - H = \varphi$ and so $cl_\omega(H) - H$ is semi-closed.

(b) $\Rightarrow$ (a) since $cl_\omega(H) - H$ is semi-closed and H is semi-gω-closed by assumption, using Proposition 3.19 (a) we have $cl_\omega(H) - H = \varphi$ and so H is ω-closed.

Theorem 3.26 Let (X, τ) be a space and $H \subseteq X$. If H is semi-gω-closed then $H = G - N$ where G is ω-closed and N contains no nonempty semi-closed set.

Proof. If H is semi-gω-closed, then by Proposition 3.19 (a), $cl_\omega(H) - H = N$ (say) contains no nonempty semi-closed set. If $G = cl_\omega(H)$, then G is ω-closed such that $G - N = cl_\omega(H) - (cl_\omega(H) - H) = H$.

Theorem 3.27 If K and L are semi-gω-closed sets, then $K \cup L$ is semi-gω-closed.

Proof. Suppose that $K \cup L \subseteq U$, where U is semi-open. Then $K \subseteq U$ and $L \subseteq U$. Since K and L are semi-gω-closed sets, $cl_\omega(K) \subseteq U$ and $cl_\omega(L) \subseteq U$. Now $cl_\omega(K \cup L) = cl_\omega(K) \cup cl_\omega(L) \subseteq U$. It shows $K \cup L$ is semi-gω-closed.

Theorem 3.28 Let K and L be subsets of a space (X, τ) such that $K \subseteq L \subseteq cl_\omega(K)$ and K is a semi-gω-closed set. Then L is also semi-gω-closed.

Proof. Let $L \subseteq U$, where U is semi-open. Then $K \subseteq L \subseteq U$ and K is semi-gω-closed implies that $cl_\omega(K) \subseteq U \Rightarrow cl_\omega(L) \subseteq cl_\omega(cl_\omega(K)) = cl_\omega(K) \subseteq U$. Therefore L is semi-gω-closed.

Proposition 3.29 The intersection of a semi-closed set and a closed set is semi-closed.

Proof. Let H be a semi-closed set and G be a closed set in X. Then $int(cl(H)) \subseteq H$ and $cl(G) = G$. We have $G \cap H \supseteq int(G) \cap int(cl(H)) = int(G \cap cl(H)) = int(cl(G) \cap cl(H)) \supseteq int(cl(G \cap H))$. This shows that $G \cap H$ is semi-closed.

Theorem 3.30 If (X, τ) is any space and $H \subseteq X$ such that $cl_\omega(H) = cl(H)$ then the following are equivalent.

(a) H is semi-gω-closed.

(b) $cl_\omega(H) \subseteq G$ whenever $H \subseteq G$ and G is semi-open in X.

(c) For all $x \in cl_\omega(H)$, $scl(\{x\}) \cap H \neq \varphi$.

(d) $cl_\omega (H) - H$ contains no nonempty semi-closed set.

Proof. (a) ⇒ (b): If H is semi-gω-closed, then $cl_\omega(H) \subseteq G$ whenever $H \subseteq G$ and G is semi-open in X. This proves (b).

(b) ⇒ (c): Suppose $x \in cl_\omega(H)$. If $scl(\{x\}) \cap H = \varphi$, then $H \subseteq X - scl(\{x\})$. By (b), $cl_\omega(H) \subseteq X - scl(\{x\})$, a contradiction, since $x \in cl_\omega(H)$.

(c) ⇒ (d): Suppose $F \subseteq cl_\omega(H) - H$, where F is semi-closed and $\neq$ **φ**. Let $x \in F$. Since $F \subseteq X - H$ and F is semi-closed, $scl(\{x\}) \cap H = \varphi$. Since $x \in cl_\omega(H)$ by (c) $scl(\{x\}) \cap H \neq \varphi$, a contradiction which proves (d).

