Tripled Coincidence Point Theorem for Compatible Maps in Partially Ordered Metric Spaces

P.P.Murthy*, R.M.Kenvat, B. S. Selukar**, G. S. Kamble**, M. V. Jagtap****

*Dept. of Pure and Applied Mathematics, Guru Ghasidas University, Bilaspur, India
**First Year Engineering Dept., Aanatrao Pawar College of Engineering & Research, Parvati, Pune India

ppmurthy@gmail.com (Murthy) rashmi.kenvat@abmspcoerpune.org (Kenvat)

balaji.selukar@abmspcoerpune.org (Selukar)

Article Info
Page Number:1952-1960
Publication Issue:
Vol 70 No. 2 (2021)

Article History
Article Received*: 05 September 2021*
Revised*: 09 October 2021*
Accepted*: 22 November 2021*
Publication*: 26 December 2021*

Abstract

The present study introduce the notion of compatibility of maps in partially ordered metric spaces and use this perception to establish a tripled coincidence point result for mixed g-monotone mappings. Our effort extend the recent work of Borcut and Berinde [M. Borcut, V. Berinde, Tripled fixed point theorem for contractive type mappings in partially ordered metric spaces, Nonlinear Anal., Accepted (2011)] and refrences therein. We support the result by establishing an illustrative example.

Keywords: Partially ordered set; Tripled coincidence point; Mixed monotone property; compatible mappings.

Introduction

In recent years, the study of fixed point for mappings that possess monotonicity type properties, in the context of partially ordered metric spaces which combine method of contraction principle with method of monotone iterations and method of lower and upper solution has been the focus of vigorous research activity. In particular the approach to weaken the requirement of contraction by considering that the operator assumed is monotone was initiated by Ran and Ruering in [12] and it was refined and extended in [14, 15] and was applied to periodic boundary value problem.

Specifically, Bhaskar and Lakshmikanthan [3] established coupled fixed point for mixed monotone operator in partially ordered metric spaces. Afterward, Lakshmikanthan and Ciric [8] extended the results of [3] by furnishing coupled coincidence and coupled fixed point theorem for two commuting mappings having mixed g-monotone property. In a subsequent series, B. S. Choudhary and A. kundu [6] introduced the concept of compatibility and proved the result of [8] under different set of condition. Very recently Borcut and Berinde [18] introduce tripled fixed point theorem for contractive type mapping in partially ordered metric spaces.

The purpose of this work is to generalize results of [18] by introducing weaker variant as compatibility of mappings and using control φ-function. The result obtained can be applied to study of several nonlinear problems.

1. Preliminaries

In what follows, we collect some relevant definitions, results, examples for our further use.

Let $(X, \leq)$ be partially ordered set and $F: X \rightarrow X$ be a mapping from X to itself. The mapping F is said to be non-decreasing if for all $x_1, x_2 \in X$, $x_1 \leq x_2$ implies $F(x_1) \leq F(x_2)$ and non-increasing, if for all $x_1, x_2 \in X$, $x_1 \leq x_2$ implies $F(x_1) \geq F(x_2)$.

Definition 1.1 ([3]). Let $(X, \leq)$ be partially ordered set and the mapping $F: X \times X \rightarrow X$ is said to have mixed monotone property if F is monotone non-decreasing in its fist argument and is monotone non-increasing in its second argument, that is, if for any $x_1, x_2 \in X$, $x_1 \leq x_2$ implies $F(x_1, y) \leq F(x_2, y)$ for $y \in X$ and for all $y_1, y_2 \in X$, $y_1 \leq y_2$ implies $F(x, y_1) \geq F(x, y_2)$.

Definition 1.2 (Mixed g-monotone property [8]) Let $(X, \leq)$ be partially ordered set and $F: X \times X \rightarrow X$ and $g: X \rightarrow X$ be two self mappings. F has mixed g-monotone property if F is monotone g-non-decreasing in its fist argument and is monotone g-non-increasing in its second argument, that is, if for any $x_1, x_2 \in X$, $gx_1 \leq gx_2$ implies $F(x_1, y) \leq F(x_2, y)$ for $y \in X$ and for all $y_1, y_2 \in X$, $gy_1 \leq gy_2$ implies $F(x, y_1) \geq F(x, y_2)$.

