Coefficient Bounds for Subclasses of Analytic Univalent Functions Using Quasi- Subordination

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Abstract: In this paper, we determine the coefficient estimates for $Ƕ_q^{\alpha}(\delta,\lambda,\dot{n},\mathrm{e},\Theta)$ and $Ꙩ_q(\beta,\tau,ŧ,ħ,\Theta)$, the class of analytic and univalent functions associated with quasi-subordination. We find estimates coefficients $|a_2|, |a_3| \, and \, |a_3-\mu a_2^2|, \mu\in\mathbb{R}$ for functions in these classes.

Keywords : Univalent functions; Starlike; Convex functions; Subordination and quasi-subordination.

1-Introduction and Definitions

Let Ƀ be the class of analytic functions defined on an open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$,with the normalized condition $f(0)=0=f'(0)-1$ .Let $Ƕ^*$ be the class of all functions $f\in$ Ƀ which are a univalent in U. So $f(z)\in Ƕ^*$ has the form:

$$f(z)=z+\sum_{i=2}^{\infty}a_i z^i \qquad (z\in U) \qquad (1.1)$$

Let f and g are analytic functions in Ƀ . f is said to be subordinate to g, or g is said to be superordinate to f in U and write $f\prec g$, if there exists a Schwarz function $\dot{w}(z)$ in U , which with $\dot{w}(0)=0$, and $|\dot{w}(z)|<1$, $(z\in U)$,where $f(z)=g(\dot{w}(z))$ .If g is univalent in U, then $f\prec g$ if and only if $f(0)=g(0)$ and $f(U)\subset g(U)$.

Definition (1.2)[18]: The function f is said to be quasi-subordination to g in U and written as follows $f(z)\prec_q g(z)$ $(z\in U)$,if there exist $Q(z)$ and $\dot{w}(z)$ be two analytic functions in U, with $\dot{w}(0)=0$,such that $|Q(z)|<1$,$|\dot{w}(z)|<1$,and $f(z)=Q(z)\dot{g}(\dot{w}(z))$ .If $Q(z)=1$,then $f(z)=g(\dot{w}(z))$,so that $f(z)\prec g(z)$ *in* U .If $\dot{w}(z)=\dot{w}$,then $f(z)=Q(z)g(z)$,and it is said that f is majorized by g and written $f(z)\ll g(z)$ in U .Hence it's abvious that quasi-subordination is a generalization of subordination and marjorization .

Ma and Manda (see[1,5,10,14]) defined a class of starlike and convex functions by using the method of subordination and studied class $ꞪA(\Theta)$ *and* $Æ(\Theta)$, which is defined by :

$$Ꝇ(\Theta) = \{f \in \mathbb{R}: \frac{zf'(z)}{f(z)} \prec \Theta(z)\ z \in U\} \text{ and}$$

$$Æ(\Theta) = \left\{f \in \mathbb{R}: \frac{zf''(z)}{f(z)} \prec \Theta(z)\ , z \in U\right\}.$$

In the sequal ,it assumed that Θ of the form :

$$\Theta(z) = E_0 + E_1 z + E_2 z^2 + \cdots, \qquad (1.2)$$

where Q(0)=1 ,and Q'(0)>0 ,also

$$Q(z) = 1 + A_1 z + A_2 z^2 + \cdots. \qquad (1.3)$$

Now, consider the following:

$$\dot{w}(z) = 1 + \dot{w}_1 z + \dot{w}_2 z^2 + \cdots, \qquad (1.4)$$

which are analytic and bounded in U .For more information (see[2,14,17,23,24]) ,for works related to quasi-subordination . Many auther have been investigated the coefficient bounded $|\dot{a}_2|\ and\ |\dot{a}_3|$ of Fekete-Szegö (see[3,4,7,11,14,17,19,25,27]).

In Fekete and Szegö found the maximum value of the coefficient function $|\dot{a}_3 - \mu\dot{a}_2^2|, \mu \in \mathbb{R}$,for functions of the form (1.1) belonging to the class Ƀ (see[4,6,8,9,12,16,20,21,26]) .