(d) ⇒ (a): Let $H \subseteq U$ where U is semi-open. Then $X - U \subseteq X - H$ and $cl_\omega(H) \cap (X - U) \subseteq cl_\omega(H) \cap (X - H) = cl_\omega(H) - H$. This shows $cl(H) \cap (X - U) \subseteq cl_\omega(H) - H$. Since cl(H) is closed, $X - U$ is semi-closed and by Proposition 3.27, $cl(H) \cap (X - U)$ is semi-closed contained in $cl_\omega (H) - H$. $cl(H) \cap (X - U) = \varphi$ by (d). Thus $cl(H) \subseteq U$ and $cl_\omega(H) \subseteq cl(H) \subseteq U$. Therefore H is semi-g$\omega$-closed.

Definition 3.31 A space (X, τ) is a semi-gω-$T_{1/2}$ if every semi-gω-closed set in (X, τ) is ω-closed in (X, τ).

Theorem 3.32 For a space (X, τ), the following are equivalent.

(a) X is semi-gω-$T_{1/2}$.

(b) Every singleton is either semi-closed or ω-open.

Proof. (a) ⇒ (b): Suppose $\{x\}$ is not a semi-closed subset for some $x \in X$. Then $X - \{x\}$ is not semi-open and hence X is the only semi-open set containing $X - \{x\}$. Therefore $X - \{x\}$ is semi-gω-closed. Since (X, τ) is semi-gω-$T_{1/2}$ -space, $X - \{x\}$ is ω-closed and thus $\{x\}$ is ω-open.

(b) ⇒ (a): Let H be a semi-gω-closed subset of (X, τ) and $x \in cl_\omega(H)$. We show that $x \in H$.

Case (i) If $\{x\}$ is semi-closed and $x \notin H$, then $x \in (cl_\omega(H) - H)$. Thus $\bar{c}l_\omega(H) - H$ contains a nonempty semi-closed set $\{x\}$, a contradiction to Proposition 3.19 (a). So $x \in H$.

Case (ii) If $\{x\}$ is ω-open, since $x \in cl_\omega(H)$, for every ω-open set U containing x, we have $U \cap H \neq \varphi$. But $\{x\}$ is ω-open. So $\{x\} \cap H \neq \varphi$ and $x \in H$. Thus in both cases, $x \in H$. Therefore H is ω-closed.

SEMI-gω -OPEN SETS

Definition 4.1 A subset H of a space (X , τ) is called semi-gω-open if its complement $X - H$ is semi-gω-closed in (X , τ).

Theorem 4.2 Let (X, τ) be a space and $H \subseteq X$. Then H is semi-gω-open if and only if $G \subseteq int_\omega (H)$ whenever G is semi-closed and $G \subseteq H$.

Proof. Suppose H is semi-gω-open. If G is semi-closed and $G \subseteq H$, then $X - H \subseteq X - G$ and so $cl_\omega(X - H) \subseteq X - G$. Therefore $G \subseteq int_\omega (H)$. Conversely, Suppose the condition holds. Let U be a semi-open set such that $X - H \subseteq U$. Then $X - U \subseteq H$ and so by assumption $X - U \subseteq int_\omega (H)$ which implies that $cl_\omega(X - H) \subseteq U$. Therefore $X - H$ is semi-gω-closed and so H is semi-gω-open.

Theorem 4.3 Let K and L be subsets of a space (X, τ) such that $int_\omega (K) \subseteq L \subseteq K$. If K is semi-g$\omega$-open, then L is also semi-gω-open.

The proof follows from Theorem 3.28, by taking complements.

Theorem 4.4 If H is a semi-gω-closed subset of (X, τ), then $cl_\omega (H) - H$ is semi-gω-open.

Proof. Let H be a semi-gω-closed subset of (X, τ) and let U be a semi-closed subset such that U ⊆ $cl_{\omega}(H) - H$. By Proposition 3.19 (a), U = φ and thus U ⊆ $int_{\omega}(cl_{\omega}(H) - H)$. By Theorem 4.2 $cl_{\omega}(H) - H$ is semi-gω-open.

Lemma 4.5 If every semi-open set is closed, then all subsets of (X, τ) are semi-gω-closed (and hence all are semi-gω-open).

Proof. Let A be any subset of X such that A ⊆ U and U is semi-open, then $cl_{\omega}(A) \subseteq cl_{\omega}(U) \subseteq cl(U) = U$. Therefore A is semi-gω-closed.