Definition 1.3 ([3]). An element $(x, y) \in X \times X$, is called a coupled fixed point of mapping $F: X \times X \rightarrow X$ if $F(x, y) = g(x)$ and $F(y, x) = y$.

Definition 1.4 ([3]). An element $(x, y) \in X \times X$, is called a coupled coincident point of mapping $F: X \times X \rightarrow X$ and $g: X \rightarrow X$ if $F(x, y) = g(x)$ and $F(y, x) = y$.

Definition 1.5 ([6]) The mappings F and g where $F: X \times X \rightarrow X$ and $g: X \rightarrow X$, are said to be compatible if

$$\lim_{n\to\infty} d(g(F(x_n, y_n)), F(g(x_n), g(y_n))) = 0$$

and

$$\lim_{n\to\infty} d(g(F(y_n, x_n)), F(g(y_n), g(x_n))) = 0.$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$, for all $x, y \in X$ are satisfied.

Now, we are ready to prove our results which are of three folds:

(i) We use compatibility which is more general variant.

(ii) We proceed with φ-contraction which more general.

(iii) We use g-mixed monotone property which is more general mixed monotone property.

2. MAIN RESULTS

Let $(X, \leq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Consider on the product $X \times X \times X$ the following partial order: for $(x, y, z), (u, v, w) \in X \times X \times X$,

$$(u, v, w) \leq (x, y, z) \Leftrightarrow x \geq u, y \leq v, z \geq w.$$

Definition 2.1 Let $(X, \leq)$ be a partially ordered set and $F: X \times X \times X \to X$ and $g: X \to X$. We say that F has the g-mixed monotone property if F(x, y, z) is monotone non decreasing in x and z, and is monotone non increasing in y, that is, for any $x, y, z \in X$,

$$x_1, x_2 \in X,\ g(x_1) \leq g(x_2) \Rightarrow F(x_1, y, z) \leq F(x_2, y, z),$$

$$y_1, y_2 \in X, g(y_1) \leq g(y_2) \Rightarrow F(x, y_1, z) \geq F(x, y_2, z),$$

and

$$z_1, z_2 \in X, g(z_1) \leq g(z_2) \Rightarrow F(x, y, z_1) \leq F(x, y, z_2).$$

Now, we introduce the concept of compatible mapping for trivariate mapping F and self mapping g akin to compatible mapping as introduce by Choudhary and Kundu [6] for bivariate mapping F and self mapping g.

Definition 2.1 The mapping F and g where $F: X \times X \times X \to X$ and $g: X \to X$ are said to be compatible if

$$\lim_{n\to\infty} d(g(F(x_n, y_n, z_n)), F(g(x_n), g(y_n), g(z_n))) = 0,$$

$$\lim_{n\to\infty} d(g(F(y_n, x_n, z_n)), F(g(y_n), g(x_n), g(z_n))) = 0$$

and $$\lim_{n\to\infty} d(g(F(z_n, y_n, x_n)), F(g(z_n), g(y_n), g(x_n))) = 0$$

whenever $\{x_n\}$,$\{y_n\}$ and $\{z_n\}$ are sequences in X, such that $\lim_{n\to\infty} F(x_n, y_n, z_n) = \lim_{n\to\infty} g(x_n) = x$, $\lim_{n\to\infty} F(y_n, x_n, z_n) = \lim_{n\to\infty} g(y_n) = y$ and $\lim_{n\to\infty} F(z_n, y_n, x_n) = \lim_{n\to\infty} g(z_n) = z$ for all $x, y, z \in X$ are satisfied.

We establish the main result of this section.