Definition (1.2) **:** A function $f \in Ƀ$ defined by (1.1) is said to be in the class $Ѣ_q^\alpha(\delta, \lambda, \dot{n}, ẹ, \Theta)$,if the following quasi-subordination condition is hold :

$$\frac{\gamma}{\delta}\left[\left(\frac{zf'(z)}{f(z)}\right)^\alpha + (2\dot{n} + 5ẹ)f(z)\left(\frac{z}{f(z)}\right)' + (4\dot{n} + 3ẹ)\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right] \prec_q (Q(z) - 1), \qquad (1.5)$$

where $(0 \leq \dot{n}, ẹ \leq 1, \alpha \geq 0\ , \gamma, \delta \in \mathbb{C}\ /\ \{0\}), z \in U$.

Remark (1.1) : For $\dot{n} = ẹ = 0$ in definition (1-3), we obtain definition (1.3) obtained by Zayed et al. [22] .

Definition (1.3) : A function $f \in Ƀ$ defined by (1.1) is said to be in the class $Ѣ_q^\alpha(\delta, \lambda, \dot{n}, ẹ, \Theta)$,if the following quasi-subordination condition is hold:

$$\frac{\gamma}{\delta}\left[\left(\frac{zf'(z)}{f(z)}\right)^\alpha - 1\right] \prec_q (Q(z) - 1). \qquad (1.6)$$

Definition (1.4): A function $f \in Ƀ$ defined by (1.1) is said to be in the class $Ꙍ_q(\beta, \tau, ƫ, ħ, \Theta)$,if the following quasi-subordination condition is hold:

$$\frac{\tau}{\beta}\left[(\mathfrak{t}+\hbar)\left(\frac{zf''(z)}{f'(z)}\right)+(\mathfrak{t}-\hbar)z^2\left(\frac{f(z)}{z}\right)'+(2\mathfrak{t}-2\hbar)\left(\frac{\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right)-1\right]\prec_q (Q(z)-1), \qquad (1.7)$$

where $(0\le \mathfrak{t},\hbar\le 1,\beta,\tau\in\mathbb{C}\setminus\{0\}), z\in U$.

Lemma (1.1) [13]; Let $\dot{w}(z)=1+\dot{w}_1 z+\dot{w}_2 z^2+\cdots$ be in the clas B ,then for any complex number μ

$$|\dot{w}_3-\mu\dot{w}_2^2|\le \boldsymbol{max}\{\mathbf{1},|\mu|\}. \qquad (1.8)$$

The result is sharp for the function $\dot{w}(z)=z^2\ \ or\ \dot{w}(z)=z$.

Lemma (1.2) [15]; Let $\Theta(z)=E_0+E_1 z+E_2 z^2+\cdots$ be an Analytic function in U ,such that $|\Theta(z)|\le 1. Then\ |E_0|\le 1$,

and

$$|E_n|\le 1-|E_0^2|\le 1\quad ,n=\{1,2,\dots\}. \qquad (1.9)$$

2-Main Results :

Theorem (2.1) ; A function f is given by (1.1) belong to the class $\mathrm{Ћ}_q^{\alpha}(\delta,\lambda,\dot{n},e,\Theta)$,then

$$|a_2|\le\frac{\delta A_1}{\gamma(\alpha+\dot{n}-e)}\ , \qquad (2.1)$$

and

$$|a_3|\le\frac{\delta A_1}{\gamma(4\alpha+6\dot{n}+2e)}\left[1+max\left\{1,\frac{\delta(\alpha^2-3\alpha+5\dot{n}+2e)}{2\gamma(\alpha+\dot{n}-e)^2}A_1\ +\left|\frac{A_2}{A_1}\right|\right\}\right], \qquad (2.2)$$

and for some $\mu\in\mathbb{C}$,we have

$$\left|\dot{a}_3-\mu\dot{a}_2^2\right|\le\frac{\boldsymbol{\delta}A_1}{\boldsymbol{\gamma}(\boldsymbol{\alpha}+\dot{n}-e)}\left[\mathbf{1}+\mathbf{max}\left\{\mathbf{1},\frac{\boldsymbol{\delta}(\boldsymbol{\alpha}^2-\mathbf{3}\boldsymbol{\alpha}+\mathbf{5}\dot{n}+\mathbf{2}e)}{\mathbf{2}\boldsymbol{\gamma}(\boldsymbol{\alpha}+\dot{n}-e)^2}A_1+\left|\frac{A_2}{A_1}\right|-\frac{\boldsymbol{\delta}(\mathbf{4}\boldsymbol{\alpha}+\mathbf{6}\dot{n}+\mathbf{2}e)}{\boldsymbol{\gamma}(\boldsymbol{\alpha}+\dot{n}-e)^2}A_1|\mu|\right\}\right].(\mathbf{2.3})$$