Theorem 4.6 If K and L are open, semi-gω-open, and separated sets, then K ∪ L is semi- gω-open.

Proof. Let F be a semi-closed subset of K ∪ L. Then F ⊆ K ∪ L ⊆ cl(K ∪ L). Now F ∩ cl(K) ⊆ (K ∪ L) ∩ cl(K) = [K ∩ cl(K)] ∪ [L ∩ cl(K)] = K ∪ φ = K. Since F is semi-closed and cl(K) is closed, by Proposition 3.27, F ∩ cl(K) is semi-closed. By Theorem 4.2, we have F ∩ cl(K) ⊆ $int_{\omega}(K)$. Similarly, F ∩ cl(L) ⊆ $int_{\omega}(L)$. Since F ⊆ cl(K ∪ L), F = F ∩ cl(K ∪ L) = F ∩ [cl(K) ∪ cl(L)] = [F ∩ cl(K)] ∪ [F ∩ cl(L)] ⊆ $int_{\omega}(K) \cup int_{\omega}(L) \subseteq int_{\omega}(K \cup L)$. Hence K ∪ L is semi-gω-open.

GENERALIZATION OF LOCALLY CLOSED SETS

Definition 5.1 A subset H of a space (X, τ) is called semi-gω-locally closed if H = M ∩ N where M is semi-gω-open and N is ω-closed.

Proposition 5.2 Let (X, τ) be a space and H ⊆ X. The following hold.

(a) If H is semi-gω-open, then H is semi-gω-locally closed.

(b) If H is ω-closed, then H is semi-gω-locally closed.

Remark 5.3 None of the converses of the statements in the above Proposition need be true as shown in the following Examples.

Example 5.4 In the space (R, τ_u), the set H=(0,1) is ω-open and hence semi-gω-open. Thus H is semi-gω-locally closed. But H is not ω-closed since 0 is a condensation point of H and 0 ∉ H.

Example 5.5 In R with the topology τ = {φ, R, R − Q}, the set H = Q is closed and hence ω-closed. Thus H is semi-gω-locally closed. But H is not semi-gω-open since R\H is not semi-gω-closed which is proved in the Example 3.4.

Definition 5.6 A subset H of a space (X, τ) is called locally ω-closed [13] if H = M ∩ N where M is ω-open and N is closed.

Theorem 5.7 Every locally ω-closed set is semi-gω-locally closed.

Proof. It follows from the fact that (a) Every ω-open set is semi-gω-open and

(b) Every closed set is ω-closed.

The converse of the above theorem need not be true as seen from the following Example.

Example 5.8. In R with usual topology τ_u, the set H = Q is ω-closed since Q is countable. Hence H is semi-gω-locally closed set. But H is not locally ω-closed. Suppose H is locally ω- closed. Then H = M ∩ N where M ∈ τ_{ω} and N is closed. Then we have H ⊆ N and cl(H) ⊆ cl(N) = N by assumption. Thus cl(H) = R ⊆ N and so R = N. Hence H = M ∩ R = M ∈ τ_{ω} which is a contradiction since H = Q is not ω-open. This proves that H = Q is not locally ω-closed.

Theorem 5.9 Every locally closed set is semi-gω-locally closed.

Proof. It follows from the fact that (a) Every open set is ω-open, which is semi-g_{ω}-open and (b) Every closed set is ω-closed.

The converse of the above theorem need not be true as seen from the following Example.

Example 5.10 In R with usual topology τ_u, the set H = R −Q is ω-open and hence semi-gω-open. Therefore H is semi-gω-locally closed. But H = R − Q is not locally closed. Suppose H = R − Q is locally closed. Then H = M ∩ N where M is open and N is closed. Thus H ⊆ N ⇒ R = cl(H) ⊆ cl(N) = N. Hence N = R and H = R∩ M = M which is open. This is a contradiction since H is not open. This proves that H is not locally closed.

Definition 5.11 A subset H of a space (X, τ) is called locally ω-semi-closed if H = M ∩ N where M is semi-open and N is ω-closed.