Theorem 2.1 Let $(X, \leq)$ be partially ordered set and let there be a metric d on X such that (X, d) be a metric space. Let $\varphi: [0, +\infty) \to [0, +\infty)$ be such that $\varphi(t) < t$ and $\lim_{r\to t^+} \varphi(r) < t$ for all $t > 0$. Let $F: X \times X \times X \to X$ and $g: X \to X$ are such that F has mixed g-monotone property and satisfy

$$d(F(x, y, z), F(u, v, w)) \leq \varphi\left(\frac{d(gx,gu) + d(gy,gv) + d(gz,gw)}{3}\right) \tag{2.1}$$

for all $x, y, z, u, v, w \in X$, with $gx \leq gu$, $gy \geq gv$ and $gz \leq gw$. Let $F(X \times X \times X) \subseteq g(X)$, g be continuous and monotone increasing and F and g be compatible mappings. Also suppose

(a) F is continuous or

(b) X has following properties

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all $n \geq 0$,

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \leq y$ for all $n \geq 0$,

(iii) if a non-decreasing sequence $\{z_n\} \to z$, then $z_n \leq z$ for all $n \geq 0$.

If there exist $x_0, y_0, z_0 \in X$, such that $g(x_0) \leq F(x_0, y_0, z_0)$, $g(y_0) \geq F(y_0, x_0, z_0)$ and $g(z_0) \leq F(z_0, y_0, x_0)$, there exist $x, y, z \in X$ such that $g(x) = F(x, y, z)$, $g(y) = F(y, x, z)$ and $g(z) = F(z, y, x)$, that is, F and g have coupled coincidence point in X.

Proof: Let $x_0, y_0, z_0 \in X$, be such that $g(x_0) \leq F(x_0, y_0, z_0)$, $g(y_0) \geq F(y_0, x_0, z_0)$ and $g(z_0) \leq F(z_0, y_0, x_0)$. Since $F(X \times X \times X) \subseteq g(X)$, we can define $x_1, y_1, z_1 \in X$ such that $g(x_1) = F(x_0, y_0, z_0)$, $g(y_1) = F(y_0, x_0, z_0)$ and $g(z_1) = F(z_0, y_0, x_0)$. In the same way, we construct $g(x_2) = F(x_1, y_1, z_1)$, $g(y_2) = F(y_1, x_1, z_1)$ and $g(z_2) = F(z_1, y_1, x_1)$. Continuing like this, we construct three sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that for all $n \geq 0$

(2.2) $\quad g(x_{n+1}) = F(x_n, y_n, z_n),\ g(y_{n+1}) = F(y_n, x_n, z_n),\ g(z_{n+1}) = F(z_n, y_n, x_n).$

Now, it is obvious by mathematical induction that for all $n \geq 0$,

(2.3) $\quad g(x_n) \leq g(x_{n+1}),\ g(y_n) \geq g(y_{n+1})$ and $g(z_n) \leq g(z_{n+1})$.

Let, $\delta_n = d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) + d(gz_n, gz_{n+1})$.

Next we prove that

(2.4) $\quad \delta_n = 3\varphi\left(\frac{\delta n-1}{3}\right)$.

Since for all $n \geq 0$, $g(x_{n-1}) \leq g(x_n)$, $g(y_{n-1}) \geq g(y_n)$ and $g(z_{n-1}) \leq g(z_n)$, using (2.1), (2.2) and (2.5), we have $\quad d(gx_n, gx_{n+1}) = d(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_n, y_n, z_n))$

$$\leq \varphi\left(\frac{d(gx_{n-1},gx_n)+ d(gy_{n-1},gy_n)+ d(gz_{n-1},gz_n)}{3}\right)$$

$$= \varphi\left(\frac{\delta n-1}{3}\right).$$

Similarly, we have from (2.1) and (2.2),

(2.6) $\quad d(gy_n, gy_{n+1}) \leq \varphi\left(\frac{\delta n-1}{3}\right),\ d(gz_n, gz_{n+1}) \leq \varphi\left(\frac{\delta n-1}{3}\right).$

Combining (2.5) and (2.6), we obtain (2.4).