Proof ; Let $f\in \mathrm{Ћ}_q^{\alpha}(\delta,\lambda,\dot{n},e,\Theta)$ and ,then ,there are analytic functions Q(z) in U with |Q(z)|<1 and $\dot{w}$:U→U ,with $\dot{w}(0)=0$ and $|\dot{w}(z)|<1$, such that

$$\frac{\gamma}{\delta}\left[\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}+(2\dot{n}+5e)f(z)\left(\frac{z}{f(z)}\right)'+(4\dot{n}+3e)\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right]\prec_q=(\Theta(z)[Q(\dot{w}(z))-1]). \qquad (2.4)$$

Since

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha} = 1 + \alpha \dot{a}_2 z + \left[4\alpha \dot{a}_3 - \frac{1}{2}(\alpha^2 - 3\alpha)\dot{a}_2^2\right] z^2 + \cdots, \qquad (2.5)$$

and

$$(2\dot{n} + 5e) f(z)\left(\frac{z}{f(z)}\right)' + (4\dot{n} + 3e)\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1$$
$$= (2\dot{n} + 5e)[-\dot{a}_2 z - (2\dot{a}_3 - \dot{a}_2^2) z^2 + \cdots]$$
$$+ (4\dot{n} + 3e)[1 + 2\dot{a}_2 z + (6\dot{a}_3 - 4\dot{a}_2^2) z^2 + \cdots]. \qquad (2.6)$$

Then implies that

$$\frac{\gamma}{\delta}\left[\left(\frac{zf'(z)}{f(z)}\right)^{\alpha} + (2\dot{n} + 5e) f(z)\left(\frac{z}{f(z)}\right)' + (4\dot{n} + 3e)\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right]$$
$$= \frac{\gamma}{\delta}\Bigg[(1 + 2\dot{n} + e) + (\alpha + \dot{n} - e) a_2 z$$
$$+ \left[(4\alpha + 6\dot{n} + 2e) a_3 - \left(\frac{1}{2}(\alpha^2 - 3\alpha) + 5\dot{n} + 2e\right) a_2^2\right] z^2$$
$$+ \cdots\Bigg], \qquad (2.7)$$

and since

$$Q(\dot{w}(z)) = 1 + \dot{w}_1 A_1 z + (\dot{w}_2 A_1 + \dot{w}_1^2 A_2) z^2 + \cdots \qquad , A_1 > 0$$

$$\Theta(z)[Q(\dot{w}(z)) - 1] = E_0 \dot{w}_1 A_1 z + (E_1 A_1 \dot{w}_1 + E_0 \dot{w}_2 A_1 + E_0 \dot{w}_1^2 A_2) z^2 + \cdots. \qquad (2.8)$$

Putting (2.7) and (2.8) in (2.4) ,and equating ceofficients in both sides, we get

$$a_2 = \frac{\delta E_0 \dot{w}_1 A_1}{\gamma(\alpha + \dot{n} - e)},$$

and

$$a_3 = \frac{\delta A_1 E_0}{\gamma(4\alpha + 6\dot{n} + 2e)}\left[, \frac{\delta(\alpha^2 - 3\alpha + 5\dot{n} + 2e)}{2\gamma(\alpha + \dot{n} - e)^2} A_1 E_0 \dot{w}_1^2 + \dot{w}_1 \frac{E_1}{E_0} + \dot{w}_2 + \dot{w}_1^2 \frac{A_2}{A_1}\right].$$

Thus

$$\dot{a}_3 - \mu \dot{a}_2^2 = \frac{\delta A_1 E_0}{\gamma(4\alpha + 6\dot{n} + 2e)}\Bigg[\frac{\delta(\alpha^2 - 3\alpha + 5\dot{n} + 2e)}{2\gamma(\alpha + \dot{n} - e)^2} A_1 E_0 \dot{w}_1^2 + \dot{w}_1 \frac{E_1}{E_0} + \dot{w}_2$$
$$+ \dot{w}_1^2\left(\frac{A_2}{A_1} - \frac{\delta(4\alpha + 6\dot{n} + 2e)}{\gamma(\alpha + \dot{n} - e)^2} A_1 E_0 \mu\right)\Bigg]$$