Proposition 5.12 Let (X, τ) be a space and H ⊆ X. The following hold.
(a) If H is ω-open, then H is locally ω-closed.
(b) If H is closed, then H is locally ω-closed.
(c) If H is semi-open, then H is locally ω-semi-closed.
(d) If H is ω-closed, then H is locally ω-semi-closed.

Remark 5.13 None of the converses of the statements in the above proposition need be true as shown in the following Examples.

Example 5.14 In R with the topology $\tau = \{\varphi, R, Q\}$, the set H = Q is ω-open and hence locally ω-closed. But H is not closed.

Example 5.15 In R with usual topology τ_u, the set H = [0,1] is closed and hence locally ω-closed. But H is not ω-open, since $int_\omega(H) \neq H$.

Example 5.16 In Example 5.5, the set H=Q is ω-closed and hence locally ω-semi-closed. But H is not semi-open, since cl(int(H)) = φ.

Example 5.17 In R with usual topology τ_u, the set H = (0,1) is semi-open and hence H is locally ω-semi-closed. But H is not ω-closed, since 0 is a condensation point of H and $0 \notin H$.

Theorem 5.18 Every locally closed set is locally ω-semi-closed.

Proof. It follows from the fact that
(a) Every open set is semi-open and
(b) Every closed set is ω-closed.

The converse of the above theorem need not be true as seen from the following Example.

Example 5.19 In R with usual topology τ_u, the set H = Q is ω-closed since Q is countable. Hence H is locally ω-semi-closed. But H = Q is not locally closed. Suppose H = Q is locally closed. Then H = M ∩ N where M is open and N is closed. Thus H ⊆ N and cl(H) ⊆ cl(N) = N. But cl(H) = cl(Q) = R which implies R ⊆ N and so R = N. Then H = M ∩ N = M ∩ R = M which means Q=H=M is open. This is a contradiction, for Q is not open. This proves that H is not locally closed.

Theorem 5.20 For a subset H of a space (X, τ), the following are equivalent.
(a) H is ω-closed.
(b) H is semi-gω-closed and locally ω-semi-closed.

Proof. (a) ⇒ (b): It follows from the fact that
(1) Every ω-closed set is semi-gω-closed and
(2) Every ω-closed set is locally ω-semi-closed.

(b) ⇒ (a): Given H is locally ω-semi-closed. So H = M ∩ N where M is semi-open and N is ω-closed. Since H ⊆ N where M is semi-open, $cl_\omega(H) \subseteq cl_\omega(N) = N$. Given H is semi-g$\omega$-closed. Since H ⊆ M where M is semi-open, $cl_\omega(H) \subseteq M$. We have $cl_\omega(H) \subseteq M \cap N = H$ and hence H is ω-closed.

The following Examples show that the concepts of semi-gω-closedness and locally ω- semi-closedness are independent.

Example 5.21 In R with usual topology τ_u, the set H = (0,1) is semi-open and H⊆ H. But $cl_\omega(H) = [0,1]$. Thus H is not semi-gω-closed. But H is locally ω-semi-closed since H is semi-open.

Example 5.22 In Example 3.7, in R with the topology $\tau = \{\varphi, R, F\}$, where F = {A/0 ∈A and A is non-empty open subset of R in τ_u} is a topology. In (R, τ), H=R\{0} is not semi-open since int(H) = φ with no non-empty open subsets in it. Thus R is the only semi-open set containing H. Hence H is semi-gω-closed. If H is locally ω-semi-closed, then H = M ∩ N where M is semi open and N is ω-closed. So H ⊆ N and $cl_\omega(H) \subseteq cl_\omega(N)=N$. But $cl_\omega(H)=R$. Thus R⊆N and so N=R. This gives H = M ∩ N = M ∩ R =M and H is semi-open which is a contradiction. So H is not locally ω-semi-closed.

Theorem 5.23 Every locally closed set is locally ω-closed.