Since $\varphi(t) < t$ for $t > 0$, it follows from (2.4) that the sequence $\{\delta_n\}$ is a monotone decreasing sequence of non-negative real numbers. Hence there exist $\delta \geq 0$ such that $\lim_{n\to\infty} \delta_n = \delta$. If possible, let $\delta > 0$, Taking the limit as $n \to \infty$ in (2.4) and using $\lim_{r\to t^+} \varphi(r) < t$ for all $t > 0$.

$$\delta = \lim_{n\to\infty} \delta_n \leq 3 \lim_{n\to\infty} \varphi\left(\frac{\delta n-1}{3}\right) = 3 \lim_{\delta_{n-1}\to\delta^+} \varphi\left(\frac{\delta n-1}{3}\right) < 3\,\frac{\delta}{3} = \delta,$$

which is a contradiction. Thus $\delta = 0$. Hence we have

(2.7) $\quad \lim_{n\to\infty} [d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) + d(gz_n, gz_{n+1})] = \lim_{n\to\infty} d_n = 0.$

Thus,

(2.8) $\quad \lim_{n\to\infty} d(gx_{n+1}, gx_n) = \lim_{n\to\infty} d(gy_{n+1}, gy_n) = \lim_{n\to\infty} d(gz_{n+1}, gz_n) = 0.$

Next we show that $\{gx_n\}$, $\{gy_n\}$ and $\{z_n\}$ are Cauchy sequences. Let at least one of $\{gx_n\}$, $\{gy_n\}$ and $\{z_n\}$ be not a Cauchy sequence. Then there exist $\varepsilon > 0$ and the sequence of natural numbers $\{m(k)\}$ and $\{l(k)\}$ such that for every natural number k

$$m(k) > l(k) \geq k$$

and

(2.9) $\quad d_k = d(g(x_{l(k)}), g(x_{m(k)}) + d(g(y_{l(k)}), g(y_{m(k)}) + d(g(z_{l(k)}), g(z_{m(k)}) \geq \varepsilon.$

Now corresponding to l(k) we can choose m(k) to be smallest positive integer for which (2.9) holds. Then,

(2.10) $\quad d(g(x_{l(k)}), g(x_{m(k)-1}) + d(g(y_{l(k)}), g(y_{m(k)-1}) + d(g(z_{l(k)}), g(z_{m(k)-1}) < \varepsilon.$

Further from (2.9), (2.10) and triangle inequality, for all $k \geq 0$, we have

$$\begin{aligned}\varepsilon \leq d_k &\leq d(g(x_{l(k)}), g(x_{m(k)-1})) + d(g(x_{m(k)-1}), g(x_{m(k)})) + d(g(y_{l(k)}), g(y_{m(k)-1})) \\ &\quad + d(g(y_{m(k)-1}), g(y_{m(k)-1})) + d(g(z_{l(k)}), g(z_{m(k)-1})) + d(g(z_{m(k)-1}), g(z_{m(k)})) \\ &= d(g(x_{l(k)}), g(x_{m(k)-1})) + d(g(y_{l(k)}), g(y_{m(k)-1})) + d(g(z_{l(k)}), g(z_{m(k)-1})) + d_{m(k)-1} \\ &< \varepsilon + d_{m(k)-1}.\end{aligned}$$

Taking the limit as $k \to \infty$, we have by (2.8),

(2.11) $\lim\limits_{n\to\infty} d_k = \varepsilon.$

Again, for all $k \geq 0$

$$\begin{aligned}d_k &= d(g(x_{l(k)}), g(x_{m(k)})) + d(g(y_{l(k)}), g(y_{m(k)})) + d(g(z_{l(k)}), g(z_{m(k)})) \\ &\leq d(g(x_{l(k)}), g(x_{l(k)+1})) + d(g(x_{l(k)+1}), g(x_{m(k)+1})) + d(g(x_{m(k)+1}), g(x_{m(k)})) \\ &\quad + d(g(y_{l(k)}), g(y_{l(k)+1})) + d(g(y_{l(k)+1}), g(y_{m(k)+1})) + d(g(y_{m(k)+1}), g(y_{m(k)})) \\ &\quad + d(g(z_{l(k)}), g(z_{l(k)+1})) + d(g(z_{l(k)+1}), g(z_{m(k)+1})) + d(g(z_{m(k)+1}), g(z_{m(k)})) \\ &= d(g(x_{l(k)}), g(x_{l(k)+1})) + d(g(y_{l(k)}), g(y_{l(k)+1})) + d(g(z_{l(k)}), g(z_{l(k)+1})) \\ &\quad + d(g(x_{l(k)+1}), g(x_{m(k)+1})) + d(g(y_{l(k)+1}), g(y_{m(k)+1})) + d(g(z_{l(k)+1}), g(z_{m(k)+1})) \\ &\quad + d(g(x_{m(k)+1}), g(x_{m(k)})) + d(g(x_{m(k)+1}), g(x_{m(k)})) + d(g(z_{m(k)+1}), g(z_{m(k)}))\end{aligned}$$