Since Q(z) is analytic and bounded in U ,by using lemma (1.2) ,we have

$$|\text{Ę}_n| \leq 1 - |\text{Ę}_0^2| \leq \mathbf{1}, \qquad n \geq 0.$$

By using this fact and the well-known inequality $|\dot{w}(z)|<1$,we get

$$|\dot{a}_3 - \mu \dot{a}_2^2| \leq \frac{\delta A_1}{\gamma(4\alpha + 6\dot{n} + 2ẹ)}\left[\frac{\delta(\alpha^2 - 3\alpha + 5\dot{n} + 2ẹ)}{2\gamma(\alpha + \dot{n} - ẹ)^2} A_1 + \left|\dot{w}_2 + \dot{w}_1^2\left(\frac{A_2}{A_1} - \frac{\delta(4\alpha + 6\dot{n} + 2ẹ)}{\gamma(\alpha + \dot{n} - ẹ)^2} A_1 \text{Ę}_0 \mu\right)\right| + 1\right]$$

Now, we shall use lemma (1.1) to

$$\left[\frac{\delta(\alpha^2 - 3\alpha + 5\dot{n} + 2ẹ)}{2\gamma(\alpha + \dot{n} - ẹ)^2} A_1 + \left|\dot{w}_2 + \dot{w}_1^2\left(\frac{A_2}{A_1} - \frac{\delta(4\alpha + 6\dot{n} + 2ẹ)}{\gamma(\alpha + \dot{n} - ẹ)^2} A_1 \text{Ę}_0 \mu\right)\right| + 1\right],$$

yields the

$$|\dot{a}_3 - \mu \dot{a}_2^2| \leq \frac{\delta A_1}{\gamma(\alpha + \dot{n} - ẹ)}\left[1 + \max\left\{1, \frac{\delta(\alpha^2 - 3\alpha + 5\dot{n} + 2ẹ)}{2\gamma(\alpha + \dot{n} - ẹ)^2} A_1 + \left|\frac{A_2}{A_1}\right| - \frac{\delta(4\alpha + 6\dot{n} + 2ẹ)}{|\mu|\gamma(\alpha + \dot{n} - ẹ)^2} A_1 |\mu|\right\}\right].$$

If we put $\mu = 0$,in above inequality ,we get derived estimate

$$|ạ_3| \leq \frac{\delta A_1}{\gamma(4\alpha + 6\dot{n} + 2ẹ)}\left[1 + max\left\{1, \frac{\delta(\alpha^2 - 3\alpha + 5\dot{n} + 2ẹ)}{2\gamma(\alpha + \dot{n} - ẹ)^2} A_1 + \left|\frac{A_2}{A_1}\right|\right\}\right].$$

Theorem (2.2) ; A function $f \in B$ satisfies:

$$\frac{\gamma}{\delta}\left[\left(\frac{zf'(z)}{f(z)}\right)^{\alpha} + (2\dot{n} + 5ẹ) f(z)\left(\frac{z}{f(z)}\right)' + (4\dot{n} + 3ẹ)\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right] \ll (Q(z) - 1).$$

Then

$$|ạ_2| \leq \frac{\delta A_1}{\gamma(\alpha + \dot{n} - ẹ)},$$

and

$$|ạ_3| \leq \frac{\delta A_1}{\gamma(4\alpha + 6\dot{n} + 2ẹ)}\left[\frac{\delta(\alpha^2 - 3\alpha + 5\dot{n} + 2ẹ)}{2\gamma(\alpha + \dot{n} - ẹ)^2} A_1 + \left|\frac{A_2}{A_1}\right|\right],$$

and for some $\mu \in \mathbb{C}$, we have

$$|\dot{a}_3 - \mu \dot{a}_2^2| \leq \frac{\delta A_1}{\gamma(4\alpha + 6\dot{n} + 2ẹ)}\left[\frac{\delta(\alpha^2 - 3\alpha + 5\dot{n} + 2ẹ)}{2\gamma(\alpha + \dot{n} - ẹ)^2} A_1 + \left|\frac{A_2}{A_1}\right| - \frac{\delta(4\alpha + 6\dot{n} + 2ẹ)}{\gamma(\alpha + \dot{n} - ẹ)^2} A_1 |\mu|\right].$$

Proof: The results following by taking $\dot{w}(z)=z$,in the proof of theorem (2.1) .