Proof. It follows from the fact that every open set is ω-open.
The converse of Theorem 5.23 need not be true as seen from the following Example.
Example 5.24 In R with usual topology τ_u, the set H = R − Q is locally ω-closed but not locally closed.
Solution: H = R − Q is ω-open and hence locally ω-closed.
Suppose H is locally closed. Then H = M ∩ N where M is open and N is closed. Thus H ⊆ N and cl(H) ⊆ cl(N) = N since N is closed. But cl(H) = R and so N=R. This means H = M ∩ R = M and H = R − Q is open. This is a contradiction which proves that H is not locally closed.
Remark 5.25 We have the following implications for a subset H of (X, τ).

Locally closed set $\rightleftarrows$ *Locally ω-Semi-closed set*

$\upharpoonleft\downarrow$

Locally ω-closed set $\rightleftarrows$ *Semi-gω-locally closed set*

SEMI-gω -NORMAL SPACES

Definition 6.1 A space (X, τ) is said to be semi-gω-normal space if for every pair of disjoint closed sets P and Q, there exist disjoint semi-gω-open sets K and L such that P ⊆ K and Q ⊆ L.
Remark 6.2 Every normal space is semi-gω-normal.
Proof. If P and Q is a pair of disjoint closed sets of a normal space (X, τ), there exist disjoint open sets K and L such that P ⊆ K and Q ⊆ L. Since K and L are open, they are semi-gω-open and thus (X, τ) is semi-gω-normal.
The following Example shows that a semi-gω-normal space is not necessarily a normal space.
Example 6.3 Let X = {a, b, c} and τ = {φ, X, {b}, {a, b}, {b, c}}. Here every semi-open set is a subset of X and is ω-closed and so by Theorem 3.20, every subset of X is semi-gω-closed and so semi-gω-open also. If P and Q are disjoint closed sets then they are semi-gω-open also. This implies (X, τ) is semi-gω-normal. Now {a} and {c} are disjoint closed subset of X which are not separated by disjoint open sets and so (X, τ) is not normal.
Theorem 6.4 Let (X, τ) be a space. Then the following are equivalent.
(a) X is semi-gω-normal.
(b) For every pair of disjoint closed sets P and Q, there exist disjoint semi-gω-open sets K and L such that P ⊆ K and Q ⊆ L.
(c) For every closed set P and an open set M containing P, there exists a semi-gω-open set N such that P ⊆ N ⊆ cl_ω(N) ⊆ M.
Proof. (a) ⇒ (b): The proof follows from the definition of a semi-gω-normal space.
(b) ⇒ (c): Let P be a closed set and M be an open set containing P. Since P and X − M are disjoint closed sets, there exist disjoint semi-gω-open sets N and W such that P ⊆ N and X − M ⊆ W . Again N ∩ W = φ implies that N ∩ int_ω(W) = φ and so cl_ω(N) ⊆ X − int_ω (W). Since X − M is semi-closed and W is semi-gω-open, X − M ⊆ W implies that X − M ⊆ int_ω(W) and so X − int_ω(W) ⊆ M. Thus we have P ⊆ N ⊆ cl_ω(N) ⊆ X −int_ω(W) ⊆ M which proves (c).
(c) ⇒ (a): Let P and Q be two disjoint closed subsets of X. By hypothesis, there exists a semi-gω-open set K such that P ⊆ K ⊆ cl_ω (K) ⊆ X − Q. If L = X − cl_ω (K), then K and L are the required disjoint semi-gω-open sets containing P and Q respectively. So (X, τ) is semi-gω-normal.
Theorem 6.5 Let (X, τ) be a semi-gω-normal space. If F is a closed subset of X and A is a g-closed set such that A ∩ F = φ, then there exist disjoint semi-gω-open sets K and L such that A ⊆ K and F ⊆ L.

Proof. Since A ∩ F = φ, A ⊆ X − F where X − F is open. Therefore, by hypothesis, cl(A) ⊆ X − F. Since cl(A) ∩ F = φ and X is semi-gω-normal, there exist disjoint semi-gω-open sets K and L such that A ⊆ cl(A) ⊆ K and F ⊆ L.

Theorem 6.6 Let (X, τ) be a semi-gω-normal space. Then the following hold.

(a) For every closed set P and every g-open set Q containing P, there exists a semi-gω-open set U such that P ⊆ int_{ω}(U) ⊆ U ⊆ Q.