Hence, for all $k \geq 0$

(2.12) $$\begin{aligned}d_k &= d_{l(k)} + d_{m(k)} + d(g(x_{l(k)+1}), g(x_{m(k)+1})) \\ &\quad + d(g(y_{l(k)+1}), g(y_{m(k)+1})) + d(g(z_{m(k)+1}), g(z_{m(k)}))\end{aligned}$$

From (2.1)-(2.3) and (2.9), for all $k \geq 0$, we obtain

(2.13) $$\begin{aligned}d(g(x_{l(k)+1}), g(x_{m(k)+1})) &= d(F(x_{l(k)}, y_{l(k)}, z_{l(k)}), F(x_{m(k)}, y_{m(k)}, z_{m(k)})) \\ &\leq \varphi\left(\frac{d(gx_{l(k)}, gx_{m(k)}) + d(gy_{l(k)}, gy_{m(k)}) + d(gz_{l(k)}, gz_{m(k)})}{3}\right) \\ &= \varphi\left(\frac{dk}{3}\right)\end{aligned}$$

Similarly, from (2.1) – (2.3) and (2.9), for all $k \geq 0$, we get

(2.14) $d(g(y_{l(k)+1}), g(y_{m(k)+1})) = \varphi\left(\frac{dk}{3}\right)$ and $d(g(z_{l(k)+1}), g(z_{m(k)+1})) = \varphi\left(\frac{dk}{3}\right)$.

Putting (2.13) and (2.14) in (2.12), for all $k \geq 0$, we obtain $d_k = d_{l(k)} + d_{m(k)} + 3\,\varphi\left(\frac{dk}{3}\right)$.

Letting $n \to \infty$ in the above inequality and using (2.8) - (2.11), we have

$$\varepsilon \leq 3 \lim_{k\to\infty} \varphi\left(\frac{dk}{3}\right) = 3 \lim_{d_k\to\varepsilon^+} \varphi\left(\frac{dk}{3}\right) < 3\,\frac{\varepsilon}{3} = 3,$$

which is a contradiction. Therefore, $\{gx_n\}$, $\{gy_n\}$ and $\{z_n\}$ are Cauchy sequence in X and hence they are convergent in the complete metric space (X, d). Let

(2.15) $\lim\limits_{n\to\infty} F(x_n, y_n, z_n) = gx_n = x,\ \lim\limits_{n\to\infty} F(y_n, x_n, z_n) = gy_n = y,\ \lim\limits_{n\to\infty} F(z_n, y_n, x_n) = gz_n = z.$

Since F and g are compatible mappings, we have by (2.15)

(2.16) $\lim\limits_{n\to\infty} d(g(F(x_n, y_n, z_n)), F(gx_n, gy_n, gz_n)) = 0,$

(2.17) $\lim_{n\to\infty} d(g(F(y_n, x_n, z_n)), F(gy_n, gx_n, gz_n)) = 0,$

(2.18) $\lim_{n\to\infty} d(g(F(z_n, y_n, x_n)), F(gz_n, gy_n, gx_n)) = 0.$

Next we prove that $gx = F(x, y, z)$, $gy = F(y, x, z)$ and $gz = F(z, y, x)$.

Let (a) holds.

For all $n \geq 0$, we have

$$d(gx, F(gx_n, gy_n, gz_n)) \leq d(gx, g(F(x_n, y_n, z_n)) + d(g(F(x_n, y_n, z_n), F(gx_n, gy_n, gz_n)).$$

Taking the limit as $n \to \infty$, using (2. 3), (2.15) and (2.16) and the fact that F and g are continuous, we have $d(gx, F(x, y, z)) = 0$.