Setting $\alpha = 0$ in the theorem (2.1), we get the following corollary

Corollary (2.1) ; A function f is given by (1.1) in the class $Ꙃ_q^{\alpha}(\delta, \lambda, \dot{n}, ẹ, \Theta)$, then

$$|ạ_2| \leq \frac{\delta A_1}{\gamma(\dot{n} - ẹ)},$$

and

$$|a_3| \leq \frac{\delta A_1}{\gamma(6\dot{n}+2e)}\left[1+max\left\{1,\frac{\delta(5\dot{n}+2e)}{2\gamma(\dot{n}-e)^2}A_1 + \left|\frac{A_2}{A_1}\right|\right\}\right],$$

and for some $\mu \in \mathbb{C}$,we have

$$|\dot{a}_3-\mu\dot{a}_2^2| \leq \frac{\delta A_1}{\gamma(\dot{n}-e)}\left[1+\max\left\{1,\frac{\delta(5\dot{n}+2e)}{2\gamma(\dot{n}-e)^2}A_1+\left|\frac{A_2}{A_1}\right|-\frac{\delta(6\dot{n}+2e)}{\gamma(\dot{n}-e)^2}A_1|\mu|\right\}\right]$$

Setting $\alpha = \dot{n} = e = 1$ in theorem (2.1), we get:

Corollary (2.2) ; A function f is given by (1.1) in the class $Ƅ_q^{\alpha}(\delta,\lambda,1,1,\Theta)$,then

$$|a_2| \leq \frac{\delta A_1}{\gamma},$$

and

$$|a_3| \leq \frac{\delta A_1}{12\gamma}\left[1+max\left\{1,\frac{5\delta}{2\gamma}A_1 + \left|\frac{A_2}{A_1}\right|\right\}\right],$$

and for some $\mu \in \mathbb{C}$, we have

$$|\dot{a}_3-\mu\dot{a}_2^2| \leq \frac{\delta A_1}{12\gamma}\left[1+\max\left\{1,\frac{5\delta}{2\gamma}A_1+\left|\frac{A_2}{A_1}\right|-\frac{12\delta}{\gamma}A_1|\mu|\right\}\right].$$

Theorem (2.3) ; A function f is given by (1.1) belong to the class $Ƅ_q^{\alpha}(\delta,\lambda,\Theta)$, then

$$|a_2| \leq \frac{\delta A_1}{\gamma\alpha}, \tag{2.9}$$

and

$$|a_3| \leq \frac{\delta A_1}{4\gamma\alpha}\left[1+max\left\{1,\frac{\delta(\alpha-3)}{2\alpha\gamma}A_1 + \left|\frac{A_2}{A_1}\right|\right\}\right], \tag{2.10}$$

and for some $\mu \in \mathbb{C}$, we have

$$|\dot{a}_3-\mu\dot{a}_2^2| \leq \frac{\delta A_1}{4\gamma\alpha}\left[1+\max\left\{1,\frac{\delta(\alpha-3)}{\alpha\gamma}A_1+\left|\frac{A_2}{A_1}\right|-\frac{\delta}{\gamma\alpha}A_1|\mu|\right\}\right]. \tag{2.11}$$

Proof ; Let $f \in Ƅ_q^{\alpha}(\delta,\lambda,\Theta)$ and ,then ,there are analytic functions Q(z) in U with |Q(z)|<1 and $\dot{w}$:U→U ,with $\dot{w}(0)=0$ and $|\dot{w}(z)|<1$, such that

$$\frac{\gamma}{\delta}\left[\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}-1\right] \prec_q = (\Theta(z)[Q(\dot{w}(z))-1]). \tag{2.12}$$

Since

$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}$ is defined by (2.5) ,then implies

$$\frac{\gamma}{\delta}\left[\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}-1\right]=\frac{\gamma}{\delta}\left[\alpha a_2 z+\left[4\alpha a_3-\left(\frac{1}{2}(\alpha^2-3\alpha)\right)a_2^2\right]z^2+\cdots\right]. \quad (2.13)$$

It follows (2.8) and (2.13) that

$$a_2=\frac{\delta E_0 w_1 A_1}{\gamma\alpha}, \quad (2.14)$$

and

$$a_3=\frac{\delta A_1 E_0}{4\alpha\gamma}\left[,\frac{\delta(\alpha-3)}{\gamma\alpha}A_1 E_0 w_1^2+w_1\frac{E_1}{E_0}+w_2+w_1^2\frac{A_2}{A_1}\right]. \quad (2.15)$$