(b) For every g-closed set P and every open set Q containing P, there exists semi-gω-closed set U such that P ⊆ U ⊆ cl_{ω}(U) ⊆ Q.

Proof. (a) Let P be a closed set and Q be a g-open set containing P. Then P ∩ (X − Q) = φ, where P is closed and X − Q is g-closed. By Theorem 6.5, there exist disjoint semi-gω- open sets U and V such that P ⊆ U and X − Q ⊆ V. Since U is semi-gω-open and P is semi-closed being closed by Theorem 4.2, P ⊆ int_{ω}(U). Again U ∩ V = φ implies U ⊆ X – V. Therefore P ⊆ int_{ω}(U) ⊆ U ⊆ X −V ⊆ Q. This proves (a).

(c) Let P be a g-closed set and Q be an open set containing P. Then X − Q is a closed set contained in the g-open set X − P. By (a), there exists a semi-gω-open set V such that X − Q ⊆ int_{ω}(V) ⊆ V ⊆ X − P. Therefore P ⊆ X − V ⊆ cl_{ω}(X − V) ⊆ Q. If U = X − V, then P ⊆ U ⊆ cl_{ω} (U) ⊆ Q and so U is the required semi-gω-closed set.

DECOMPOSITION OF ω –CONTINUITY

Definition 7.1 A function f : X → Y is said to be ω-continuous [7] (resp. semi-gω- continuous, locally ω-semi-continuous) if f^{-1}(G) is ω-closed (resp. semi-gω-closed, lo cally ω-semi-closed) in X for each closed subset G of Y.

Theorem 7.2 For a function f : X → Y, the following are equivalent.

(a) f is ω-continuous

(b) f is semi-gω-continuous and locally ω-semi-continuous.

Proof. This is an immediate consequence of Theorem 5.20.

REFERENCES

[1] Al-Omari. A and Noorani. M. S. M : “Regular generalized ω-closed sets”, Intern. J. Math. Math. Sci., (2007), Article ID 16292, 11 pages, doi: 10.1155/2007/16292.

[2] Al-Zoubi. K. Y: “On generalized ω-closed sets”, Intern. J. Math. Math Sci., 13(2005), 2011–2021.

[3] Dunham. W: “T1/2-spaces”, Kyungpook Mathematical Journal, 17(2)(1977), 161-169.

[4] Engelking. R: “General Topology”, Heldermann Veriag Berlin (1989), 2nd edition.

[5] Ganster. M and Reilly. I. L: “Locally closed sets and LC-continuous functions", Intern. J. Math. Math. Sci., 12(3)(1989), 417-424.

[6] Hdeib. H. Z: “ω-closed mappings", Revista Colomb. De Matem., 16(1-2)(1982), 65-78.

[7] Hdeib. H. Z: “ω-continuous functions", Dirasat., 16(1989), 136-142

[8] Levine. N: “A decomposition of continuity in topological spaces”, Amer. Math. Monthly, 68(1961), 44-46.

[9] Levine. N: “Generalized closed sets in topology”, Rend. Cir. Mat. Palermo., (2)(1970), 89–96.

[10] Levine. N: “Semi-open sets and semi-continuity in topological spaces", Amer. Math. Monthly., 70(1) (1963), 36-41.

[11] Nagadevi. M and Ravi. O: “Another generalization of ω-closed sets", Mathematical Statistician and Engineering Applications, 71(4)(2022), 12752-12763.

[12] Njastad. O: “On some classes of nearly open sets", Pacific J. Math., 15(1965), 961-970.

[13] Noiri. T., Al-Omari. A. and Noorani. M. S. M: “Weak forms of ω-open sets and decompositions of continuity”, Eur. J. Pure Appl. Math., 2(1)(2009), 73-84.

[14] Umamaheswari. R., Nethaji. O and Ravi. O: “gω-continuity and its decompositions", South Asian Journal of Mathematics, 7(2)(2017), 62-72.

[15] Tong. J: “A decomposition of continuity”, Acta Math. Hungar., 48(1986), 11-15].