Similarly, from (2. 3), (2.15) and (2.16) and continuity of F and g, we have $d(gy, F(y, x, z)) = 0$ and $d(gz, F(z, y, x)) = 0$.

Combining the above three result we get $g(x) = F(x, y, z)$, $g(y) = F(y, x, z)$ and $g(z) = F(z, y, x)$.

Next we suppose that (b) holds.

By (2.3), (2.15) and (2.16), we have $\{gx_n\}$, $\{gz_n\}$ are non-decreasing sequence, $gx_n \to x$, $gz_n \to z$ respectively and $\{gy_n\}$ is non-increasing sequence, $gy_n \to y$ as $n \to \infty$. The by (i), (ii) and (iii) of (b), we have for $n \geq 0$,

(2.19) $gx_n \leq x$, $gy_n \geq y$ and $g(z_n) \leq z$.

Since, F and g are compatible mappings and g is continuous, by (2.16) – (2.18) we obtain

(2.20) $\lim_{n\to\infty} g(gx_n) = gx = \lim_{n\to\infty} g(F(x_n, y_n, z_n)) = \lim_{n\to\infty} F(gx_n, gy_n, gz_n)$

(2.21) $\lim_{n\to\infty} g(gy_n) = gy = \lim_{n\to\infty} g(F(y_n, x_n, z_n)) = \lim_{n\to\infty} F(gy_n, gx_n, gz_n)$

(2.22) $\lim_{n\to\infty} g(gz_n) = gz = \lim_{n\to\infty} g(F(z_n, y_n, x_n)) = \lim_{n\to\infty} F(gz_n, gy_n, gx_n).$

Now, we have using triangle inequality

$$d(gx, F(x, y, z)) \leq d(gx, g(gx_{n+1})) + d(g(gx_{n+1}), F(x, y, z)).$$

Taking the limit as $n \to \infty$, in the above inequality and using (2.2) and (2.20) we have,

$$\begin{aligned} d(gx, F(x, y, z)) &\leq \lim_{n\to\infty} d(gx, g(gx_{n+1})) + \lim_{n\to\infty} d(g(F(x_n, y_n, z_n)), F(x, y, z)) \\ &= \lim_{n\to\infty} d(F(gx_n, gy_n, gz_n), F(x, y, z)) \end{aligned}$$

Since the mapping g is monotone increasing, by (2.1). (2.19) and the above inequality, we have $n \geq 0$,

$$d(gx, F(x, y, z)) \leq \lim_{n\to\infty} \varphi\left(\frac{d(g(g\,x_n),gx)+ d(g(g\,y_n),gy)+ d(g(g\,z_n),gz)}{3}\right)$$

Using (3.20) and the property of φ-function we obtain, $d(gx, F(x, y, z)) \leq 0$.

Thus is

$$gx = F(x, y, z)$$

and similarly by virtue of (2.1), (2.21) and (2.22) we obtain

$$gy = F(y, z, x) \text{ and } gz = F(z, y, x).$$

Thus we have proved that F and g have coupled coincidence point in X.

Remark 2.2 The results of [3], [6] and [8] are deduced from the results discussed here, by the following choice. Set $F(x, y, z) = F_1(x, y)$ and $F(y, z, x) = F_1(y, x)$. Also by further setting $F_1(x, y) = fx$ and $F_1(y, x) = fy$ the result of [10], [11] and references there in.

Corollary 2.3 Let $(X, \leq)$ be partially ordered set and let there be a metric d on X such that (X, d) be a metric space. Let $F: X \times X \times X \to X$ and $g: X \to X$ are such that F has mixed g-monotone property and for $p \in [0, 1)$ satisfy

(2.23) $$d(F(x, y, z), F(u, v, w)) \leq \frac{p}{3}(d(gx, gu) + d(gy, gv) + d(gz, gv))$$

for all $x, y, z, u, v, w \in X$, with $gx \leq gu$, $gy \geq gv$ and $gz \leq gw$. Let $F(X \times X \times X) \subseteq g(X)$, g be continuous and monotone increasing and F and g be compatible mappings. Also suppose

(a) F is continuous or

(b) X has following properties

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all $n \geq 0$,

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \geq y$ for all $n \geq 0$,

(iii) if a non-decreasing sequence $\{z_n\} \to z$, then $z_n \leq z$ for all $n \geq 0$.