Thus

$$a_3-\mu a_2^2=\frac{\delta A_1 E_0}{4\alpha\gamma}\left[\frac{\delta(\alpha-3)}{\gamma\alpha}A_1 E_0 w_1^2+w_1\frac{E_1}{E_0}+w_2+w_1^2\left(\frac{A_2}{A_1}-\frac{\delta}{\gamma\alpha}A_1 E_0\mu\right)\right], \quad (2.16)$$

with this fact ,and using lemma (1.1) and lemma (1.2) ,we obtain

$$|a_2|\leq\frac{\delta A_1}{\gamma\alpha},$$

and

$$|a_3|\leq\frac{\delta A_1}{4\gamma\alpha}\left[1+max\left\{1,\frac{\delta(\alpha-3)}{\alpha\gamma}A_1+\left|\frac{A_2}{A_1}\right|\right\}\right].$$

Further

$$|a_3-\mu a_2^2|\leq\frac{\delta A_1}{4\alpha\gamma}\left[1+\max\left\{1,\frac{\delta(\alpha-3)}{\gamma\alpha}A_1+\left|\frac{A_2}{A_1}\right|-\frac{\delta}{\alpha\gamma}A_1|\mu|\right\}\right].$$

Theorem (2.4) ; A function $f\in B$ satisfies

$$\frac{\gamma}{\delta}\left[\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}-1\right]\ll(Q(z)-1).$$

Then

$$|a_2|\leq\frac{\delta A_1}{\gamma\alpha},$$

and

$$|a_3| \leq \frac{\delta A_1}{4\gamma\alpha}\left[1\frac{\delta(\alpha-3)}{\alpha\gamma}A_1 + \left|\frac{A_2}{A_1}\right|\right],$$

and for some $\mu \in \mathbb{C}$,we have

$$|a_3 - \mu a_2^2| \leq \frac{\delta A_1}{4\gamma\alpha}\left[\frac{\delta(\alpha-3)}{\alpha\gamma}A_1 + \left|\frac{A_2}{A_1}\right| - \frac{\delta}{\gamma\alpha}A_1|\mu|\right].$$

Proof: The results following by taking $w(z)=z$ in the proof of theorem (2.3).

Theorem (2.5) ; A function f is given by (1.1) belong to the class $\mathrm{Ю}_q(\beta,\tau,t,\hbar,\Theta)$, then

$$|a_2| \leq \frac{\beta A_1}{2\tau(t-\hbar)}, \tag{2.17}$$

and

$$|a_3| \leq \frac{\beta A_1}{3\tau(t-3\hbar)}\left[1 + max\left\{1, \frac{2\beta(2\hbar-t)}{\tau(t-\hbar)^2}A_1 + \left|\frac{A_2}{A_1}\right|\right\}\right], \tag{2.18}$$

and for some $\mu \in \mathbb{C}$, we have

$$|a_3 - \mu a_2^2| \leq \frac{\beta A_1}{3\tau(t-3\hbar)}\left[1 + \max\left\{1, \frac{2\beta(2\hbar-t)}{\tau(t-\hbar)^2}A_1 + \left|\frac{A_2}{A_1}\right| - \frac{3\beta(t-3\hbar)}{4\tau(t-\hbar)^2}A_1|\mu|\right\}\right]. \tag{2.19}$$

Proof ; Let $f \in \mathrm{Ю}_q(\beta,\tau,t,\hbar,\Theta)$ and ,then ,there are analytic functions $Q(z)$ in U with $|Q(z)|<1$ and $w: U \to U$,with $w(0)=0$ and $|w(z)|<1$, such that

$$\frac{\tau}{\beta}\left[(t+\hbar)\left(\frac{zf''(z)}{f'(z)}\right) + (t-\hbar)z^2\left(\frac{f(z)}{z}\right)' + (2t-2\hbar)\left(\frac{\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right) - 1\right]$$

$$= (\Theta(z)[Q(w(z)) - 1]). \tag{2.20}$$

Since

$$\frac{\tau}{\beta}\left[(t+\hbar)\left(\frac{zf''(z)}{f'(z)}\right) + (t-\hbar)z^2\left(\frac{f(z)}{z}\right)' + (2t-2\hbar)\left(\frac{\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right) - 1\right]$$

$$= \frac{\tau}{\beta}[(t-\hbar) + (2t+2\hbar)a_2 z + [(3t+9\hbar)a_3 - (16\hbar - 8t)a_2^2]z^2 + \cdots]. \tag{2.21}$$