If there exist $x_0, y_0, z_0 \in X$, such that $g(x_0) \leq F(x_0, y_0, z_0)$, $g(y_0) \geq F(y_0, x_0, z_0)$ and $g(z_0) \leq F(z_0, y_0, x_0)$, there exist $x, y, z \in X$ such that $g(x) = F(x, y, z)$, $g(y) = F(y, x, z)$ and $g(z) = F(z, y, x)$, that is, F and g have coupled coincidence point in X.

Proof: By setting $\varphi(t) = pt$, the proof follows easily from theorem 2.1.

Remark 2.4 If we put $g(x) = x$ for all $x \in X$ in corollary (2.1) it generalizes theorem 7 and 8 of [18] (By setting $i = j = k = p / 3$).

Example 2.1 Let $X = [0, 1]$ be endowed with Euclidean metric $d(x, y) = |x - y|$, for all $x, y \in X$.

Then, $(X, \leq)$ is a partial ordered set with natural ordering of real numbers. Let $F: X \times X \times X \to X$ and $g: X \to X$ defined as $g(x) = x$ for all $x \in X$ and

$$F(x, y, z) = \begin{cases} \left(\frac{x+z-y}{3}\right)^2, & \text{if } x, y, z \in [0, 1], x \geq z \geq y \\ 0, & \text{if } x < y \; 0r \; z < y \end{cases}, \text{ respectively.}$$

Clearly, $F(X \times X) \subseteq g(X)$, also F obeys mixed g-monotone property.

Let $\varphi: [0, +\infty) \to [0, +\infty)$ be defined as $\varphi(t) = \frac{2}{3}t$ for $t \in [0, +\infty)$.

We define the sequences $\{x_n\} = \frac{1}{n}$, $\{y_n\} = \frac{1}{n^2}$ and $\{z_n\} = \frac{1}{2n}$.

Obviously the pair {F, g} is compatible.

Also, $x_0 = 0$, $y_0 = 2c$, $z = c$ (> 0) are two point in X such that

$$g(x_0) = g(0) = 0 = F(0, 2c, c) = F(x_0, y_0, z_0),$$

$$g(y_0) = g(2c) = 2c \geq c^2 = F(2c, c, 0) = F(y_0, z_0, x_0)$$

and $$g(z_0) = g(c) = c \geq 0 = F(z_0, y_0, x_0)$$

We next verify inequality (2.1) of theorem 2.1. We take x, y, u, v ∈ X such that $x \geq u$, $z \geq w$, $y \leq v$ or $(x, y, w) \geq (u, v, w)$. We have the following cases

Case-I If (x, y, z) = (u, v, w) or (x, y, z) = (0, 0, 0), (u, v, w) = (0, 1, 1) or (x, y, z) = (1, 1, 1), (u, v, w) = (0, 1, 1), One can easily see

$$d(F(x, y, z), F(u, v, w)) \leq \varphi\left(\frac{d(gx,gu,)+ d(gy,gv)+ d(gz,gw)}{3}\right).$$ Hence inequality (2.1) holds.

Case-II If (x, y, z) = (1, 1, 0), (u, v, w) = (0, 0, 0), then

$$d(F(x, y, z), F(u, v, w)) = d(F(1, 1, 0), F(0, 0, 0)) = \frac{4}{9} = \varphi(\frac{2}{3})$$
$$= \varphi\left(\frac{d(g1,g0\ )+ d(g1,g0)+ d(g0,g0)}{3}\right)$$

Hence inequality (2.1) holds.

Case-III If(x, y, z) = (1, 1, 0), (u, v, w) = (0, 1, 1), then, we obtain

$$d(F(x, y, z), F(u, v, w)) = d(F(1, 1, 0), F(0, 1,1))) = \frac{4}{9}$$
$$= \frac{4}{9} = \varphi(\frac{2}{3}) = \varphi\left(\frac{d(g1,g0\ )+ d(g1,g1)+ d(g0,g1)}{3}\right)$$

Therefore, all the condition of theorem 2.1 is satisfied. Hence (0, 0, 0) is the coupled coincidence point of F and g.

Remark 2.21 It is obvious that results of papers [18] are not applicable to this example which proves the generality of our result.

Remark 2.3 As an application of theorem 2.1, the existence and uniqueness of common solution of periodic boundary value problem can be established as in [1, 3, 8, 11, 12] and references therein.

Acknowledgement: We would like to express our sincere gratitude to Dr. Pradeep J. Awasare (Adjunct Professor), Er. Ashok Saraf (Innovation Club Member) for their invaluable guidance and support throughout the research process. We also wish to thank Dr. Sunil B. Thakare (Principal, APCOER, Pune) for their support. Finally, we are grateful to all of the research participants who generously gave their time and effort to this project.

References

1. R. P. Agarwal, M. Meehan, D. ORegan, Fixed point theory and application, Cambridge University Press, 2001.
2. R. P. Agarwal, M.A. EI-Gebeily, ORegan, Generalized contraction in partially ordered metric spaces, Appl. Anal. 87(2008) 1-8.
3. T. Gana Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric space and applications, Nonlinear anal. TMA 65 (2006) 1379-1393.
4. I. Beg, A, R. Butt, Fixed point for set-valued mappings satisfying an implicit relation in partially ordered metric spaces, Nonlinear Anal. 71(2009) 3699-3704.

5. L. Ciric, V. Lakshmikantham, Coupled random fixed point theorems in partially ordered metric spaces, Stoch. Anal. 27(6) (2009) 1246-1259.
6. B. S. Choudhary, A. Kundu, A coupled coincidence point result in partially ordered metric space for compatible mappings, Nonlinear anal. 73 (2010) 2524-2531.
7. S. Hong, Fixed point of multivalued operator in ordered metric space with applications, Nonlinear Anal. 72 (2010) 3929-3942.
8. V. Lakshmikantham, L. Ciric, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear anal. 70 (12) (2009) 4341-4349.
9. N. V. Luong, N. X. Thuan, Coupled fixed point theorem in partially ordered metric space, Bull. Math. Anal., and Appl., vol. 2(2010) 16-24.
10. J. J. Neito, R. Rodriguez-Lopez, Contractive mapping theorem in partially ordered sets and application to ordinary differential equation, Order 22(2005) 223-239.
11. J. J. Neito, R. R. Lopez, Existence and uniqueness of fixed point in partially ordered sets and application to ordinary differential equation, Acta Math. Sinica Engl. Ser. 23 (12) (2007) 2205-2212.
12. A. C. M. Ran, M. C. B. Reurings, A fixed point theorem in partially ordered sets and some application to matrix equation, Proc. Amer. Math. Soc. 132 (2004) 1435-1443.
13. B. Samet, Coupled fixed point theorems for generalized Meir-Keeler contraction in partially ordered metric spaces, Nonlinear Anal., 72(2010) 4508- 4517.
14. M. D. Rus, Fixed point theorem for generalized contractions in partially ordered metric spaces with semi-monotone metric, Nonlinear. Anal. 74 (2011) 1804-1813.
15. X, Zhang, Fixed point theorem of multivalued monotone mappings in partially ordered metric spaces, Appl. Math. Lett., 23(2010) 253-240.
16. F. Sabetghadam, H. P. Masiha, A. H. Sanatpour, Some coupled fixed point theorem in cone metric spaces, Fixed Point Theory Appl. 2009, Art. ID 125426, 8 pp.
17. B. Samet, Coupled fixed point theorem for generalized Meir-Keeler contaction in partially ordered metric space, Nonlinear Anal., 72 (12) (2010) 4508-45017.
18. M. Borcut, V. Berinde, Tripled fixed point theorem for contractive type mappings in partially ordered metric spaces, Nonlinear Anal., Accepted (2011).