It follows (2.8) and (2.12) that

$$a_2 = \frac{\beta E_0 w_1 A_1}{2\tau(\hbar - t)},$$

and $$\text{ạ}_3 = \frac{\beta A_1 E_0}{3\tau(t-3\hbar)}\left[\frac{2\beta(2\hbar-t)}{\tau(t-\hbar)^2}A_1E_0\dot{w}_1^2+\dot{w}_1\frac{E_1}{E_0}+\dot{w}_2+\dot{w}_1^2\frac{A_2}{A_1}\right]. \quad (2.23)$$

Thus

$$\dot{a}_3-\mu\dot{a}_2^2 = \frac{\beta A_1 E_0}{3\tau(t-3\hbar)}\left[\frac{2\beta(2\hbar-t)}{\tau(t-\hbar)^2}A_1E_0\dot{w}_1^2+\dot{w}_1\frac{E_1}{E_0}+\dot{w}_2+\dot{w}_1^2\left(\frac{A_2}{A_1}-\frac{3\beta(t-3\hbar)}{4\tau(t-\hbar)^2}A_1E_0\mu\right)\right], (2.24)$$

with this fact and using lemma (1.1) and lemma (1.2), we obtain

$$\left|\dot{a}_3-\mu\dot{a}_2^2\right| \le \frac{\beta A_1}{3\tau(t-3\hbar)}\left[1+max\left\{1,\frac{2\beta(2\hbar-t)}{\tau(t-\hbar)^2}A_1+\frac{A_2}{A_1}-\frac{3\beta(t-3\hbar)}{4\tau(t-\hbar)^2}A_1|\mu|\right\}\right].$$

If we put $\mu = 0$,in above inequality ,we get derived estimate

$$|\text{ạ}_3| \le \frac{\beta A_1}{3\tau(t-3\hbar)}\left[1+max\left\{1,\frac{2\beta(2\hbar-t)}{\tau(t-\hbar)^2}A_1+\frac{A_2}{A_1}\right\}\right].$$

Theorem (2.6) ; A function $f \in B$ satisfies

$$\frac{\tau}{\beta}\left[(t+\hbar)\left(\frac{zf''(z)}{f'(z)}\right)+(t-\hbar)z^2\left(\frac{f(z)}{z}\right)'+(2t-2\hbar)\left(\frac{\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right)-1\right] \ll (Q(z)-1).$$

Then

$$|\text{ạ}_2| \le \frac{\beta A_1}{2\tau(t-\hbar)},$$

and

$$|\text{ạ}_3| \le \frac{\beta A_1}{3\tau(t-3\hbar)}\left[\frac{2\beta(2\hbar-t)}{\tau(t-\hbar)^2}A_1+\left|\frac{A_2}{A_1}\right|\right],$$

and for some $\mu \in \mathbb{C}$, we have

$$|\dot{a}_3-\mu\dot{a}_2^2| \le \frac{\beta A_1}{3\tau(t-3\hbar)}\left[\frac{2\beta(2\hbar-t)}{\tau(t-\hbar)^2}A_1+\left|\frac{A_2}{A_1}\right|-\frac{3\beta(t-3\hbar)}{4\tau(t-\hbar)^2}A_1|\mu|\right].$$

Proof: The results following by taking $\dot{w}(z)=z$ in the proof of theorem (2.5).

Setting t=1, and $\hbar$=0 in the theorem (2.5), we get the following corollary

Corollary (2.3); A function f is given by (1.1) in the class $\text{Ю}_q(\beta,\tau,1,\Theta)$, then

$$|a_2| \leq \frac{\beta A_1}{2\tau},$$

and

$$|a_3| \leq \frac{\beta A_1}{3\tau}\left[1 + max\left\{1, \left|\frac{A_2}{A_1}\right| - \frac{2\beta}{\tau} A_1\right\}\right],$$

and for some $\mu \in \mathbb{C}$, we have

$$|a_3 - \mu a_2^2| \leq \frac{\beta A_1}{3\tau}\left[1 + \max\left\{1, \left|\frac{A_2}{A_1}\right| - \frac{2\beta}{\tau} A_1 - \frac{3\beta}{4\tau} A_1 |\mu|\right\}\right].$$

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